Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 30, nº 2 (1980), p. 97-107

<http://www.numdam.org/item?id=AIF_1980__30_2_97_0>

Ann. Inst. Fourier, Grenoble
30, 2 (1980), 97-107.

APPROXIMATION OF HARMONIC FUNCTIONS

by Björn E. J. DAHLBERG

1. Introduction.

In this note we shall study the following approximation problem : Let u be harmonic in a domain D that has a regular boundary. When is it possible to find functions f_j of bounded variation in D (that is functions whose gradients are bounded in D) such that $\sup_D |f - f_j| \to 0$ as $j \to +\infty$? The main result of this paper is that this approximation is always possible if u is the Poisson integral of a function $f \in L^p(\sigma)$, $p \geqslant 2$, where σ denotes the surface measure of ∂D and is not always possible if $f \in L^p(\sigma)$, $p < 2$.

This type of approximation appears implicity in the main step of the proof of the Corona theorem, see Carleson [1, 2], for the case when u is a bounded and holomorphic function. For the case when u is the Poisson integral of a function of bounded mean oscillation BMO this type of approximation has been carried out by Varopoulos [9] and Garnett [5]. In these cases it is required that the approximands f_j have gradients that are Carleson measures.

THEOREM 1. — *Suppose u is harmonic in a bounded Lipschitz domain $D \subset \mathbf{R}^n$, $n \geqslant 2$. Then for every $\varepsilon > 0$ there is a function φ such that $|u - \varphi| < \varepsilon$ in D and for all $P \in \partial D$ we have that*

$$\int_{\beta(r)} |\nabla\varphi| \, dQ \leqslant C[\varepsilon^{-1} \int_{\beta(Cr)} |\nabla u|^2 \operatorname{dist}\{Q,\partial D\} \, dQ + \varepsilon r^{n-1}].$$

Here $\beta(r) = \{Q \in D : |Q-P|<r\}$ and $\nabla\varphi$ denotes the gradient of φ. The constant C only depends on D.

We remark that this result means that φ is of bounded variation if $\int_D |\nabla u|^2 \operatorname{dist}\{Q,\partial D\} \, dQ < \infty$. It's known that this happens if and only if u

is the Poisson integral of a function $f \in L^2(\sigma)$, see Stein [8] for the case of domains with smooth boundaries and Dahlberg [3] for the case of Lipschitz domains.

We recall that a measure μ is called a Carleson measure if $|\mu|(\beta(P,r)) \leqslant Cr^{n-1}$ for all $P \in \partial D$. It's known that a harmonic function u is the Poisson integral of a function of bounded mean oscillation if and only if $|\nabla u|^2 \operatorname{dist}\{Q,\partial\}$ is a Carleson measure, see Fefferman-Stein [4] for the case of smooth domains and this has recently been shown to hold for Lipschitz domains by E. Fabes and U. Neri (unpublished). Therefore $|\nabla\varphi|\, dQ$ is a Carleson measure if and only if u is the Poisson integral of a BMO-function, see Varopoulos [9].

THEOREM 2. – *Let* U *denote the unit disk in* $\mathbf{R}^2$. *If* $p < 2$ *then there is an* $f \in L^p(\sigma,\partial U)$ *such that if* $u = Pf$ *then*

$$\sup_{U} |u - \varphi| = \infty$$

for all φ *that are of bounded variation in* U.

In addition to this exemple it's known that there are bounded holomorphic functions that are not of bounded variation, see Rudin [7].

2. The method of approximation.

We start by recalling that a bounded domain $D \subset \mathbf{R}^n$ is called a Lipschitz domain if ∂D can be covered by finitely many open right circular cylinders whose bases have a positive distance from ∂D and corresponding to each cylinder L there is a coordinate system (x,y) with $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, with the y-axis parallel to the axis of L and a function $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ satisfying a Lipschitz condition (i.e. $|\varphi(x)-\varphi(z)| \leqslant M|x-z|$) such that

