ANNALES DE L'INSTITUT FOURIER

STEPHEN M. J. WILSON

Some counter-examples in the theory of the Galois module structure of wild extensions

Annales de l'institut Fourier, tome 30, nº 3 (1980), p. 1-9

<http://www.numdam.org/item?id=AIF_1980__30_3_1_0>

Ann. Inst. Fourier, Grenoble
30, 3 (1980), 1-9

SOME COUNTER-EXAMPLES IN THE THEORY OF THE GALOIS MODULE STRUCTURE OF WILD EXTENSIONS

by Stephen M. J. WILSON (*)

Let $\mathfrak{M}$ be a maximal order (over a Dedekind ring) in a (finite dimensional) semisimple algebra A then (with $\widetilde{K}_0$ denoting rankless objects in K_0, C a picard group and Z the centre of an order) we have a homomorphism

$$\mathrm{Nrd}_0 : \widetilde{K}_0(\mathfrak{M}) \to C(Z(\mathfrak{M}))$$

by $$(\alpha\mathfrak{M}) \mapsto (\nu(\alpha)\, Z(\mathfrak{M})) \qquad [4]$$

where α is an idele of A and $\nu(\alpha)$ is its reduced norm. (If M is an $\mathfrak{M}$-module and $M \otimes_{\mathfrak{M}} A \cong A^{(n)}$ then we put

$$(M) = [M] - n[\mathfrak{M}].).$$

We note that if A is simple and split (so Nrd_0 is an isomorphism), P a minimal projective of $\mathfrak{M}$ and $\mathfrak{a}$ a non-zero ideal of $Z(\mathfrak{M})$ then

$$\mathrm{Nrd}_0([\mathfrak{a}P] - [P]) = \mathfrak{a}\ . \qquad (1)$$

We consider here the following problem. Let $\mathcal{O}$ be the ring of integers in a number field. Let G be a group of automorphisms of $\mathcal{O}$ and $\mathfrak{M}$ a maximal order containing $\mathbf{Z}G$. It was thought possible that one of

$$x = \mathrm{Nrd}_0(\mathcal{O} \otimes_{\mathbf{Z}G} \mathfrak{M}) \quad \text{or} \quad y = \mathrm{Nrd}_0(\mathcal{O} \otimes_{\mathbf{Z}G} \mathfrak{M}/\text{torsion})$$

was zero or, at least, independent of $\mathfrak{M}$. (This is certainly true when $\mathcal{O}/\mathcal{O}^G$ is tamely ramified see [5] Theorem 3). Cougnard [2] has

(*) This paper was written while the author, supported by a followship from the Science Research Council of Great Britain, was enjoying the hospitality of the University of Bordeaux.

shown that y need not be zero and that x is often zero [2], [3]. However, we here present an example (with $\mathcal{O}^G = \mathbf{Z}$) where neither x nor y is even independent of the choice of $\mathfrak{M}$.

The presentation of this example falls into four parts. We use a group G (in fact $C_{23} \times D_{18}$) such that the images of $\mathbf{Z}G$ in the semisimple components of $\mathbf{Q}G$ are twisted group rings (with trivial cocycle) with the group action tamely ramified and faithful. Lemma 1 investigates the maximal orders containing such rings and lemmas 1 and 2 calculate the 'difference' between the extensions of a module to two different maximal orders, showing that this depends only on the local structure of the module at the primes where the maximal orders differ. Lemma 3 shows that in certain circumstances the local structure of a module over one twisted group ring may be determined from its local structure over another. (The point here is that the modules which we investigate are expressed as quotients of $\mathbf{Z}G$-modules which are not themselves modules over the image of $\mathbf{Z}G$ in question. They are, however, modules over a different twisted group ring with the same group.) Lemma 4 and the work which follows construct the example. Lastly, the Theorem uses the four lemmas to expose the desired properties of the example.

If M is a subset of an A-module then $\Lambda_A(M)$ denotes $\{a \in A$ such that $Ma \subseteq M$.

LEMMA 1. – *Let* S *be a Dedekind ring with field of quotient* L *and group of automorphisms* Γ. *Suppose that* S *is tamely ramified over* $S^\Gamma = R$. *Let* I *and* J *be ambiguous ideals of* S.

