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VIKTOR LOSERT

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A CHARACTERIZATION OF THE MINIMAL STRONGLY CHARACTER INVARIANT SEGAL ALGEBRA

by Viktor LOSERT

Let G be a locally compact abelian group. It is well known that any dense $L^1(G)$-Banach-module S in $L^1(G)$ contains all functions whose Fourier transform has compact support (see e.g. [13], p. 20). If S is also a multiplicative A(G)-module (A(G) denotes the Fourier algebra), then H.G. Feichtinger [6] has shown that S necessarily contains a larger space $S_0(G)$. These spaces $S_0(G)$ have been studied by Feichtinger in [6] and it turned out that they have a number of remarkable properties. Some of them resemble those of the Schwartz-space of test functions and could make $S_0(G)$ valuable for the theory of multipliers and distributions ($S_0(G)$ has the advantage of being a Banach space).

Another aspect, which will be studied more closely in this paper, are the 'functorial' properties of the family $\{S_0(G) : G \text{ abelian}\}$. It has been shown in [6] that $S_0(G)$ is invariant under the Fourier transform, that restrictions of functions from S_0 to closed subgroups belong again to S_0 and finally that $S_0(G_1 \times G_2) = S_0(G_1) \hat{\otimes} S_0(G_2)$. We will show that these stability properties actually characterize the family $\{S_0(G)\}$ (Theorem 1).

The definition of $S_0(G)$ is based on a construction which was also used by other authors (see e.g. the so-called "amalgams" of Holland [9], and the spaces $l^p(L^{p'})$ of Bertrandias et al. [1], [2]). This raises the question whether or not $S_0(G)$ coincides with one of these spaces, or more generally, what can be said about functions belonging to $S_0(G)$. For exemple, it can be shown that $S_0(G)$ contains the space of Schwartz-test-functions. We will prove two results which give alternative descriptions of the elements of $S_0(G)$ (Prop. 1,2). On the other hand, let $W(G)$ be the Wiener algebra on G ([12]) and put $W_0(G) = \{f \in W(G) : f^\wedge \in W(G^\wedge)\}$. $W_0(G)$ shares some of the functorial properties with $S_0(G)$. Nevertheless we will prove that $S_0(G) \subsetneqq W_0(G)$ except for trivial cases (Theorem 2). This proof uses a construction, which enables one to get functions satisfying prescribed norm properties and at the same time certain estimates for the Fourier transform. This construction could also be used to construct non trivial elements in other function spaces of similar type as $W_0(G)$.

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Notations and definitions. – G will always be a locally compact, abelian group (written additively), $G^\wedge$ its dual group. Integration on G will always refer to a fixed Haar measure λ on G, if Fourier transforms are considered we will implicitely assume that the measure on $G^\wedge$ is chosen in such a way that the inversion theorem holds. $L^1(G)$ denotes the space of integrable (complex valued) functions on G, $\|f\|_1 = \int_G |f(x)|\,dx$. The Fourier algebra $A(G)$ is defined as the space of functions f on G, for which there exists $g \in L^1(G^\wedge)$ such that $f = g^\wedge$, the Fourier transform of g. We put $\|f\|_A = \|g\|_1$. If $x \in G$, L_x shall denote the translation operator for functions on G: $L_x f(z) = f(z - x)$. Similarly for $y \in G^\wedge$, M_y shall denote the multiplication operator for functions on G.

A Banach space $(S(G), \|\ \|_S)$ is called a *Segal algebra* on G if the following holds : (i) $S(G)$ is a dense subspace of $L^1(G)$ and the inclusion mapping is continuous, (ii) $S(G)$ is invariant under L_x and $\|L_x f\|_S = \|f\|_S$ for all $f \in S(G)$, $x \in G$, (iii) $\lim_{x \to 0} L_x f = f$ holds in $S(G)$ for all $f \in S(G)$ (see [13] for details). $S(G)$ is called *strongly character invariant* if it is also invariant under M_y and $\|M_y f\|_S = \|f\|_S$ for all $f \in S(G)$, $y \in G^\wedge$ (equivalently one can say that $S(G)$ is an $A(G)$-module, i.e. $g \,.\, f \in S(G)$ for $g \in A(G)$, $f \in S(G)$ and $\|g \,.\, f\|_S \leqslant \|g\|_A \,.\, \|f\|_S$).