$$L \cap D = \{(x,y) : y > \varphi(x)\} \cap L$$

and

$$D \cap L = \{(x,y) : y = \varphi(x)\} \cap L .$$

We recall that a Lipschitz domain D is starshaped with star center P^* and with standard inner cone Γ if $P^* \in \Gamma(P) \subset D$ for all $P \in \partial D$, where $\Gamma(P)$ denotes the cone with vertex P having its axis along the line through P and P^* and being congruent to Γ. (With a cone we mean an open, non empty, convex and possibly truncated cone). If u is harmonic in D and $u(P^*) = 0$ we have the following result from Dahlberg [4] : Let γ be a cone with the same vertex P_0 as Γ and assume that $\bar{\gamma} - \{P_0\} \subset \Gamma$. Let $\gamma(P)$ be constructed as $\Gamma(P)$ and put

$$M(P) = \sup \{|u(Q)| : Q \in \gamma(P)\}.$$

Then

$$(2.1)\quad C^{-1} \int_{\partial D} M^2 \, d\sigma \leqslant \int_D |\nabla u|^2 \operatorname{dist}\{Q,\partial D\} \, dQ \leqslant C \int_{\partial D} M^2 \, d\sigma,$$

where C only depends on γ and Γ.

We shall first suppose that u is a function in the cube

$$U = \{(x,y) : 0<x_i<1,\ i=1,2\ldots,n-1,\ 0<y<1\}.$$

We let Ω_m denote the collection of all dyadic cubes of side 2^{-m} in $\{x \in \mathbf{R}^{n-1} : 0<x_i<1\}$. If $Q \in \Omega_m$ we put $T(Q) = \{(x,y) : x \in Q, 2^{-m-1} \leqslant y < 2^{-m}\}$. The collection of all $T(Q)$, when Q runs over $\bigcup_{m\geqslant 0} \Omega_m$ is denoted by Λ. If $T_1, T_2 \in \Lambda$ and $T_i = T(Q_i)$ we say that $T_1 < T_2$ if $Q_1 \subset Q_2$ and the side of Q_2 is twice the side of Q_1. We shall fix the number $a > 0$ and put $\Gamma = \{(x,y) : |x|<ay\}$. For $P \in \mathbf{R}^n$ we set $\Gamma_p = P + \Gamma = \{P+Q : Q \in \Gamma\}$. For $T \in \Lambda$ we put

$$L(T) = \left\{V \in \Lambda : V \cap \left[\bigcup_{p\in T} \Gamma_p\right] \neq \varnothing\right].$$

We observe that if $T_1 < T_2$ and $T_1 \in L(T)$ then $T_2 \in L(T)$ also.

We shall next describe the method for approximating u. We say that a $T \in \Lambda$ is red if $\operatorname{diam}(T) \sup_T |\mathcal{O}u| \geqslant k\varepsilon$. Otherwise it's called blue. (Here k is a small number to be chosen later.) The main step now is to put together the blue intervals into domains of Lipschitz character, where the oscillation of u is $\leqslant \varepsilon$.

Let $S = \left\{(x,y) : 0<x_i<1, \frac{1}{2} < y < 1\right\}$ and suppose that S is blue. We shall now define $K(S) \subset \Lambda$ inductively as follows : First $S \in K(S)$ and a $T \in \Lambda$ is added to $K(S)$ provided there is a $T' \in K(S)$ such that $T < T'$, all elements of $L(T)$ are blue and $|u(P_S)-u(P_T)| \leqslant m\varepsilon$, where P_T is the center of T. Put $H(S) = \underset{T\in K(S)}{L}(T)$ and let $D(S)$ denote the interior of the closure of $\bigcup_{T\in H(S)} T$. Suppose now that $T \in \Lambda$, $T \subset U - D(S)$, and $\partial T \cap d(S) \neq \varnothing$, where $d(S) = U \cap \partial D(S)$. Let T_i, $0 \leqslant i \leqslant N$, be such that $T = T_0 < T_1 < \ldots < T_N = S$ and let j be the smallest integer such that $T_j \in K(S)$. Since $T_{j-1} \notin K(S)$ there are two cases to consider. If $L(T_{j-1})$ contains a red interval R we say that $T \in A(S)$ and if this is not the case we say that $T \in B(S)$. Also, we define $\alpha(S)$ and $\beta(S)$ as $U(\partial T \cap \partial D(S))$ whereT runs over A(S) and B(S) respectively. We observe that there is a number $M > 0$ only depending on Γ such that the projection T' of T into $\mathbf{R}^{n-1}$ is contained in R^*, where $R^* \subset \mathbf{R}^{n-1}$ is the cube with the same center as R^1 but with a side that is M times the side of R'. (Here R is the red interval contained in $L(T_{j-1})$.) Also there is a $v \in H(S)$ such that-diam $R \leqslant$ diam $V \leqslant 2$ diam R and $|P_R - P_V| \leqslant M$ diam R (we'll say that R touches D(S)). Let $|E|$ denote the $(n-1)$-dimensional Hausdorff measure of a set $E \subset \mathbf{R}^n$. The Lipschitz character of D(S) implies that $|\alpha(S)| \leqslant C \left| \bigcup_{T \in A(S)} T' \right|$, which together with the above observations show that

