Some counter-examples in the theory of the Galois module structure of wild extensions


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SOME COUNTER-EXAMPLES IN THE THEORY
OF THE GALOIS MODULE STRUCTURE
OF WILD EXTENSIONS

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Let \( \mathfrak{M} \) be a maximal order (over a Dedekind ring) in a (finite dimensional) semisimple algebra \( A \) then (with \( \tilde{K}_0 \) denoting rankless objects in \( K_0 \), \( C \) a picard group and \( Z \) the centre of an order) we have a homomorphism

\[
Nrd_0 : \tilde{K}_0(\mathfrak{M}) \to C(Z(\mathfrak{M}))
\]

by

\[
(\alpha \mathfrak{M}) \mapsto (\nu(\alpha) Z(\mathfrak{M}))
\]

where \( \alpha \) is an idele of \( A \) and \( \nu(\alpha) \) is its reduced norm. (If \( M \) is an \( \mathfrak{M} \)-module and \( M \otimes \mathfrak{M} \cong A^n \) then we put

\[
(M) = [M] - n[\mathfrak{M}].
\]

We note that if \( A \) is simple and split (so \( Nrd_0 \) is an isomorphism), \( P \) a minimal projective of \( \mathfrak{M} \) and \( a \) a non-zero ideal of \( Z(\mathfrak{M}) \) then

\[
Nrd_0 ([aP] - [P]) = a.
\]  

We consider here the following problem. Let \( \mathcal{O} \) be the ring of integers in a number field. Let \( G \) be a group of automorphisms of \( \mathcal{O} \) and \( \mathfrak{M} \) a maximal order containing \( ZG \). It was thought possible that one of

\[
x = Nrd_0(\mathcal{O} \otimes ZG \mathfrak{M}) \quad \text{or} \quad y = Nrd_0(\mathcal{O} \otimes ZG \mathfrak{M}/\text{torsion})
\]

was zero or, at least, independent of \( \mathfrak{M} \). (This is certainly true when \( \mathcal{O}/\mathcal{O}^G \) is tamely ramified see [5] Theorem 3). Cougnard [2] has

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shown that \( y \) need not be zero and that \( x \) is often zero \([2],[3]\). However, we here present an example (with \( O^G = \mathbb{Z} \)) where neither \( x \) nor \( y \) is even independent of the choice of \( \mathcal{M} \).

The presentation of this example falls into four parts. We use a group \( G \) (in fact \( C_{23} \times D_{18} \)) such that the images of \( \mathbb{Z} G \) in the semisimple components of \( Q G \) are twisted group rings (with trivial cocycle) with the group action tamely ramified and faithful. Lemma 1 investigates the maximal orders containing such rings and lemmas 1 and 2 calculate the ‘difference’ between the extensions of a module to two different maximal orders, showing that this depends only on the local structure of the module at the primes where the maximal orders differ. Lemma 3 shows that in certain circumstances the local structure of a module over one twisted group ring may be determined from its local structure over another. (The point here is that the modules which we investigate are expressed as quotients of \( \mathbb{Z} G \)-modules which are not themselves modules over the image of \( \mathbb{Z} G \) in question. They are, however, modules over a different twisted group ring with the same group.) Lemma 4 and the work which follows construct the example. Lastly, the Theorem uses the four lemmas to expose the desired properties of the example.

If \( M \) is a subset of an \( A \)-module then \( \Lambda_A(M) \) denotes \( \{ a \in A \text{ such that } Ma \subseteq M \} \).

**Lemma 1.**—Let \( S \) be a Dedekind ring with field of quotient \( L \) and group of automorphisms \( \Gamma \). Suppose that \( S \) is tamely ramified over \( S^\Gamma = R \). Let \( I \) and \( J \) be ambiguous ideals of \( S \).

(i) \( \mathcal{M}(I) = \Lambda_{L\Gamma}(I) \) is a maximal order containing \( SF \) and

(ii) all such orders arise in this way. (Here \( SF \) and \( LF \) denote the appropriate twisted group rings.)

(iii) The minimal projectives of \( \mathcal{M}(I) \) are (isomorphic to) the \( aI \) where \( a \) is an ideal of \( R \).