(i) $\mathfrak{M}(I) = \Lambda_{L\Gamma}(I)$ *is a maximal order containing* $S\Gamma$ *and*

(ii) *all such orders arise in this way. (Here* $S\Gamma$ *and* $L\Gamma$ *denote the appropriate twisted group rings.)*

(iii) *The minimal projectives of* $\mathfrak{M}(I)$ *are (isomorphic to) the* $\mathfrak{a}I$ *where* $\mathfrak{a}$ *is an ideal of* R.

(iv) $\mathfrak{M}(I) = \mathfrak{M}(J)$ *if and only if* IJ^{-1} *is extended from an ideal of* R.

(v) $J\mathfrak{M}(I) = J \otimes_{S\Gamma} \mathfrak{M}(I) = \mathfrak{a}I$ *where* $\mathfrak{a}$ *is the minimal* R-*ideal such that* $\mathfrak{a}I \geqslant J$. i.e. $\mathfrak{a}^{-1} = (IJ^{-1}) \cap K$, *where* $K = L^\Gamma$.

Proof. – (i) Considering $L\Gamma$ as $\mathrm{End}_K(L)$, we have $\Lambda_{L\Gamma}(I) = \mathrm{End}_R(I)$ and so, locally, $\Lambda_{L\Gamma}(I) = \mathrm{Mat}_{|\Gamma|}(R)$.

(ii) Let $\mathfrak{M}$ be a maximal order containing $S\Gamma$. A minimal projective of $\mathfrak{M}$ may be considered as a sublattice (and hence an ambiguous S-ideal) I of L. Clearly $\mathfrak{M}(I) = \mathfrak{M}$. Moreover if $J \subseteq L$ is another minimal projective of $\mathfrak{M}$ the local $\mathfrak{M}$-isomorphisms from I to J extend to local $L\Gamma$-isomorphisms from L to L. Together these may be expressed as an idele α of K and so $J = \mathfrak{a} I$ where $\mathfrak{a}$ is the ideal of R corresponding to α. As, clearly, $\mathfrak{M}(I) = \mathfrak{M}(\mathfrak{a} I)$ for any R-ideal $\mathfrak{a}$ we have (iii).

(iv) Is immediate as, from the proof of (i), we have that I is a minimal projective of $\mathfrak{M}(I)$.

(v) $J\mathfrak{M}(I)$ is a minimal projective of $\mathfrak{M}(I)$ containing J. . . .

With notation as in Lemma 1 we define $\det^I : K_0(S\Gamma) \to C(R)$ by $\det^I([M]) = (\mathfrak{a})$ where $M \otimes_{S\Gamma} \mathfrak{M}(I) \cong \oplus\, \mathfrak{a}_i I$ (where the $\mathfrak{a}_i$ are R-ideals) and $\mathfrak{a} = \Pi\, \mathfrak{a}_i$. (In fact $M \otimes \mathfrak{M}(I) \cong \mathfrak{a} I \oplus I \ldots \oplus I$ in this case. We can also use the injection $C(R) \to C(S, \Gamma)$ to describe $\mathfrak{a}$ as the equivariant Steinitz class : $\det_S(M \otimes_{S\Gamma} \mathfrak{M}(I) . I^{-1}) \in C(S, \Gamma)$, see [7]). We define $\delta_I(x) = \det^I(x) \det^S(x)^{-1}$.

LEMMA 2. – *Let* I *and* J *be ambiguous ideals of* S.

(i) *If* $x \in \widetilde{K}_0(S\Gamma)$ *then* $\mathrm{Nrd}_0(x \otimes_{S\Gamma} \mathfrak{M}(I)) = \det^I(x)$

(ii) $\det^I(J) = ((IJ^{-1}) \cap K)^{-1}$

For every prime ideal $\mathfrak{p} \neq 0$ *of* R *let* $\phi(\mathfrak{p})$ *be the maximal ambiguous* S-*ideal lying over* $\mathfrak{p}$. *Then* $\mathfrak{p} S = \phi(\mathfrak{p})^{e(\mathfrak{p})}$.

(iii) $\delta_I(J) = \prod_{\phi(\mathfrak{p})|I} \mathfrak{p}^{\left[\frac{j(\mathfrak{p}) - i(\mathfrak{p})}{e(\mathfrak{p})}\right] - \left[\frac{j(\mathfrak{p})}{e(\mathfrak{p})}\right]}$ where $I_\mathfrak{p} = \phi(\mathfrak{p})_\mathfrak{p}^{i(\mathfrak{p})}$ and $J_\mathfrak{p} = \phi(\mathfrak{p})_\mathfrak{p}^{j(\mathfrak{p})}$.