We recall now the definition of $S_0(G)$ from [6]: Let Q be a fixed compact subset of G with nonempty interior. Then

$S_0(G) = \{f : \exists (f_n)_{n=1}^{\infty} \subseteq A(G),\ (x_n) \subseteq G$ such that $\operatorname{supp} f_n \subseteq Q\ \forall n$, $f = \sum_{n=1}^{\infty} L_{x_n} f_n$ pointwise, $\Sigma \| f_n \|_A < \infty\}$ with the norm

$$\| f \|_{S_0} = \inf \{\Sigma \| f_n \|_A : f = \Sigma L_{x_n} f_n \text{ as above}\}.$$

It is easy to see that another compact set Q gives an equivalent norm on $S_0(G)$. $S_0(G)$ is a Segal algebra (even for arbitrary locally compact groups). We mention another important property from [6] : $S_0(G)$ is the minimal strongly character invariant Segal algebra on G, i.e. if S is any strongly character invariant Segal algebra on G, then $S_0(G) \subseteq S$ and the embedding mapping is continuous.

PROPOSITION 1. – *Assume that a function* $g \in A(G)$ *with compact support is given and that there exists a discrete subgroup* M *of* G *such that* $\sum_{x \in M} L_x g = 1$ *pointwise on* G. *Then a function* f *belongs to* $S_0(G)$ *iff* $\sum_{x \in M} \| f . L_x g \|_A < \infty$ *and the last expression defines an equivalent norm on* $S_0(G)$.

Remark. – This result is particularly useful in the case $G = \mathbf{R}^n$ with $M = \mathbf{Z}^n$. For g one can take e.g. a suitably adjusted trapezoid function. See also [11], p. 293 where a similar construction has been used for the spaces W^p.

Proof. – The condition $\Sigma \| f . L_x g \|_A < \infty$ is clearly sufficient to ensure that $f \in S_0(G)$, since g has compact support and $f = \Sigma f . L_x g$ pointwise. If we choose $Q = \operatorname{supp} g$ as the fixed compact set in the definition of $S_0(G)$, we have also $\| f \|_{S_0} \leqslant \Sigma \| f . L_x g \|_A$. For the converse, note that the set $M_1 = M \cap (Q + Q - Q)$ is finite, since M is discrete. If $x_n \in G$, then $x_n = x'_n + x''_n$ with $x'_n \in Q$, $x''_n \in M$, since $M + Q = G$. If $f_n \in A(G)$ satisfies $\operatorname{supp} f_n \subseteq Q$, then $L_{x_n} f_n . L_x g = 0$ if $x \in M \backslash (x''_n + M_1)$. This gives

$$\sum_{x \in M} \| L_{x_n} f_n . L_x g \|_A \leqslant \operatorname{card} M_1 . \| g \|_A . \| f_n \|_A .$$

The result follows now from the definition of $S_0(G)$.

PROPOSITION 2. – *Assume that a non zero function* $g \in S_0(G)$ *is given. The following statements are equivalent for a function* f *on* G :

a) $f \in S_0(G)$

b) $\int_G \|f \,.\, L_x g\|_A \, dx < \infty$

c) $\int_G \|f \,.\, L_x g\|_{S_0} \, dx < \infty$.

The expressions in b) *and* c) *define norms on* $S_0(G)$ *which are equivalent to the original one.*

Remark. – This is a continuous analogon of Prop. 1. It was first proved in a different fashion (using structure theory) by D. Poguntke ([15], Satz 2) and by H.G. Feichtinger [6]. Our proof is valid even for arbitrary locally compact topological groups, if g has compact support and $L_x g$ is replaced by $L_{x^{-1}} g$.