(iv) \( \mathcal{M}(I) = \mathcal{M}(J) \) if and only if \( IJ^{-1} \) is extended from an ideal of \( R \).

(v) \( J \mathcal{M}(I) = J \otimes_{ST} \mathcal{M}(I) = aI \) where \( a \) is the minimal \( R \)-ideal such that \( aI \supseteq J \). i.e. \( a^{-1} = (IJ^{-1}) \cap K, \) where \( K = L^\Gamma \).
Proof. — (i) Considering \( L \Gamma \) as \( \text{End}_K(L) \), we have \( \Lambda_{L\Gamma}(I) = \text{End}_R(I) \) and so, locally, \( \Lambda_{L\Gamma}(I) = \text{Mat}_{|I\Gamma|}(R) \).

(ii) Let \( \mathcal{A} \) be a maximal order containing \( S\Gamma \). A minimal projective of \( \mathcal{A} \) may be considered as a sublattice (and hence an ambiguous \( S \)-ideal) \( I \) of \( L \). Clearly \( \mathcal{A}(I) = \mathcal{A} \). Moreover if \( J \subseteq L \) is another minimal projective of \( \mathcal{A} \) the local \( \mathcal{A} \)-isomorphisms from \( I \) to \( J \) extend to local \( L \Gamma \)-isomorphisms from \( L \) to \( L \). Together these may be expressed as an idele \( \alpha \) of \( K \) and so \( J = \alpha I \) where \( \alpha \) is the ideal of \( R \) corresponding to \( \alpha \). As, clearly, \( \mathcal{A}(I) = \mathcal{A}(\alpha I) \) for any \( R \)-ideal \( \alpha \) we have (iii).

(iv) Is immediate as, from the proof of (i), we have that \( I \) is a minimal projective of \( \mathcal{A}(I) \).

(v) \( J\mathcal{A}(I) \) is a minimal projective of \( \mathcal{A}(I) \) containing \( J \).

With notation as in Lemma 1 we define \( \det^1 : K_0(S\Gamma) \to C(R) \) by \( \det^1([M]) = (\alpha) \) where \( M \otimes_{S\Gamma} \mathcal{A}(I) \cong \alpha_i I \) (where the \( \alpha_i \) are \( R \)-ideals) and \( \alpha = \prod \alpha_i \). (In fact \( M \otimes \mathcal{A}(I) \cong \alpha I \oplus \ldots \oplus I \) in this case. We can also use the injection \( C(R) \to C(S,\Gamma) \) to describe \( \alpha \) as the equivariant Steinitz class : \( \det_S(M \otimes_{S\Gamma} \mathcal{A}(I), I^{-1}) \in C(S,\Gamma) \), see [7]). We define \( \delta_1(x) = \det^1(x) \det^S(x)^{-1} \).

Lemma 2. — Let \( I \) and \( J \) be ambiguous ideals of \( S \).

(i) If \( x \in \tilde{K}_0(S\Gamma) \) then \( \text{Nrd}_0(x \otimes_{S\Gamma} \mathcal{A}(I)) = \det^1(x) \)

(ii) \( \det^1(J) = ((IJ^{-1}) \cap K)^{-1} \)

For every prime ideal \( \mathfrak{p} \neq 0 \) of \( R \) let \( \phi(\mathfrak{p}) \) be the maximal ambiguous \( S \)-ideal lying over \( \mathfrak{p} \). Then \( \mathfrak{p} S = \phi(\mathfrak{p})^e(\mathfrak{p}) \).

(iii) \( \delta_1(J) = \prod_{\mathfrak{p} \mid \mathfrak{p}} \mathfrak{p} \left[ \begin{array}{c} j(\mathfrak{p}) - \ell(\mathfrak{p}) \\ \epsilon(\mathfrak{p}) \end{array} \right] - \left[ \begin{array}{c} j(\mathfrak{p}) \\ \epsilon(\mathfrak{p}) \end{array} \right] \) where \( I_p = \phi(\mathfrak{p})I_p(\mathfrak{p}) \)

and \( J_p = \phi(\mathfrak{p})J_p(\mathfrak{p}) \).