(*Here* $[x]$ *denotes the greatest integer not greater than* x.)

(iv) *In particular if* $I = \phi(\mathfrak{p})^i$ *then* $\delta_I(x)$ *depends only on the image of* x *in* $K_0(S_\mathfrak{p}\Gamma)$ *–or, indeed, that in* $K_0(S_\mathfrak{P}\Gamma(\mathfrak{P}))$, *where* $\mathfrak{P}$ *is a prime of* S *over* $\mathfrak{p}$ *and* $\Gamma(\mathfrak{P})$ *its decomposition group.*

(v) *Specifically, if* $|\Gamma| = e = 2$ *and* x *goes to* $n[S_\mathfrak{p}] + m[\phi(\mathfrak{p})_\mathfrak{p}]$ *in* $K_0(S_\mathfrak{p}\Gamma)$ *then* $\delta_{\phi(\mathfrak{p})}(x) = (\mathfrak{p}^{-n})$.

Proof. – (i) If $x \in \widetilde{\mathrm{K}}_0(\mathrm{S}\Gamma)$ then

$$x \otimes \mathfrak{M}(\mathrm{I}) = \sum_{i=j}^{n} n_i[\mathfrak{a}_i \mathrm{I}] \quad \text{where} \quad \Sigma n_i = 0$$
$$= \sum_{j}^{n} n_i([\mathfrak{a}_i \mathrm{I}] - [\mathrm{I}]) .$$

The first expression gives $\det^{\mathrm{I}}(x) = \Pi \mathfrak{a}^{n_i}$ and by (1) the second gives $\mathrm{Nrd}(x \otimes \mathfrak{M}(\mathrm{I})) = \Pi \mathfrak{a}_i^{n_i}$.

(ii) Is immediate from Lemma 1 (v).

(iii) $(\mathrm{IJ}^{-1}) = (\phi(\mathfrak{p})^{i-j})_{\mathfrak{p}}$ and $\phi(\mathfrak{p})^n \cap \mathrm{K} = \mathfrak{p}^{-\left[\frac{-n}{e(\mathfrak{p})}\right]}$.

(iv) From (iii), if $\mathrm{I} = \phi(\mathfrak{p})^i$ then $\delta_{\mathrm{I}}(\mathrm{J})$ depends only on $j(\mathfrak{p}) \bmod e$ and as the distinct irreducible projectives of $\mathrm{S}_{\mathfrak{p}}\Gamma$ are $\phi(\mathfrak{p})^i_{\mathfrak{p}}$, $i = 0, \dots\ e-1$ and those of $\mathrm{S}_{\mathfrak{P}}\Gamma(\mathfrak{P})$ are $\mathfrak{P}^i_{\mathfrak{P}}$, $i = 0, \dots\ e-1$, see [6], $\delta_{\mathrm{I}}(\mathrm{J})$ is determined by the image of J in $\mathrm{K}_0(\mathrm{S}_{\mathfrak{p}}\Gamma)$ or in $\mathrm{K}_0(\mathrm{S}_{\mathfrak{P}}\Gamma(\mathfrak{P}))$. The general result follows.

(v) $$\delta_{\phi(\mathfrak{p})}(n[\mathrm{S}] + m[\phi(\mathfrak{p})]) = \mathfrak{p}^{\left(n\left(\left[\frac{0-1}{2}\right] - \left[\frac{0}{2}\right]\right) + m\left(\left[\frac{1-1}{2}\right] - \left[\frac{1}{2}\right]\right)\right)}$$
$$= (\mathfrak{p}^{-n}) .$$

LEMMA 3. – *Let* S *be a complete discrete valuation ring with maximal ideal* $\mathfrak{P}$. *Let* Γ *be a finite group of automorphisms of* S *with inertia subgroup* $\mathrm{C} = \langle \tau \rangle$ *of order* e *prime to* $|\mathrm{S}/\mathfrak{P}|$. *Then* $\mathrm{P}_i = \mathfrak{P}^i$, $i = 0, \dots e-1$ *are the distinct minimal projectives of* $\mathrm{S}\Gamma$ *and* $\mu_i = \mathfrak{P}^i/\mathfrak{P}^{i+1}$, $i = 0, \dots e-1$ *are the distinct simple* $\mathrm{S}\Gamma$-*modules. Choose a generator* π *of* $\mathfrak{P}$ *such that* $\pi^\tau/\pi = \eta$ *is an* e^{th} *root of* 1. *Let* R_0 *be a sub-valuation-ring of* S^{C} *containing* η *with* $\mathfrak{p}_0 = \mathrm{R}_0 \cap \mathfrak{P}$ *and suppose that* $r = \mathrm{rk}_{\mathrm{R}_0}(\mathrm{S})$ *and* $f = \dim_{\mathrm{R}_0/\mathfrak{p}_0}(\mathrm{S}/\mathfrak{P})$ *are finite.*