$$|\alpha(S)| \leqslant C\Sigma |\partial R|, \tag{2.2}$$

where the sum is taken over all red intervals that touch D(S). Let $b > a$ be sufficiently large and put $\gamma = \{(x,y) : |x| < -by\}$. If $\Omega = \bigcup_{P \in d(S)} \gamma_p$, then $D_1 = D(S) - \bar{\Omega}$ is again a Lipschitz domain. It's easily seen that if $a > 0$ has been chosen sufficiently small then b can be chosen so that D_1 is a starshaped Lipschitz domain with starcenter P_S and a standard inner cone P' that only depends on a and b. We have also that

$$\left| \bigcup_{T \in B(S)} \partial T \cap d(S) \cap \partial D_1 \right| \geqslant C|\beta(S)|$$

where $c > 0$ only depends on a and b.

For $P \in \partial D_1$ we put $M_S(P) = \sup |u(Q) - u(P_S)|$, where Q runs over all points on the line segment between P and P_S. Suppose now that $T \in B(S)$ and $T = T_0 < T_1 < \ldots < T_N = S$. If j is the smallest index for which $T_j \in K(S)$ it follows that $L(T_{j-1})$ does not contain any red cube. If P_{j-1} denotes the center of T_{j-1} it follows that $|u(P_{j-1}) - u(P_S)| \geqslant m\varepsilon$.

If $j = 1$ it follows that $|u(P) - u(P_S)| \geqslant (m-k)\varepsilon$ for all $P \in T = T_0$ and hence $M_S(P) \geqslant (m-k)\varepsilon$ for all $P \in \partial T \cap \partial D_1$. Suppose now that $j > 1$ and $P \in \partial T \cap d(S) \cap \partial D_1$. Let Q denote the point on the line segment between P and P_S that has the same y-coordinate as P_{j-1}. Since the line segment between P_{j-1} and Q is contained in D(S) it follows that

$$|u(P_{j-1}) - u(Q)| \leqslant k\varepsilon |P_{j-1} - Q| (\text{diam } T_{j-1})^{-1} < m\varepsilon/2$$

if k has been chosen sufficiently small. Hence we have in all cases that

$$(2.3)\qquad |\beta(S)| \leqslant C|\{P \in \partial D_1 : M(P) > m\varepsilon/2\}|.$$

If there is an interval in $\Lambda - H(S)$ that's not red let S_1 denote one with maximal diameter. After making a change of scale we construct $H(S_1)$ as above and in this way we get a decomposition $\Lambda = \Lambda_R \cup \left[\bigcup_j H(S_j)\right]$ into pairwise disjoint sets, where Λ_R denotes the collection of all red intervals in Λ. We claim that if u is harmonic and $L_j = |\partial D(S_j)|$ then

$$(2.4)\qquad \Sigma L_j \leqslant C\left[1 + \varepsilon^{-2} \iint_{\tilde{U}} y|\nabla u|^2\, dx\, dy\right]$$

where C is independent of u and ε, $\tilde{U} = \{(x,y) : -1 < x_i < 2,\ 0 < y < 2\}$. Following Garnett [5] we first observe that if $R \in \Lambda$ is red then