(Here \([x]\) denotes the greatest integer not greater than \( x \).)

(iv) In particular if \( I = \phi(\mathfrak{p})^i \) then \( \delta_1(x) \) depends only on the image of \( x \) in \( K_0(S_p\Gamma) \) — or, indeed, that in \( K_0(S_\mathfrak{p}\Gamma(\mathfrak{p})) \), where \( \mathfrak{p} \) is a prime of \( S \) over \( \mathfrak{p} \) and \( \Gamma(\mathfrak{p}) \) its decomposition group.

(v) Specifically, if \( |\Gamma| = e = 2 \) and \( x \) goes to \( n[S_p] + m[\phi(\mathfrak{p})] \)

in \( K_0(S_p\Gamma) \) then \( \delta_{\phi(\mathfrak{p})}(x) = (\mathfrak{p}^{-n}) \).
Proof. — (i) If \( x \in \tilde{K}_0(S^\Gamma) \) then
\[
x \otimes K(1) = \sum_{i=j}^n n_i[a_i \bar{1}] \quad \text{where} \quad \Sigma n_i = 0
\]
\[
= \sum_{i}^n n_i([a_i \bar{1}] - [1]).
\]
The first expression gives \( \det^f(x) = \Pi a^i \) and by (1) the second gives \( \text{Nrd}(x \otimes K(1)) = \Pi a^i \).

(ii) Is immediate from Lemma 1 (v).

(iii) \( (\Pi^{-1}) = (\phi(\varphi)^{i-j}) \) and \( \phi(\varphi) \cap K = \varphi \cdot \left[ \frac{-n}{e(\varphi)} \right] \).

(iv) From (iii), if \( 1 = \phi(\varphi)^i \) then \( \delta_i(J) \) depends only on \( \varphi \) mod \( e \) and as the distinct irreducible projectives of \( S_p \) are \( \phi(\varphi)^i \), \( i = 0, \ldots, e - 1 \) and those of \( \Psi \) are \( \Psi^i \), \( i = 0, \ldots, e - 1 \), see [6], \( \delta_i(J) \) is determined by the image of \( J \) in \( K_0(S_p \) or in \( K_0(S_p \Gamma) \). The general result follows.

(v) \( \delta_{\psi}(n[S] + m[\phi(\varphi)]) = \varphi \left( n\left( \left[ \frac{0-1}{2} \right] - \left[ \frac{0}{2} \right] \right) + m\left( \left[ \frac{1-1}{2} \right] - \left[ \frac{1}{2} \right] \right) \right) 
= (\varphi^{-n}) \).

Lemma 3. — Let \( S \) be a complete discrete valuation ring with maximal ideal \( \mathfrak{p} \). Let \( \Gamma \) be a finite group of automorphisms of \( S \) with inertia subgroup \( C = \langle \tau \rangle \) of order \( e \) prime to \( |S/\mathfrak{p}| \). Then \( P_i = \mathfrak{p}^i \), \( i = 0, \ldots, e - 1 \) are the distinct minimal projectives of \( S^\Gamma \) and \( \mu_i = \mathfrak{p}^{i}/\mathfrak{p}^{i+1} \), \( i = 0, \ldots, e - 1 \) are the distinct simple \( S^\Gamma \)-modules. Choose a generator \( \pi \) of \( \mathfrak{p} \) such that \( \pi^e/\pi = \eta \) is an \( e^{th} \) root of 1. Let \( R_0 \) be a sub-valuation-ring of \( S^C \) containing \( \eta \) with \( \varphi_0 = R_0 \cap \mathfrak{p} \) and suppose that \( r = \text{rk}_{R_0}(S) \) and \( f = \dim_{R_0/V_0}(S/\mathfrak{p}) \) are finite.

Let \( S', \mathfrak{p}', \Gamma', C', P'_i, \mu'_i, \pi', \eta', R'_0, r' \) be another such set of data with \( C' = C \) and \( R'_0 = R_0 \).