Proof. – a) $\Rightarrow$ c)

This is similar to Prop. 1. Let Q be the compact subset of G used in the definition of $S_0(G)$ and put $c = \lambda(Q - Q)$. If $\operatorname{supp} f$, $\operatorname{supp} g \subseteq Q$, then $\operatorname{supp} f. L_x g \subseteq Q$ and $f \,.\, L_x g = 0$ for $x \notin Q - Q$. Consequently

$$\int_G \|f \,.\, L_x g\|_{S_0} \, dx \leqslant \int_G \|f \,.\, L_x g\|_A \, dx = \int_{Q-Q} \|f \,.\, L_x g\|_A \, dx \leqslant c \,.\, \|f\|_A \,.\, \|g\|_A \,.$$

The same argument holds for translates of f and g, and by bilinearity we get $\int \|f \,.\, L_x g\|_{S_0} \, dx \leqslant c \, \|f\|_{S_0} \,.\, \|g\|_{S_0}$ for arbitrary $f, g \in S_0(G)$.

c) $\Rightarrow$ b) is trivial.

b) $\Rightarrow$ a)

Since A(G) is a Banach algebra, we may replace g by $h \,.\, g$ with $h \in A(G)$. In particular, we may assume that g has compact support and satisfies $\int g(x) \, dx = 1$. We take $Q = \operatorname{supp} g$. The support of $f. L_x g$ lies in a translate of Q, therefore $\|f \,.\, L_x g\|_{S_0} \leqslant \|f \,.\, L_x g\|_A$ for all $x \in G$ and consequently $\int \|f \,.\, L_x g\|_{S_0} \, dx \leqslant \int \|f \,.\, L_x g\|_A \, dx < \infty$ holds. This shows that the continuous function $u_{f,g} : G \to S_0(G)$ defined by $u_{f,g}(x) = f \,.\, L_x g$ is Bochner integrable. Since $S_0(G)$ is complete, it follows that $\int_G u_{f,g}(x) \, dx \in S_0(G)$. $S_0(G)$ is continuously embedded into

A(G), consequently all point functionals are continuous and we may evaluate the integral pointwise. For $z \in G$ we get :

$$\left(\int u_{f,g}(x)\,dx\right)(z) = \int u_{f,g}(x)\,(z)\,dx = \int f(z)\,g(z-x)\,dx$$
$$= f(z)\int g(x)\,dx = f(z).$$

This shows that $f = \int u_{f,g}(x)\,dx \in S_0(G)$ and

$$\|f\|_{S_0} = \left\|\int u_{f,g}(x)\,dx\right\|_{S_0} \leqslant \int \|u_{f,g}(x)\|_{S_0}\,dx$$
$$= \int \|f\,.\,L_x g\|_{S_0}\,dx \leqslant \int \|f\,.\,L_x g\|_A\,dx\,.$$

THEOREM 1. – *Assume that* $\{S(G)\}$ *is a family of Segal algebras, defined for all locally compact abelian groups. We consider the following "functiorial" properties for* $\{S(G)\}$:

(i) *every continuous automorphism of* G *induces an automorphism of* S(G) *(in the canonical way)*

(ii) $S(G^{\wedge}) = \{f^{\wedge} : f \in S(G)\}$ *for all* G *(invariance under Fourier transforms)*

(iii) *if* H *is a closed subgroup of* G, $f \in S(G)$ *then* $f\,|\,H \in S(H)$ *(restriction property)*

(iv) *if* $f_i \in S(G_i)$ $(i = 1, 2)$ *then* $f_1 \otimes f_2 \in S(G_1 \times G_2)$ *(here* $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)\,.\,f_2(x_2)$ – *multiplication property).*

(v) *if* H *is an open subgroup of* G, $f \in S(G)$ *and* f_x *denotes the restriction of* f *to the coset* $x + H$ ($f_x = 0$ *outside* $x + H$), $T \subseteq G$ *is a set of representatives for* G/H, *then* $f = \sum_{x \in T} f_x$ *and the series converges absolutely in* S(G) *(decomposition with respect to open subgroups).*

If (i) – (v) *hold then* $S(G) = S_0(G)$ *for all* G.

Remark. – Condition (i) is only needed to ensure that (iii) makes sense : if H is an abstract group, then the statement "H is a closed subgroup of G" means that there exists a continuous monomorphism from H onto a closed subgroup of G. This monomorphism is unique only up to an automorphism of H and in general there is no "canonical" way to select such a monomorphism.

In the case of $S_0(G)$ the properties (i) – (iv) are valid even in a stronger form (see [6]). For exemple in (iii) the restriction operator is surjective and (iv), (v) are special cases of the more general formula $S_0(G_1 \times G_2) = S_0(G_1) \hat{\otimes} S_0(G_2)$.