Let S', $\mathfrak{P}'$, Γ', C', P'_i, μ'_i, π', η', R'_0, r', f' *be another such set of data with* $\mathrm{C}' = \mathrm{C}$ *and* $\mathrm{R}'_0 = \mathrm{R}_0$.

(i) *Choose* $x \in \mathrm{G}_0^t(\mathrm{S}\Gamma)$ *and* $x' \in \mathrm{G}_0^t(\mathrm{S}'\Gamma')$ *such that their images in* $\mathrm{G}_0^t(\mathrm{R}_0\mathrm{C})$ *agree. If* $x = \Sigma n_i[\mu_i]$ *then* $x' = \Sigma \dfrac{f}{f'} n_{it}[\mu'_i]$ *where* $\eta' = \eta^t$ *(suffixes are taken modulo* e*).*

(ii) *Let* M *be an* $S\Gamma$- *and an* $S'\Gamma'$- *module with identical* R_0C *actions and commuting* S *and* S′ *actions.*

If $[M] = \sum_0^{e-1} m_i[P_i]$ *in* $G_0(S\Gamma)$ *and* $\sum_0^{e-1} m_i'[P_i']$ *in* $G_0(S'\Gamma')$ *then* $m_i' = m_{i-1}' + \frac{f}{f'}(m_{it} - m_{(i-1)t})$.

(iii) *In particular if* $|\Gamma| = |\Gamma'| = 2 = e$ *and*

$$[M]_{S\Gamma} = a([P_0] + [P_1]) + b[P_1]$$

then
$$[M]_{S'\Gamma'} = a'([P_0'] + [P_1']) + \frac{bf}{f'}[P_1']$$

where
$$a' = a\frac{r}{r'} + \frac{b}{2}\left(\frac{r}{r'} - \frac{f}{f'}\right).$$

Proof. – (i) Let $\nu_i = (R_0/\mathfrak{p}_0)\,C/(\tau - \eta^i)$, $i = 0, \dots e-1$, be the simple R_0C-modules. Then we have restriction isomorphisms

$$G_0^t(S\Gamma) \xrightarrow{\sim} G_0^t(R_0C) \xleftarrow{\sim} G_0^t(S'\Gamma'),$$

where $[\mu_i] \mapsto f[\nu_i]$ and $[\mu_i'] \mapsto f'[\nu_{it}]$. The result follows.

(ii) Let $M = M_1/M_2$ where M_1, M_2 are torsion-free $S \otimes_R S'C$-modules (we assume $\Gamma = \Gamma' = C$ in view of the restriction isomorphisms between $K_0(S\Gamma)$ and $K_0(SC)$ etc.) and put $\widetilde{M} = M_1\pi'/M_2\pi'$.

If $N \subseteq M_i$, $N \underset{SC}{\cong} P_j$ then $N\pi' \underset{SC}{\cong} P_{j+t}$. Hence in $G_0(SC)$

$$\begin{aligned}[M] - [M] = [M_1] - [M_1\pi'] - [M_2] + [M_2\pi'] &= \Sigma m_i([P_i] - [P_{i+1}]) \\ &= \Sigma(m_i - m_{i-t})[P_i].\end{aligned}$$

Now if $[M_1/M_1\pi'] - [M_2/M_2\pi'] = \Sigma x_i[\mu_i]$ in $G_0^t(SC)$ then $[M] - [\widetilde{M}] = \Sigma x_i([P_i] - [P_{i+1}]) = \Sigma(x_i - x_{i-1})[P_i]$ in $G_0(SC)$.

But, in $G_0^t(S'C)$, $[M_1/M_1\pi'] - [M_2/M_2\pi']$ is the 'semisimplification' of $[M_1] - [M_2]$ and so is $\Sigma m_i'[\mu_{i'}]$.

So $$m_i' = \frac{f}{f'}x_{it} = m_{i-1}' + \frac{f}{f'}(m_{it} - m_{(i-1)t})$$

(iii) follows.