$$(2.5)\qquad |\partial R| \leqslant C\varepsilon^{-2} \iint_{R^*} y|\nabla u|^2\, dx\, dy,$$

where $\quad R^* = \bigcup_{P \in R} B(P,\delta/2), \qquad \delta = \operatorname{dist}\{R, \mathbf{R}^{n-1}\}$

and $\quad B(P,r) = \{Q : |P-Q| < r\}.$

To see (2.5), we first observe that there is a number c_n only depending on n such that there is $P \in \bar{R}$ with $|\nabla u(P)| \geqslant c_n k\varepsilon\delta^{-1}$. Since $|\nabla u|^2$ is subharmonic it follows that

$$\iint_{R^*} |\nabla u|^2 y\, dx\, dy \geqslant \frac{1}{2}\delta \iint_{B(P,\delta/2)} |\nabla u|^2\, dx\, dy \geqslant c\varepsilon^2 |\partial R,$$

which gives (2.5). We also observe that from Cauchy's inequality follows that $\left(\iint_R |\nabla u| dx\, dy\right)^2 \leqslant C|\partial R| \iint_R |\nabla u|^2 y\, dx\, dy$ which together with (2.5) gives

$$(2.6)\qquad \iint_R |\nabla u|\, dx\, dy \leqslant C\varepsilon^{-1} \iint_{R^*} |\nabla u|^2 y\, dx\, dy.$$

Let $\theta > 0$ be a small fixed number and let I denote those $j : s$ for which $|\partial D(S_j) \cap \mathbf{R}^{n-1}| \geqslant \theta L_j$. Since the domains $D(S_j)$ are pairwise disjoint it follows that

$$(2.7)\qquad \sum_I L_j \leqslant \theta^{-1}.$$

Let II denote those $j : s$ for which $|\alpha(S_j)| \geqslant \theta L_j$. Since the domains $\{R^*\}_{R \in \Lambda_R}$ have uniformly bounded overlap and there is a fixed number N such that no red interval $R \in \Lambda_R$ touches more than N of the domains $D(S_j)$ it follows from (2.2) and (2.5) that

$$\sum_{\mathrm{II}} L_j \leqslant \theta^{-1} \Sigma |\alpha(S_j)| \leqslant C\varepsilon^{-2} \iint_{\tilde{U}} y|\nabla u|^2 \, dx \, dy. \tag{2.8}$$

Finally let III be those $j : s$ for which $|\beta(S_j)| \geqslant \theta L_j$. From (2.1) and (2.3) follows that

$$|\beta(S_j)| \leqslant C\varepsilon^{-2} \int_{D_j} \mathrm{dist}\,\{Q, \partial D_j\} \, |\nabla u|^2 \, dQ \leqslant C\varepsilon^{-2} \iint_{D_j} y|\nabla u|^2 \, dx \, dy$$

so we have that

$$\sum_{\mathrm{III}} L_j \leqslant C\varepsilon^{-2} \iint_{U} y|\nabla u|^2 \, dx \, dy. \tag{2.9}$$

If the constant θ has been chosen small enough then each $D(S_j)$ belongs to one of the categories I, II or III. Hence (2.4) follows from (2.7-9).

We now define $\varphi = uh + \Sigma u(P_j)h_j$, where h is the characteristic function of $\bigcup_{R \in \Lambda_R} \bar{R}$, h_j is the characteristic function of $D(S_j)$ and P_j is the center of S_j. Clearly $|u - \varphi| \leqslant \varepsilon$. It remains to estimate $|\nabla\varphi|$. To this end let λ_j be the surface measure of $\partial D(S_j)$ and if $\{R_j\}_{j=1}^{\infty} = \Lambda_R$ we let σ_j denote the surface measure of ∂R_j. With this notation we have that $|\nabla\varphi| \leqslant C[|\nabla u|h + \varepsilon\Sigma(\sigma_j + \lambda_j)]$, where the ε in front of the sum appears because the jump of φ at a common boundary point of domains of the form $D(S_j)$ or R_k is less than ε.