Proof. – An application of the closed graph theorem shows that the map $f \longrightarrow f\hat{}$ from $S(G)$ to $S(G\hat{})$ is continuous, hence an isomorphism. This shows that $S(G)$ is strongly character invariant (since the operator M_y on $S(G)$ corresponds to L_y on $S(G\hat{})$). By the minimality of $S_0(G)$, we get $S_0(G) \subseteq S(G)$. For the proof of the opposite inclusion, note that property (v) implies that $S(G)$ is uniquely determined by $S(H)$, if H is an open subgroup of G. By the general structure theory ([8], 24.30), this reduces the problem to the case $G = \mathbf{R}^n \times K$ (K compact). If we apply (ii) once again, we arrive at $G = \mathbf{R}^n$.

Observe that by (ii) $S(G) \subseteq A(G)$, consequently the point functionals are continuous. Another application of the closed graph theorem shows that the restriction mappings in (iii) and the bilinear mapping in (iv) are always continuous. Now we want to apply Prop. 1 in order to show that $S(\mathbf{R}^n) \subseteq S_0(\mathbf{R}^n)$. Put $M = \mathbf{Z}^n$ and choose a compactly supported function $g \in A(\mathbf{R}^n)$ such that $\sum_{x \in \mathbf{Z}^n} L_x g = 1$ pointwise. If $f \in S(\mathbf{R}^n)$, then $f \otimes g \in S(\mathbf{R}^n \times \mathbf{R}^n)$ by (iv). Now we consider the subgroups $H = \{(x, x+y) : x \in \mathbf{R}^n, y \in \mathbf{Z}^n\}$ and $H_1 = \{(x,x) : x \in \mathbf{R}^n\}$ of $\mathbf{R}^{2n}$. By (iii) $f \otimes g \,|\, H$ belongs to $S(H)$. $H_1 \cong \mathbf{R}^n$ is open in H and $T = \{(0,y) : y \in \mathbf{Z}^n\}$ is a set of representatives for H/H_1. The restriction of $f \otimes g$ to the coset defined by $(0, -y)$ is simply $f \,.\, L_y g$ and (v) gives now $\sum_{y \in \mathbf{Z}^n} \| f \,.\, L_y g \|_{S(\mathbf{R}^n)} < \infty$. It follows that $\sum_{y \in \mathbf{Z}^n} \| f \,.\, L_y g \|_A < \infty$ holds too and Prop. 1 gives now $f \in S_0(\mathbf{R}^n)$.

We recall the definition of Wiener's Algebra $W(G)$ on a locally compact abelian group G (see [12]): let g be a continuous function on G with compact support. Then

$$W(G) = \{f \in C(G) : \int_G \| L_x f \,.\, g \|_\infty \, dx < \infty\}$$

with the norm $\| f \|_W = \int_G \| L_x f \,.\, g \|_\infty$ ($C(G)$ denotes here the space of all bounded, continuous functions G, $\| \;\; \|_\infty$ the supremum norm, see also [5] and [3] for equivalent definitions and some properties of W(G)). W(G) is a Segal algebra on G and also a C(G)-module (by multiplication). H.G. Feichtinger has shown that it is the smallest C(G)-module which is also a Segal algebra [5]. The definition of W(G) resembles closely that of $S_0(G)$ in Prop. 2 but in general W(G) is not invariant under Fourier transforms. Therefore we put $W_0(G) = \{f \in W(G) : f^\wedge \in W(G^\wedge)\}$, with the norm $\|f\|_{W_0} = \|f\|_W + \|f^\wedge\|_W$. It can easily be proved that $W_0(G)$ has the properties (i) – (iv) of Theorem 1 and R. Bürger [4] has proved that the restriction operator in (iii) is again surjective. Nevertheless we will prove now that $W_0(G)$ and $S_0(G)$ are different in general.

First we prove a technical lemma.