(i) Choose \( x \in G'_0(S^\Gamma) \) and \( x' \in G'_0(S'^\Gamma) \) such that their images in \( G'_0(R_0C) \) agree. If \( x = \Sigma n_i[a_i \bar{1}] \) then \( x' = \Sigma f' \eta_i'[\mu'_i] \) where \( \eta'_i = \eta^i \) (suffixes are taken modulo \( e \)).
(ii) Let $M$ be an $S\Gamma$- and an $S'T'$-module with identical $R_0C$ actions and commuting $S$ and $S'$ actions.

If $[M] = \sum_{i=0}^{e-1} m_i [P_i] \in G_0(S\Gamma)$ and $[M'] = \sum_{i=0}^{e-1} m'_i [P'_i] \in G_0(S'T')$

then $m'_i = m'_{i-1} + \frac{f}{f'} (m_{it} - m_{(i-1)t})$.

(iii) In particular if $|\Gamma| = |\Gamma'| = 2 = e$ and

$$[M]_{S\Gamma} = a([P_0] + [P_1]) + b[P_1]$$

then

$$[M]_{S'T'} = a'( [P'_0] + [P'_1]) + \frac{bf}{f'} [P'_1]$$

where

$$a' = a \frac{r}{r'} + b \left( \frac{r}{r'} - \frac{f}{f'} \right).$$

Proof. — (i) Let $\nu_i = (R_0/\nu_0) C/(\tau - \eta^i)$, $i = 0, \ldots, e - 1$, be the simple $R_0C$-modules. Then we have restriction isomorphisms

$$G'_0(S\Gamma) \cong G'_0(R_0C) \cong G'_0(S'T'),$$

where $[\mu_i] \mapsto f[\nu_i]$ and $[\mu'_i] \mapsto f'[\nu_{i+1}]$. The result follows.

(ii) Let $M = M_1/M_2$ where $M_1, M_2$ are torsion-free $S \otimes_R S'C$-modules (we assume $\Gamma = \Gamma' = C$ in view of the restriction isomorphisms between $K_0(S\Gamma)$ and $K_0(SC)$ etc.) and put $\tilde{M} = M_1 \pi'/M_2 \pi'$. If $N \subseteq M$, $N \cong_{S'C} P_j$ then $N\pi' \cong_{S'C} P_{j+t}$.

Hence in $G_0(SC)$

$$[M] - [N] = [M_1] - [M_1 \pi'] - [M_2] + [M_2 \pi'] = \sum m_i ([P_i] - [P_{i+1}]) = \sum (m_i - m_{i-1}) [P_i].$$

Now if $[M_1/M_1 \pi'] - [M_2/M_2 \pi'] = \sum x_i [\mu_i]$ in $G'_0(SC)$ then

$$[M] - [\tilde{M}] = \sum x_i ([P_i] - [P_{i+1}]) = \sum (x_i - x_{i-1}) [P_i]$$

in $G_0(SC)$.

But, in $G'_0(S'C)$, $[M_1/M_1 \pi'] - [M_2/M_2 \pi']$ is the 'semisimplication' of $[M_1] - [M_2]$ and so is $\sum m'_i [\mu_i]$. So

$$m'_i = \frac{f}{f'} x_{it} = m'_{i-1} + \frac{f}{f'} (m_{it} - m_{(i-1)t})$$

(iii) follows.

Let $\Delta$ be a finite group. The intersection, $\mathfrak{H}$, of all maximal orders which contain $Z\Delta$ is the minimal hereditary order containing $Z\Delta$. I am indebted to Anne-Marie Berge for the following cons-
truction of a Galois extension $\hat{L}$ of $Q_p$, $p$ an odd prime, such that, with $\Delta = \text{Gal}(\hat{L}/Q_p)$ and $\hat{S} = \text{int}(\hat{L})$, $\hat{S}\not\subseteq \hat{S}_p$. (The fact that this is so will emerge later although my proof, using the preceeding lemma, is somewhat different from that of Mlle Bergé.)