Let $Q \subset \mathbf{R}^{n-1}$ be a cube and put

$$S(Q) = \{(x,y) : x \in Q, \ 0 < y < \text{side of } Q\}.$$

We shall now estimate $\iint_{S(Q)} |\nabla\varphi| \, dx \, dy$. Let M be a large positive number and let $V \subset \mathbf{R}^{n-1}$ be the largest dyadic cube that contains Q for which $|V| \leqslant 6^n M |Q|$. If M is large enough, then it follows from (2.5) and (2.6) that

$$\iint_{S(Q)} |\nabla u| h \, dx \, dy + \varepsilon \Sigma \sigma_j(S(Q)) \leqslant C\varepsilon^{-1} \iint_{S(V)} |\nabla u|^2 \, dx \, dy.$$

From (2.4) and possibly a change of scale we see that

$$\Sigma' \lambda_j(S(Q)) \leqslant C\left[\varepsilon^{-2} \iint_{S(V)} |\nabla u|^2 y \, dx \, dy + |Q|\right],$$

where the prime denotes summation over those $j : s$ for which $S_j \subset S(V_1)$, where V_1 is the largest dyadic cube that contains Q for which $|V_1| \leqslant M|Q|$. If $\lambda_j(S(Q)) > 0$ and if S_j is not contained in $S(V_1)$ then $D(S_j)$ contains (x_Q, Ly_Q) where (x_Q, y_Q) is the center of $S(Q)$ and the constant L only depends on M and the choice of the cone Γ for the construction of $D(S_j)$. Since the domains $D(S_k)$ are pairwise disjoint there is at most one j with this property and from the Lipschitz character of $D(S_j)$ it follows that $\lambda_j(SQ)) \leqslant C|Q|$ which concludes the proof of theorem 1 for the case of smooth domains.

The case when u is harmonic in a Lipschitz domain is easily reduced to the case when u is defined in

$$U' = \{(x,y) : 0<x_i<1, f(x)<y<f(x)+1\},$$

where f is a Lipschitz function. Letting $T(x,y) = (x, y-f(x))$ we see that T maps U' onto

$$U = \{(x,y) : 0<x_i<1, 0<y<1\}.$$

Let $u_1 = u \circ T^{-1}$ and construct φ_1 in U as above, this time approximating u_1. Letting $\varphi = \varphi_1 \circ T$, it's easily seen that the methods for estimating $\nabla\varphi$ work in this case too, which yields theorem 1.

3. An example.

In this section we shall identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and we'll denote points in $\mathbf{C}$ by $z = x + iy$, $x, y \in \mathbf{R}$. We'll put $J = \{x : -1 < x < 1\}$ and $Q = \{z : |x|<2, 0<y<4\}$. If $f \in L^p(\mathbf{R})$ we let Pf denote the Poisson integral of f. We shall establish the following result.

THEOREM 3. — *For all $p < 2$ there is an $f \in L^p(\mathbf{R})$ with support in J such that* $\sup_Q |Pf - \varphi| = \infty$ *for all φ such that* $\iint_Q |\nabla\varphi| \, dx \, dy < \infty$.

We shall deduce theorem 3 from the following lemma, the proof of which is given later.

LEMMA 1. — *For $\theta \in (0,1)$ there is a function $g_\theta \in L^2(\mathbf{R})$ with support in J such that if $0 < \varepsilon < 1$ and $|u - \varphi| \leqslant \varepsilon$ in Q, then* $\iint_Q |\nabla\varphi| \, dx \, dy \geqslant c\varepsilon^{-\theta}$, *where $c > 0$ is independent of ε.*

Proof of theorem 3. — We shall first define a sequence of intervals $I_j \subset \mathbf{R}$ by putting $I_1 = [0,1]$ and requiring that I_{j+1} is to the right of I_j, $|I_j| = 2^{-j}$ and dist $\{I_j, I_{j+1}\} = j^{-2}$. Let c_j denote the center of I_j and put

$$g_j(x) = g_\theta(2^{j+1}(x-c_j)), \tag{3.1}$$

where g_θ is as in lemma 1. It's easily seen that

$$|\nabla P g_j(z)| \leqslant C 2^{-j}|z-c_j|^{-2}$$

whenever $|z-c_j| > 2^{-j}$. If $Q_j = \{z : |x-c_j| < 2^{-j},\ 0<y<2^{1-j}\}$ we therefore have

$$\sup\{|\nabla P g_k(z)| : z \in Q_j\} \leqslant C 2^{-k}k^2 (k \neq j). \tag{3.2}$$