Let Δ be a finite group. The intersection, $\mathcal{H}$, of all maximal orders which contain $\mathbf{Z}\Delta$ is the minimal hereditary order containing $\mathbf{Z}\Delta$. I am indebted to Anne-Marie Bergé for the following construction of a Galois extension $\hat{L}$ of $\mathbf{Q}_p$, p an odd prime, such that, with $\Delta = \mathrm{Gal}(\hat{L}/\mathbf{Q}_p)$ and $\hat{S} = \mathrm{int}(\hat{L})$, $\hat{S}\mathcal{H}_p \not\cong \mathcal{H}_p$. (The fact that this is so will emerge later although my proof, using the preceeding lemma, is somewhat different from that of Mlle Bergé.)

LEMMA 4. – (i) *Let* $\hat{L}_0 = \mathbf{Q}_p[\sqrt{p}]$ *and let* $\hat{L}$ *be the extension of* $\hat{L}_0$ *with norm group* $N = \langle \mathbf{Q}_0^*, \sqrt{p}, (1+\sqrt{p})^{p^2} \rangle \subseteq \hat{L}_0^*$. *Putting* $H = \mathrm{Gal}(\hat{L}/\hat{L}_0)$ *we have ramification groups* $H^0 = H^1 \simeq C_{p^2}$, $H^2 = H^3 \simeq C_p$, $H^4 = \{1\}$ *and* $\mathrm{Gal}(\hat{L}/\mathbf{Q}_p)$ *is dihedral.*

(ii) $H_0 = H_1 \simeq C_p$, $H_2 = \ldots = H_{2p+1} \simeq C_p$, $H_{2p+2} = \{1\}$.

(iii) Let σ be a generator of H and π_1 a prime of $\hat{S}_1 = \hat{S}^{\langle \sigma^p \rangle}$ *then* $\mathrm{tr}_{\sigma^p} \hat{S} = \pi_1^{2p-1} \hat{S}_1$.

Proof. – (i) $\hat{L}/\hat{L}_0$ is clearly cyclic of degree p^2 and, as $(1+\sqrt{p})(1-\sqrt{p}) = 1 - p \in N$, $\hat{L}/\mathbf{Q}_p$ is dihedral. Moreover $1+\sqrt{p} \in U^1 \backslash U^2$ and $(1+\sqrt{p})^p = 1 + p\sqrt{p} + (\sqrt{p})^p + \ldots \in U^3 \backslash U^4$ and the result follows by local class field theory.

(ii) $H^i = H_{\psi(i)}$ where $\psi(x) = \int_0^x |H^0 : H^u| \, du$ so $\psi(0) = 0$, $\psi(1) = 1$, $\psi(1+r) = 1 + rp$, $0 < r < 2$. Hence the result.

(iii) Put $H' = \mathrm{Gal}(\hat{L}/\hat{L}^{\langle \sigma^p \rangle})$. Then

$$H'_0 = H'_1 = \ldots = H'_{2p+1} = C_p, \quad H'_{2p+2} = \{1\}.$$

So the value of the different of $\hat{S}$ over $\hat{S}_1$ is

$$(2p+2)(p-1) = 2p^2 - 2.$$

Therefore $$\mathrm{tr}_{\sigma^p}(\hat{S}) = \pi_1^{\left[\frac{2p^2-2}{p}\right]} \hat{S}_1 = \pi_1^{2p-1} \hat{S}_1.$$

Choose a prime $p > 2$ and $n \in N^*$ prime to p such that, with $\theta = \sqrt[n]{1}$, the prime $\mathfrak{p}_0$ over p in $\mathbf{Z}[\theta]$ is not principal (e.g. $p = 3$, $n = 23$). We put $G = C_n \times (C_{p^2} \times \Gamma)$ where $C_{p^2} = \langle \sigma \rangle$, $\Gamma = \langle \tau \rangle$ of order 2 and $\sigma^\tau = \sigma^{-1}$. Put $T = \mathbf{Z}[\theta, \zeta]$, where $\xi = \sqrt[p^2]{1}$, and $T_1 = \mathbf{Z}[\theta, \zeta^p]$ and let $\mathfrak{P}_1$, $\mathfrak{p}_1$, $\mathfrak{P}$, $\mathfrak{p}$ be the primes over $\mathfrak{p}_0$ in T_1, T_1^Γ, T, T^Γ where $\tau : \begin{cases} \theta \mapsto \theta \\ \zeta \mapsto \zeta^{-1} \end{cases}$. We note that all these primes are non-principal.