**Lemma 4.** (i) Let $\hat{L}_0 = Q_p[\sqrt{p}]$ and let $\hat{L}$ be the extension of $\hat{L}_0$ with norm group $N = \langle Q^*_p \rangle, (1 + \sqrt{p})^2 \subseteq \hat{L}_0$. Putting $H = \text{Gal}(\hat{L}/\hat{L}_0)$ we have ramification groups $H^0 = H^1 \cong C_p, H^2 = H^3 \cong C_p, H^4 = \{1\}$ and $\text{Gal}(\hat{L}/Q_p)$ is dihedral.

(ii) $H_0 = H_1 \cong C_p, H_2 = \ldots = H_{2p+1} \cong C_p, H_{2p+2} = \{1\}$.

(iii) Let $\sigma$ be a generator of $H$ and $\pi_1$ a prime of $\hat{S}_1 = S^{(obs)}$ then $\text{tr}_{o_p} \hat{S} = \pi_1^{2p-1} \hat{S}_1$.

**Proof.** (i) $\hat{L}/\hat{L}_0$ is clearly cyclic of degree $p^2$ and, as $(1 + \sqrt{p})(1 - \sqrt{p}) = 1 - p \in N, \hat{L}/Q_p$ is dihedral. Moreover $1 + \sqrt{p} \in U^1 \setminus U^2$ and $(1 + \sqrt{p})^2 = 1 + p\sqrt{p} + (\sqrt{p})^2 + \ldots \in U^3 \setminus U^4$ and the result follows by local class field theory.

(ii) $H^1 = H^0(\phi)$ where $\psi(x) = \int_0^x H^0 : H^1 \setminus du$ so $\psi(0) = 0, \psi(1) = 1, \psi(1 + r) = 1 + rp, 0 < r < 2$. Hence the result.

(iii) Put $H' = \text{Gal}(\hat{L}/\hat{L}(a_p))$. Then $H'_0 = H'_1 = \ldots = H'_{2p+1} = C_p, H'_{2p+2} = \{1\}$.

So the value of the different of $\hat{S}$ over $\hat{S}_1$ is $(2p + 2)(p - 1) = 2p^2 - 2$.

Therefore $\text{tr}_{o_p}(\hat{S}) = \pi_1 \left[\frac{2p^2 - 2}{p}\right] \hat{S}_1 = \pi_1^{2p-1} \hat{S}_1$.

Choose a prime $p > 2$ and $n \in N^*$ prime to $p$ such that, with $\theta = \sqrt[n]{1}$, the prime $\nu_0$ over $p$ in $Z[\theta]$ is not principal (e.g. $p = 3, n = 23$). We put $G = C_n \times (C_p \times \Gamma)$ where $C_p = \langle \sigma \rangle, \Gamma = \langle \tau \rangle$ of order 2 and $\sigma^p = \sigma^{-1}$. Put $T = Z[\theta, \xi], \xi = \frac{\rho^2}{\sqrt{1}}$, and $T_1 = Z[\theta, \xi^p]$ and let $\psi_1, \nu_1, \psi, \nu$ be the primes over $\nu_0$ in $T_1, T^\Gamma, T, T^\Gamma$ where $\tau: \{\theta \mapsto \theta, \xi \mapsto \xi^{-1}\}$. We note that all these primes are non-principal.

Choose a $D_{2p^2} (= C_p \times \Gamma)$-extension $L$ of $Q$ such that $L_p$ is the $\hat{L}$ in lemma 4 (we can do this as $p$ is odd see [1] ch. 10
Thm 5), and an extension $M$ of $Q$ disjoint from $L$, cyclic of degree $n$ and non-ramified where $L$ is ramified. Then $E = LM$ is a galois extension of $Q$ with group $G$ and if $S$, $V$ and $W$ are the rings of integers of $L$, $M$ and $E$ respectively we have $W = SV = S \otimes Z V$.

We put $L_1 = L^{(a^p)}$ and $L_0 = L^{(a)}$ with rings of integers $S_1$, $S_0$ and we choose $\pi, \pi_1, \pi_0$, prime elements of $L, L_1, L_0$ so that $\pi^r = - \pi$ etc.