LEMMA. – *If* H *is an arbitrary discrete abelian group,* K *a subgroup of* H *and* ord H/K $> (\epsilon/2)^{-12}$, *then there exists* $\mu \in l^1(H)$ *such that* $\|\mu\|_1 = 1$, $\|\mu^\wedge\|_\infty < \epsilon$ *and* supp μ *intersects each coset of* K *in at most one point.*

Proof. – Put $n =$ ord H/K (the case of infinite H/K can be proved in a similar way). H/K can be decomposed into a product of cyclic subgroups: $H/K = H_1 \times \ldots \times H_k$ with ord $H_i = n_i$, $n = n_1 \ldots n_k$. Let x_i be an element of H such that $x_i + K$ generates H_i $(i = 1, \ldots, k)$. Put $m_i = [\log n_i/\log 2]$, $m = \Sigma\, m_i$ ($[\alpha]$ denotes the largest integer not exceding α). Then we consider the elements $y_1 = x_1$, $y_2 = 2x_1$, $y_3 = 2^2 x_1, \ldots, y_{m_1} = 2^{m_1 - 1} x_1$, $y_{m_1+1} = x_2, \ldots, y_m = 2^{m_k - 1} x_k$. Since $2^{m_i} \leqslant n_i$, it is easily seen that the points $\sum_{i=1}^{m} l_i y_i$ are pairwise different, for different choices of $l_i \in \{0, 1\}$. The number k_0 of indices for which $n_i \geqslant 3$ is at most $\log n/\log 3$. Therefore

$$m = \Sigma\, m_i > \Sigma \log n_i/\log 2 - k_0 \geqslant \log n \left(\frac{1}{\log 2} - \frac{1}{\log 3}\right) \geqslant \frac{1}{3} \log n .$$

The set of points $\left\{ \sum_{i=1}^{n} l_i y_i : l_i = 0, 1 \right\}$ forms what is called in [10] an «m-maille». We put $\mu_{x,y} = \frac{1}{4}(\delta_0 + \delta_x + \delta_y - \delta_{x+y})$ and $\mu = \mu_{y_1 y_2} * \mu_{y_3 y_4} * \ldots * \mu_{y_{m'-1} y_{m'}}$ where $m' = m$ if m is even and $m' = m - 1$ if m is odd. Then it has been shown in [10] p. 31

that $\|\mu\|_1 = 1$ and $\|\mu^\wedge\|_\infty \leqslant 2^{-\frac{m'}{4}} < 2n^{-\frac{1}{12}} < \epsilon$. The condition on the support of μ follows immediately from the choice of $x_1, \ldots, x_k$.

THEOREM 2. – $S_0(G) = W_0(G)$ *holds iff* G *is either compact or discrete.*

Proof. – If G is discrete, then $S_0(G) = W_0(G) = l^1(G)$. If G is compact, we may pass to the dual group (by property (ii)) and get the same conclusion.

For the proof of the converse, assume that G is neither compact nor discrete. If $S_0(G) = W_0(G)$ set-theoretically, it follows from the closed graph theorem that the norms on both spaces are equivalent. In order to show $S_0(G) \subsetneqq W_0(G)$, it is sufficient to prove the existence of elements $f \in S_0(G)$ such that the quotient $\|f\|_{W_0}/\|f\|_{S_0}$ becomes arbitrarily small.

We fix a compactly supported function $g \in A(G)$ which serves for the definition of the norm on $S_0(G)$ (as in Prop. 2b) and on $W_0(G)$. We may assume that $\|g\|_{S_0} = \int_G \|L_x g \cdot g\|_A \, dx = 1$. Put $Q = \operatorname{supp} g - \operatorname{supp} g$ and assume that $\epsilon > 0$ is given.

Now two cases are possible : either Q generates a compact, open subgroup of G. Since G is not compact, we can apply the Lemma to find a discrete measure $\mu = \sum_{i=1}^{n} a_i \delta_{x_i}$ on G such that $\|\mu\|_1 = \Sigma \, |a_i| = 1$ and $\|\mu^\wedge\|_\infty < \epsilon$ and the sets $x_i + Q$ $(i = 1, \ldots, n)$ are pairwise disjoint. In the other case G contains a discrete subgroup isomorphic to $\mathbf{Z}$ ([8] 9.10, 9.1). Then a simple modification of the Lemma gives the same conclusion. (This second case follows also from the classical fact that $A(T)$ is different from $C(T)$.)