Let $b_j > 0$ be defined by $b_j^{p2^{-j}} = j^{-2}$ and put $f = \Sigma b_j g_j$. Clearly $f \in L^p(\mathbf{R})$ and the support of f is bounded. From (3.2) follows $u = b_j P f_j + R_j$, where $u = Pf$ and

$$\sup\{|\nabla R_j(z) : z \in Q_j\} \leqslant C \sum_k b_k k^2 2^{-k} = M < \infty. \tag{3.3}$$

Suppose now that $|u-\varphi| \leqslant L < \infty$ in $\bigcup_{j\geqslant 1} Q_j$. We shall next show that this implies that $\sum_j \iint_{Q_j} |\nabla\varphi|\, dx\, dy = \infty$ whenever $\theta > p-1$.

If z_j denotes the center of Q_j it follows from (3.3) that

$$\sup\{|m_j - R_j(z)| : z \in Q_j\} \leqslant M \operatorname{diam}(Q_j) \to 0 \text{ as } j + \infty,$$

where $m_j = R_j(z_j)$. Therefore there is a j_0 such that if $j \geqslant j_0$ then $|Pf_j - \varphi_j| \leqslant 2Lb_j^{-1}$ in Q_j, where $\varphi_j = (\varphi - m_j)b_j^{-1}$. From lemma 1 follows now that

$$\sum_j \iint_{Q_j} |\nabla\varphi|\, dx\, dy \geqslant C \sum_{j\geqslant j_0} 2^{-j} b_j^{i+\theta} = \infty \text{ if } \theta > p-1$$

which yields the theorem.

We remark that by using a suitable conformal mapping it's easily seen that theorem 2 follows from theorem 3.

We'll need the following lemma for the proof of lemma 1.

LEMMA 2. — *Suppose u is harmonic in $B = B(z_0,5r) \subset \mathbf{C}$. If $|u-\varphi| \leqslant \varepsilon$ in B and if* $\sup\{|u(z_1)-u(z_2)| : z_1,z_2 \in B(z_0,r)\}$ *then* $\iint_B |\nabla\varphi|\, dx\, dy \geqslant c\varepsilon r$, *where $c > 0$ is a universal constant.*

Proof. – Pick $z_1, z_2 \in \mathrm{B}(z_0,r)$ such that $|u(z_1)-u(z_2)| \geqslant 7\varepsilon$. Since the function $z \to |u(z)-u(z_2)|^2$ is subharmonic it follows that

$$\int_{\mathrm{B}(z_1,r)} |u(z) - u(z_2)|^2 \, dx \, dy \geqslant 7^2\pi\varepsilon^2 r^2 .$$

Since $\mathrm{B}(z_1,r) \subset \tilde{\mathrm{B}} = \mathrm{B}(z_2,3r)$ we therefore have that

$$\int_{\tilde{\mathrm{B}}} |\varphi - \tilde{\varphi}|^2 \, dx \, dy \geqslant \pi\varepsilon^2 r^2 ,$$

where $\tilde{\varphi} = \int_{\tilde{\mathrm{B}}} \varphi \, dx \, dy \int_{\tilde{\mathrm{B}}} dx \, dy$. The Poincaré-Soboev inequality (see Meyers and Ziemer [6] for general versions) says that there is a constant C such that for all balls

$$\tilde{\mathrm{B}}\left(\int_{\tilde{\mathrm{B}}} |\varphi - \tilde{\varphi}|^2 \, dx \, dy\right)^{1/2} \leqslant \mathrm{C} \int_{\tilde{\mathrm{B}}} |\nabla\varphi| \, dx \, dy ,$$

which yields lemma 2.