Choose a D_{2p^2} $(= C_{p^2} \times \Gamma)$-extension L of $\mathbf{Q}$ such that L_p is the $\hat{L}$ in lemma 4 (we can do this as p is odd see [1] ch. 10

Thm 5), and an extension M of $\mathbf{Q}$ disjoint from L, cyclic of degree n and non-ramified where L is ramified. Then $E = LM$ is a galois extension of $\mathbf{Q}$ with group G and if S, V and W are the rings of integers of L, M and E respectively we have $W = SV = S \otimes_{\mathbf{Z}} V$.

We put $L_1 = L^{\langle \sigma^p \rangle}$ and $L_0 = L^{\langle \sigma \rangle}$ with rings of integers S_1, S_0 and we choose π, π_1, π_0, prime elements of $\hat{L}, \hat{L}_1, \hat{L}_0$ so that $\pi^\tau = -\pi$ etc.

Let χ, χ_1, be the characters of G induced from the C_{np^2} characters $\rho \mapsto \theta$, $\sigma \mapsto \zeta$ and $\rho \mapsto \theta$, $\sigma \mapsto \zeta^p$ and let $A\chi$, $A\chi_1$ be the corresponding factors of $\mathbf{Q}G$. Choose $\mathfrak{M}$, $\mathfrak{M}_1$, maximal orders containing the images of $\mathbf{Z}G$ in the complements of $A\chi$ and $A\chi_1$. Note that the projections of $\mathbf{Z}G$ into $A\chi$ and $A\chi_1$ are, respectively, $T\Gamma$ and $T_1\Gamma$. We recall that $\mathfrak{M}(T)$ is a maximal order containing $T\Gamma$ etc.

THEOREM. – *With the above data*

(i) $\mathrm{Nrd}_0(W \otimes_{\mathbf{Z}G} (\mathfrak{M} \oplus \mathfrak{M}(\mathfrak{P}))) \neq \mathrm{Nrd}_0(W \otimes_{\mathbf{Z}G} (\mathfrak{M} \oplus \mathfrak{M}(T)))$.

(ii) $\mathrm{Nrd}_0(W.(\mathfrak{M}_1 \oplus \mathfrak{M}(\mathfrak{P}_1))) \neq \mathrm{Nrd}_0(W.(\mathfrak{M}_1 \oplus \mathfrak{M}(T_1)))$.

Proof. – (i) We write $\mathrm{tr} = \mathrm{tr}_{L/L_1}$. As

$$\mathbf{Z}[\zeta] = \mathbf{Z}[\sigma]/(1 + \sigma^p + \dots \sigma^{p(p-1)}),$$

$$[S \otimes_{\mathbf{Z}C_{p^2}} \mathbf{Z}_p[\zeta]]_{\hat{S}_0\Gamma} = [\hat{S}/\mathrm{tr}\hat{S}] = [\hat{S}] - [\mathrm{tr}\hat{S}] = [\hat{S}] - [\pi_1^{2p-1} S_1]$$

by lemma 4

$$= [\hat{S}_0] + [\pi\hat{S}_0] + \dots + [\pi^{p^2-1}\hat{S}_0] - [\pi_1^{2p-1} S_0] - \dots - [\pi_1^{3p-2} S_0]$$

$$= \frac{p^2 - p + 2}{2}[\hat{S}_0\Gamma] - 2[\hat{\Phi}_0]$$

where $\hat{\Phi}_0 = \pi_0 \hat{S}_0$ as $\pi^i \hat{S}_0 \cong_{\hat{S}_0\Gamma} \pi_1^i \hat{S}_0 \cong \hat{\Phi}_0^i$ and $\hat{S}_0\Gamma \cong_{\hat{S}_0\Gamma} \hat{S}_0 \oplus \hat{\Phi}_0$.

Also $V \otimes_{\mathbf{Z}C_n} \mathbf{Z}[\theta]_{\mathfrak{p}_0} \cong \mathbf{Z}[\theta]_{\mathfrak{p}_0}$ as $V/\mathbf{Z}$ is tame at p $((n,p) = 1)$.