Let $\chi, \chi_1$, be the characters of $G$ induced from the $C_{np^2}$ characters $\rho \mapsto \theta$, $\sigma \mapsto \xi$ and $\rho \mapsto \theta$, $\sigma \mapsto \xi^p$ and let $A\chi, A\chi_1$ be the corresponding factors of $QG$. Choose $\mathfrak{N}$, $\mathfrak{N}_1$, maximal orders containing the images of $ZG$ in the complements of $A\chi$ and $A\chi_1$. Note that the projections of $ZG$ into $A\chi$ and $A\chi_1$ are, respectively, $TT$ and $T_1 \Gamma$. We recall that $\mathfrak{N}(T)$ is a maximal order containing $TT$ etc.

**THEOREM.** — *With the above data*

(i) $\text{Nrd}_0(W \otimes_{ZG} (\mathfrak{N} \oplus \mathfrak{N}(\Psi))) \neq \text{Nrd}_0(W \otimes_{ZG} (\mathfrak{N} \oplus \mathfrak{N}(\mathfrak{T}))).

(ii) $\text{Nrd}_0(W, (\mathfrak{N}_1 \oplus \mathfrak{N}(\Psi))) \neq \text{Nrd}_0(W, (\mathfrak{N}_1 \oplus \mathfrak{N}(\mathfrak{T}))).$

**Proof.** — (i) We write $tr = tr_{L/L_1}$. As $Z[\xi] = Z[a]/(1 + a^{p^2} + \ldots a^{p^{(p-1)}}),

[S \otimes_{ZP^2} Z_p[\xi]]_{S_0} = [S/\text{tr}S] = [\hat{S}] - [\text{tr}S] = [\hat{S}] - [\pi^{2p-1} S_1]

by lemma 4

$= \hat{S}_0 + [\pi \hat{S}_0] + \ldots + [\pi^{2p-1} \hat{S}_0]

- [\pi^{2p-1} S_0] - \ldots - [\pi^{3p-2} S_0]

= \frac{p^2 - p + 2}{2} \left[\hat{S}_0 \Gamma\right] - 2[\hat{\Phi}_0]

where $\hat{\Phi}_0 = \pi_0 \hat{S}_0$ as $\pi^i \hat{S}_0 \cong s_0 \Gamma, \pi^i \hat{S}_0 \cong \hat{\Phi}_0$ and $\hat{S}_0 \Gamma \cong s_0 \Gamma, \hat{S}_0 \oplus \hat{\Phi}_0$.

Also $V \otimes_{ZC_n} Z[\theta]_{p_0} \cong Z[\theta]_{p_0}$ as $V/Z$ is tame at $p$ ($(n, p) = 1$).

Hence

$[W \otimes_{ZC_{np^2}} T_\Psi]_{S_0} = [(V \otimes Z S) \otimes_{ZC_n \times C_{p^2}} (Z[\theta]_{p_0} \otimes_{Z_p} Z_p[\xi])]

= [(V \otimes_{ZC_n} Z[\theta]_{p_0}) \otimes_{Z_p} (S \otimes_{ZC_{p^2}} Z_p[\xi])]

= f \left(\frac{p^2 - p + 2}{2}\right) \left[\hat{S}_0 \Gamma\right] - 2f[\hat{\Phi}_0]$, 


where \( f = \text{rk}_Z(Z[\theta]_{p_0}) = \dim_{F_p}(Z[\theta]_{p_0}) = \dim_{F_p}(T/\mathfrak{B}) \). Now \( \dim_{F_p}(\hat{S}_0/\hat{\Phi}_0) = 1 \) so, by lemma 3 (iii),

\[
[W \otimes_{Z_{C_{np}}} T_{p_1}]/_{T_{p_1}} = s[T_{p_1}] - 2[\mathfrak{B}_{p_1}] \quad \text{for some } s
\]

\[= 2[T_{\mathfrak{B}}] \quad \text{as its } T\text{-rank is } 2.\]