We shall next prove lemma 1. Let $\alpha > 0$ be defined by $(1-2\alpha) = \theta(1+2\alpha)$ and put $a_k = k^{-1/2-\alpha}$ for $k = 1, 2 \ldots$ Let $\delta > 0$ be a given number. We claim that there is a sequence of positive integers $n_k + \infty$ such that if $f(z) = \sum^{\infty} a_k z^{n_k}$ and if

$$\mathrm{S}_k = \{z : n_k^{-1} \leqslant 1 - |z| \leqslant 4n_k^{-1}\}$$

then $f'(z) = a_k n_k z^{n_k - 1} + \mathrm{R}_k(z)$, where

$$\sup \{|\mathrm{R}_k(z)| : z \in \mathrm{S}_k\} \leqslant \delta a_k n_k .$$

To see this choose $n_1 = 100$ and if $n_1, \ldots, n_{k-1}$ have been chosen then

$$\left|\sum_{j<k} a_j n_j z^{n_j - 1}\right| \leqslant k n_{k-1} < \delta/2 a_k n_k$$

if n_k has been chosen large enough. If we also require that $n_{j+1} \geqslant n_j + 2$ and

$$(1 - n_j^{-1})^{\frac{1}{2}n_j} + 1^{-1} n_{j+1} \leqslant \min(1, a_j \delta/2)$$

we have for $z \in S_k$ that

$$\left|\sum_{j>k} a_j n_j z^{n_j-1}\right| \leqslant \sum_{j>k} a_j n_j (1-n_k^{-1})^{n_j-1}$$

$$\leqslant \sum_{j>k} (1-n_k^{-1})^{\frac{1}{2}n_j} \leqslant \sum_{s=1}^{\infty} (1-n_k^{-1})^{\frac{1}{2}n_k+s} \leqslant \frac{1}{2}\delta n_k a_k$$

and adding these extimates yields the claim.

Hence if δ has been chosen sufficiently small then whenever $B \subset S_k$ is disk of radius $(10n_k)^{-1}$ we have that

(3.4) $$\sup\{|f(z_1) - f(z_2)| : z_1, z_2 \in B\} > ca_k$$

where $c > 0$ is independent of k.

Let $u = P(fh)$, where h is the characteristic function of

$$\{z : |z|=1, \ \mathrm{Re}\, z>0\} = L.$$

Since $u - f$ has boundary values zero on L it follows that $u - f$ has a harmonic extension to all of $\{z : \mathrm{Re}\, z>0\}$. We therefore have that if B is a disk of radius $(10n_k)^{-1}$ such that

$$B \subset S_k \cap \{z : |\arg z| \leqslant \pi/3\} = S_k^*$$

then it follows from (3.4) that

(3.5) $$\sup\{|u(z_1)-u(z_2)| : z_1, z_2 \in B\} \geqslant da_k$$

for $k \geqslant k_0$, where $d > 0$ is independent of k.

Suppose now that $\varepsilon > 0$ is a small number and that

$$|u-\varphi| \leqslant \varepsilon \text{ in } \Omega = \{z : |z|<1, \ \mathrm{Re}\, z> -1/2\}.$$

There is a number $\lambda\ 0$ such that we can find more than λn_k disks $B(j,k)$ of radius $(10n_k)^{-1}$ such that $10B(j,k) \subset S_k^*$ whenever $1 \leqslant j \leqslant \lambda n_k$ and the disks $B(j,k)$ are pairwise disjoint. It's easily seen from (3.5) that there is an $m > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then

$$\sup\{|u(z_1)-u(z_2)| : z_1, z_2 \in B(j.k)\} > 10\varepsilon$$

whenever $1 \leqslant j \leqslant \lambda n_k$, $ko < k < L(\varepsilon)$, where $L(\varepsilon) = m\varepsilon^{-\beta}$, $\beta = 2(1+2\alpha)^{-1}$. From lemma 2 follows now that

$$\iint_{\Omega} |\nabla\varphi|\, dx\, dy \geqslant \sum_{k=k_0}^{L(\varepsilon)} \sum_{j=1}^{\lambda n_k} \iint_{10B(j,k)} |\nabla\varphi|\, dx\, dy \geqslant c' L(\varepsilon)\varepsilon = c\varepsilon^{-\theta}.$$

Finally, mapping the unit disk conformally onto the upper half plane yields lemma 1.

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Manuscrit reçu le 12 novembre 1979.

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