Hence

$$[W \otimes_{\mathbf{Z}C_{np^2}} T_{\mathfrak{P}}]_{\hat{S}_0\Gamma} = [(V \otimes_{\mathbf{Z}} S) \otimes_{\mathbf{Z}C_n \times C_{p^2}} (\mathbf{Z}[\theta]_{\mathfrak{p}_0} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\zeta])]$$

$$= [(V \otimes_{\mathbf{Z}C_n} \mathbf{Z}[\theta]_{\mathfrak{p}_0}) \otimes_{\mathbf{Z}_p} (S \otimes_{\mathbf{Z}C_{p^2}} \mathbf{Z}_p[\zeta])]$$

$$= f\left(\frac{p^2 - p + 2}{2}\right)[\hat{S}_0\Gamma] - 2f[\hat{\Phi}_0],$$

where $f = \mathrm{rk}_{\mathbf{Z}}\,(\mathbf{Z}[\theta]_{\mathfrak{p}_0}) = \dim_{\mathbf{F}_p}(\mathbf{Z}[\theta]/\mathfrak{p}_0) = \dim_{\mathbf{F}_p}(\mathrm{T}/\mathfrak{P})$. Now $\dim_{\mathbf{F}_p}(\hat{\mathrm{S}}_0/\hat{\Phi}_0) = 1$ so, by lemma 3 (iii),

$$\begin{aligned}[\mathrm{W} \otimes_{\mathbf{Z}C_{np^2}} \mathrm{T}_{\mathfrak{P}}]_{\mathrm{T}_{\mathfrak{P}}\Gamma} &= s[\mathrm{T}_{\mathfrak{P}}\Gamma] - 2[\mathfrak{P}_{\mathfrak{P}}] \quad \text{for some } s\\ &= 2[\mathrm{T}_{\mathfrak{P}}] \quad \text{as its T-rank is 2}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathrm{Nrd}_0(\mathrm{W} \otimes_{\mathbf{Z}G} (\mathfrak{M} \otimes \mathfrak{M}(\mathfrak{P}))) \times \mathrm{Nrd}_0(\mathrm{W} \otimes_{\mathbf{Z}G} (\mathfrak{M} \oplus \mathfrak{M}(\mathrm{T})))^{-1}\\ &= \mathrm{Nrd}_0(\mathrm{W} \otimes_{\mathbf{Z}G} \mathfrak{M}(\mathfrak{P})) \times \mathrm{Nrd}_0(\mathrm{W} \otimes_{\mathbf{Z}G} \mathfrak{M}(\mathrm{T}))^{-1}\\ &= \delta_{\mathfrak{P}}([\mathrm{W} \otimes_{\mathbf{Z}G} \mathrm{T}\Gamma] - [\mathrm{T}\Gamma]) \qquad \text{by Lemma 2 (i)}\\ &= \delta_{\mathfrak{P}}([\mathrm{W} \otimes_{\mathbf{Z}C_{np^2}} \mathrm{T}] - [\mathrm{T}\Gamma]) = \delta_{\mathfrak{P}}([\mathrm{T}] - [\mathfrak{P}]) \qquad \text{by Lemma 2 (iv)}\\ &= (\mathfrak{p}^{-1}) \qquad \text{by Lemma 2 (v)}\\ &\neq 0.\end{aligned}$$

(ii) The kernel of the epimorphism, $s \otimes 1 \mapsto (\mathrm{tr}\, s)$

$$\mathrm{S} \otimes_{\mathbf{Z}C_{np^2}} \mathbf{Z}[\zeta^p] \to \mathrm{tr}\,\mathrm{S}/(\mathrm{S}_0 \cap \mathrm{tr}\,\mathrm{S})$$

is a torsion group as the ranks of the two modules are equal. The image module is, however, torsion-free and so

$$\begin{aligned}[(\mathrm{S} \otimes_{\mathbf{Z}C_{p^2}} \mathbf{Z}[\zeta^p]/(\text{torsion})]_{\hat{\mathrm{S}}_0\Gamma} &= [\mathrm{tr}\,\hat{\mathrm{S}}/\mathrm{tr}\hat{\mathrm{S}} \cap \hat{\mathrm{S}}_0] = [\pi_1^{2p-1}\,\hat{\mathrm{S}}_1] + [\pi^2\,\hat{\mathrm{S}}_0]\\ &= \frac{p-3}{2}\,[\hat{\mathrm{S}}_0\Gamma] + 2[\hat{\Phi}_0] \ldots \qquad (2)\end{aligned}$$