Hence

\[
\text{Nrd}_0(W \otimes_{Z_G} (\mathfrak{M}_1 \oplus \mathfrak{M}_G(\mathfrak{B}))) \times \text{Nrd}_0(W \otimes_{Z_G} (\mathfrak{M}_1 \oplus \mathfrak{M}_G(T)))^{-1}
\]

\[= \delta_{\mathfrak{B}}([W \otimes_{Z_G} T[T]]) - [T[T]]) \quad \text{by Lemma } 2 \text{ (i)}
\]

\[= \delta_{\mathfrak{B}}([W \otimes_{Z_{C_{np}}} T[T]]) - [T[T]]) = \delta_{\mathfrak{B}}([T] - [\mathfrak{B}]) \quad \text{by Lemma } 2 \text{ (iv)}
\]

\[= (p^{-1}) \quad \text{by Lemma } 2 \text{ (v)}
\]

\[\neq 0.\]

(ii) The kernel of the epimorphism, \( s \otimes 1 \mapsto (\text{tr } s) \)

\[
S \otimes_{Z_{C_{np}}} Z[t^p] \twoheadrightarrow \text{tr } S/(S_0 \cap \text{tr } S)
\]

is a torsion group as the ranks of the two modules are equal. The image module is, however, torsion-free and so

\[
[(S \otimes_{Z_{C_{np}}} 2Z[t^p])/(\text{torsion})]_{S_0} = [\text{tr } \hat{S}/\text{tr } \hat{S} \cap \hat{S}_0] = [\pi_1^2 p^{-1} \hat{S}_1] + [\pi_2^2 \hat{S}_0]
\]

\[= \frac{p - 3}{2} [\hat{S}_0] + 2[\hat{\Phi}_0] \ldots \quad (2)
\]

Hence

\[
[W \otimes_{Z_{C_{np}}} T_{1,\mathfrak{B}_1}]/(\text{torsion})]_{S_0} = [Z[\theta]] \otimes_Z \left(\frac{p - 3}{2} [\hat{S}_0] + 2[\hat{\Phi}_0]\right)
\]

\[= \frac{p - 3}{2} \text{tr}[\hat{S}_0] + 2f[\hat{\Phi}_0]. \quad \text{(cf (1))}
\]

Hence

\[
[W \otimes_{Z_{C_{np}}} T_{1,\mathfrak{B}_1}]/(\text{torsion})]_{T_1} \mathfrak{B}_1 = s[(T_1^1)]_{\mathfrak{B}_1} + 2[\mathfrak{B}_1, \mathfrak{B}_1]
\]

\[= 2[\mathfrak{B}_1, \mathfrak{B}_1] \quad \text{by Lemma } 3 \text{ (iii)}
\]

Hence

\[
\text{Nrd}_0(W . (\mathfrak{M}_1 \oplus \mathfrak{M}_G(\mathfrak{B}_1))) \cdot \text{Nrd}_0(W . (\mathfrak{M}_1 \oplus \mathfrak{M}_G(T_1)))^{-1}
\]

\[= \text{Nrd}_0(W \otimes_{Z_G} \mathfrak{M}_1(\mathfrak{B}_1)/\text{torsion}) \cdot \text{Nrd}_0(W \otimes_{Z_G} \mathfrak{M}_G(T_1)/\text{torsion})^{-1}
\]
= \delta_{\psi_1} ([W \otimes \mathbb{Z}_c^{np^2} \text{torsion}] - [T \Gamma]) \quad \text{(as } \otimes_{T \Gamma} \mathcal{M}(I) \text{ is exact)}

= \delta_{\psi_1} ([\mathbb{F}_1] - [T_1]) = (p_1) \quad \text{by Lemma 2 (iv) and (v)}

\neq 0.

Note that, from (2), we deduce easily, using lemma 3, that

\text{S} \cdot \mathbb{Z} [\mathfrak{p}^p] \cong_{\mathbb{Z}[\mathfrak{p}^p] \Gamma} (1 - \xi) \oplus (1 - \xi) \not\cong \mathbb{Z}[\mathfrak{p}^p] \Gamma \quad \text{showing that, in the notation above with } \Delta = D_{2p^2}, \text{ S} \mathcal{H}_p \not\cong \mathcal{H}_p. \quad \text{Also, of course, from (3) we have that, with } \Delta = G, \text{ W} \cdot \mathcal{H}_p \not\cong \mathcal{H}_p.

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