Recall that $g^\wedge \in L^1(G^\wedge)$. By the Dunford-Pettis criterion [7] p. 220 the family of functions $\{M_x g^\wedge : x \in G\}$ is weakly-relatively compact in $L^1(G^\wedge)$. Since convolution is continuous, the same holds for the family $\{g^\wedge * M_x g^\wedge : x \in G\}$. Consequently there exists a compact set Q_ϵ in $G^\wedge$ such that

$$\int_{G^\wedge \setminus Q_\epsilon} |g^\wedge * M_x g^\wedge(y)| \, dy < \frac{1}{4\lambda(Q)}$$

for all $x \in G$. In the same way as before, we find a discrete measure $\nu = \sum_{i=1}^{m} b_i \delta_{y_i}$ on $G\hat{}$, such that $\|\nu\|_1 = 1$, $\|\nu\hat{}\|_\infty < \epsilon$ and the sets $y_i + Q_\epsilon$ are pairwise disjoint. Finally we put $f = \mu * (\nu\hat{} \,.\, g)$.

First we compute the S_0-norm of f. By Prop. 2b: $\|f\|_{S_0} = \int \|f \,.\, L_x g\|_A \, dx$. By the definition of Q and since the sets $x_i + Q$ are pairwise disjoint, this integral equals

$$\sum_{i=1}^{n} |a_i| \int_Q \|\nu\hat{} \,.\, g \,.\, L_x g\|_A \, dx = \int_Q \|\nu\hat{} \,.\, g \,.\, L_x g\|_A dx \quad (\|\mu\|_1 = 1) \,.$$

Now $\|\nu\hat{} \,.\, g \,.\, L_x g\|_A = \|\nu * g\hat{} * M_x g\hat{}\|_1$. From the fact that the sets $y_i + Q_\epsilon$ are pairwise disjoint and the properties of Q_ϵ we get:

$$\begin{aligned}
\|\nu * g\hat{} * M_x g\hat{}\|_1 &= \int_{G\hat{}} \Big| \sum_{i=1}^{m} b_i L_{y_i}(g\hat{} * M_x g\hat{})(y) \Big| \, dy \\
&\geqslant \sum_{i=1}^{m} |b_i| \Big(\int_{y_i + Q_\epsilon} |L_{y_i}(g\hat{} * M_x g\hat{})(y)| \, dy \\
&\qquad - \int_{G\hat{}\setminus(y_i + Q_\epsilon)} |L_{y_i}(g\hat{} * M_x g\hat{})(y)| \, dy \Big) \\
&\geqslant \int_{Q_\epsilon} |(g\hat{} * M_x g\hat{})(y)| \, dy - \frac{1}{4\lambda(Q)} \\
&\geqslant \|g\hat{} * M_x g\hat{}\|_1 - \frac{1}{2\lambda(Q)} \,.
\end{aligned}$$

Integration over Q gives (recall that $\int_G \|g\hat{} * M_x g\hat{}\|_1 \, dx = 1$): $\|f\|_{S_0} \geqslant 1 - \frac{1}{2} = \frac{1}{2}$. The $W(G)$-norm can be estimated as follows: $\|f\|_W = \|\mu * (\nu\hat{} \,.\, g)\|_W \leqslant \|\nu\hat{} \,.\, g\|_W \leqslant \|\nu\hat{}\|_\infty \,.\, \|g\|_W < \epsilon \|g\|_W$ (since $W(G)$ is a $C(G)$-module).

Similarly we have on the dual group:

$$\|f\hat{}\|_W = \|\mu\hat{} \,.\, (\nu * g\hat{})\|_W \leqslant \|\mu\hat{}\|_\infty \,.\, \|\nu * g\hat{}\|_W < \epsilon \|g\hat{}\|_W \,.$$

This gives combined: $\|f\|_{W_0} < \epsilon \|g\|_{W_0}$ (recall that $g \in A(G)$ with compact support, therefore $g \in S_0(G) \subseteq W_0(G)$). Since we have chosen g independently of ϵ, we get the desired conclusion.

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Viktor LOSERT,
Institut für Mathematik
der Universität Wien
Strudlhofgasse 4
A – 1090 Wien (Autriche).