Hence

$$\begin{aligned}[\mathrm{W} \otimes_{\mathbf{Z}C_{np^2}} \mathrm{T}_{1,\mathfrak{P}_1}/(\text{torsion})]_{\hat{\mathrm{S}}_0\Gamma} &= [\mathbf{Z}[\theta]] \otimes_{\mathbf{Z}} \left(\frac{p-3}{2}\,[\hat{\mathrm{S}}_0\Gamma] + 2[\hat{\Phi}_0]\right) \qquad \text{(cf (1))}\\ &= \frac{p-3}{2}\,f[\hat{\mathrm{S}}_0\Gamma] + 2f[\hat{\Phi}_0].\end{aligned}$$

Hence

$$\begin{aligned}[\mathrm{W} \otimes_{\mathbf{Z}C_{np^2}} \mathrm{T}_{1,\mathfrak{P}_1}/(\text{torsion})]_{\mathrm{T}_1\Gamma_{\mathfrak{P}_1}} &= s[(\mathrm{T}_1\Gamma)_{\mathfrak{P}_1}] + 2[\mathfrak{P}_{1,\mathfrak{P}_1}] \qquad \text{by Lemma 3 (iii)}\\ &= 2[\mathfrak{P}_{1,\mathfrak{P}_1}] \quad \text{as the rank is 2} \ldots \qquad (3)\end{aligned}$$

Hence

$$\begin{aligned}&\mathrm{Nrd}_0(\mathrm{W}\,.\,(\mathfrak{M}_1 \oplus \mathfrak{M}(\mathfrak{P}_1)))\,.\,\mathrm{Nrd}_0(\mathrm{W}\,.\,(\mathfrak{M}_1 \oplus \mathfrak{M}(\mathrm{T}_1)))^{-1}\\ &= \mathrm{Nrd}_0(\mathrm{W} \otimes_{\mathbf{Z}G} \mathfrak{M}(\mathfrak{P}_1)/\text{torsion})\,.\,\mathrm{Nrd}_0(\mathrm{W} \otimes_{\mathbf{Z}G} \mathfrak{M}(\mathrm{T}_1)/\text{torsion})^{-1}\end{aligned}$$

$$= \delta_{\mathfrak{P}_1}([W \otimes_{\mathbf{Z}C_{np^2}} T/\text{torsion}] - [T\Gamma]) \quad (\text{as } \otimes_{T\Gamma} \mathfrak{M}(I) \text{ is exact})$$

$$= \delta_{\mathfrak{P}_1}([\mathfrak{P}_1] - [T_1]) = (\mathfrak{p}_1) \quad \text{by Lemma 2 (iv) and (v)}$$

$$\neq 0 .$$

Note that, from (2), we deduce easily, using lemma 3, that $S . \mathbf{Z}[\zeta^p] \cong_{\mathbf{Z}[\zeta^p]\Gamma} (1 - \zeta) \oplus (1 - \zeta) \ncong \mathbf{Z}[\zeta^p]\Gamma$ showing that, in the notation above with $\Delta = D_{2p^2}$, $\hat{S} . \mathcal{H}_p \ncong \mathcal{H}_p$. Also, of course, from (3) we have that, with $\Delta = G$, $W . \mathcal{H}_p \ncong \mathcal{H}_p$.

BIBLIOGRAPHY

[1] E. ARTIN and J. TATE, Class Field Theory, Benjamin, New York, 1967.

[2] J. COUGNARD, Un Contre-exemple à une conjecture de J. Martinet, in 'Algebraic Number Fields', ed. Fröhlich, Acad. Press, New York, 1977.

[3] J. COUGNARD, Une propriété de l'anneau des entiers des extensions galoisiennes non-abéliennes de degré pq des rationnels, to appear in *Compositio Mathematica.*

[4] A. FROHLICH, Locally free modules over arithmetic orders, *J. Reine Angew. Math.,* 274/275 (1975), 112-138.

[5] A. FROHLICH, Galois Module Structure, in Algebraic Number Fields, ed. Fröhlich, Acad. Press 1977.

[6] I.N. ROSEN, Representations of twisted group rings, Thesis, Princeton University, 1963.

[7] S.M.J. WILSON, K-Theory for twisted group rings, *Proc. London Math. Soc.,* (3) 29 (1974), 257-271.

Manuscrit reçu le 17 décembre 1979.

Stephen M.J. WILSON,
University of Durham
Department of Mathematics
Science Laboratories
South Road
Durham DH1 3 LE (G.B.).