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1. Introduction.

Schwartz’s Theorem in the theory of mean periodic functions on the real line states that every closed, translation-invariant subspace of the space of continuous functions on $\mathbb{R}$ is spanned by the polynomial-exponential functions it contains [5]. In particular, every translation-invariant subspace contains an exponential function.

In [2] the two-sided analogue of this result was generalized to $SL_2(\mathbb{R})$. However, since [3] it is known that Schwartz's Theorem fails to hold for $\mathbb{R}^n$, $n > 1$.

Our main goal is to show that the two-sided analogue of Schwartz’s Theorem holds for the motion group $M(2)$. That is, every closed, two-sided invariant subspace of $C(M(2))$ contains an irreducible invariant subspace and every such subspace is spanned by a class of functions which replace the polynomial-exponentials on $\mathbb{R}$.

It seems remarkable that the analogue of Schwartz’s Theorem holds for the three dimensional Lie groups $SL_2(\mathbb{R})$ and $M(2)$ while it fails to hold for $\mathbb{R}^2$.

In section 3 we verify the two-sided Schwartz’s Theorem for the motion group. In section 4 we consider the problem of one-sided spectral analysis. Finally, in section 5, we study some invariant subspaces of continuous functions on $\mathbb{R}^2$. It turns out that the one-sided Schwartz's Theorem for the motion group is intimately connected with a problem of Pompeiu type [1, 4, 7].
2. Preliminaries and Notation.

Let \( M(2) \) denote the Euclidean motion group consisting of the matrices \( \begin{pmatrix} e^{i\alpha} & z \\ 0 & 1 \end{pmatrix} \), \( \alpha \in \mathbb{R}, \ z \in \mathbb{C} \).

Let \( C(M(2)) \) denote the space of all continuous functions on \( M(2) \) with the usual topology of uniform convergence on compact sets. Let \( \mathcal{S}(\mathbb{R}^n) \) be the space of infinitely differentiable functions on \( \mathbb{R}^n \) endowed with the topology of uniform convergence of functions and their derivatives on compacta. Let \( \mathcal{S}'(\mathbb{R}^n) \) be the dual of \( \mathcal{S}(\mathbb{R}^n) \), the space of Schwartz distributions on \( \mathbb{R}^n \) having compact support. The pairing between \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \) is denoted by \( \langle T, f \rangle \) for \( f \in \mathcal{S}(\mathbb{R}^n) \) and \( T \in \mathcal{S}'(\mathbb{R}^n) \), and for such \( f \) and \( T \) we denote by \( T \ast f \) the convolution of \( T \) and \( f \). For \( T \in \mathcal{S}'(\mathbb{R}^n) \), the Fourier transform of \( T \) is defined by \( \hat{T}(z) = T(e^{iz \cdot x}) \) where \( z \in \mathbb{C}^n, \ x \in \mathbb{R}^n \) and \( z \cdot x = z_1 x_1 + \ldots + z_n x_n \). By Paley-Wiener-Schwartz Theorem, the space \( \mathcal{S}'(\mathbb{R}^n) \) of Fourier transforms of elements of \( \mathcal{S}(\mathbb{R}^n) \) is identified with the space of entire functions of \( n \) complex variables of exponential type which have polynomial growth on the real subspace \( \mathbb{R}^n \). The topology of \( \mathcal{S}'(\mathbb{R}^n) \) is so defined as to make the Fourier transform a topological isomorphism.

Let \( \Pi \) denote the group of all rotations of \( \mathbb{R}^2 \). We denote by \( \mathcal{S}'(\mathbb{R}^2) \) the space of all \( T \in \mathcal{S}'(\mathbb{R}^2) \) which satisfy \( T \circ \tau = T \) for every \( \tau \in \Pi \). Let \( \mathcal{S}'(\mathbb{R}^2) \) denote the space of Fourier transforms of elements of \( \mathcal{S}'(\mathbb{R}^2) \). We notice that each \( f \in \mathcal{S}'(\mathbb{R}^2) \) is a function of \( z_1^2 + z_2^2 \) and that for any even function \( g \in \mathcal{S}'(\mathbb{R}^2) \) the function \( \tilde{g} \) where \( \tilde{g}(z_1, z_2) = g(\sqrt{z_1^2 + z_2^2}) \) belongs to \( \mathcal{S}'(\mathbb{R}^2) \).

Let \( \mathcal{S}_0(\mathbb{R}^2) \) denote the space of elements of \( \mathcal{S}(\mathbb{R}^2) \) having compact support and \( \mathcal{S}_r(\mathbb{R}^2) \) the space of radial functions in \( \mathcal{S}_0(\mathbb{R}^2) \).

Let \( C(\mathbb{R}^n) \) denote the space of continuous functions on \( \mathbb{R}^n \) with the topology of uniform convergence on compacta and \( C^r(\mathbb{R}^2) \) the radial functions in \( C(\mathbb{R}^2) \). The dual of \( C(\mathbb{R}^n) \) is the space \( M_0(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \) of all complex-valued Radon measures having compact support. Let \( M_r(\mathbb{R}^2) = M_0(\mathbb{R}^2) \cap \mathcal{S}'(\mathbb{R}^2) \).

Finally, for \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \) and \( z = x + iy \in \mathbb{C} \) let \( \langle \lambda, z \rangle = \lambda_1 x + \lambda_2 y \).
3. Two-sided spectral synthesis.

The two-sided analogue of Schwartz's Theorem in spectral analysis for the motion group is stated in the following:

**Theorem 1.** — Every closed, two-sided invariant subspace of $C(M(2))$ contains either a character of $M(2)$ or a function $g(e^{i\alpha}, z) = e^{i(\lambda_1, \lambda_2)}$ where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and $\lambda_1^2 + \lambda_2^2 \neq 0$. The two-sided invariant subspace generated by $e^{i(\lambda_1, \lambda_2)}$ where $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1^2 + \lambda_2^2 \neq 0$, is irreducible (minimal).

**Proof.** — For $f \in C(M(2))$, $f \neq 0$, let $V_f$ denote the closed subspace generated by the two-sided translates of $f$.

The subspace $V_f$ contains all the functions $g$ where
\begin{equation}
\begin{aligned}
g(e^{i\alpha}, z) &= f(e^{i(\alpha+\theta)}, u e^{i\alpha} + e^{i\theta} z + w) \\
&= e^{im\alpha} \int_0^{2\pi} f(e^{i\beta}, z) e^{-im\beta} d\beta
\end{aligned}
\end{equation}
for every $\theta, \beta \in \mathbb{R}$ and $u, w \in \mathbb{C}$. Let $u = \theta = w = 0$ in (1).

Then, for a suitable $m \in \mathbb{Z}$ the function
\[
\int_0^{2\pi} f(e^{i(\alpha+\theta)}, z) e^{-im\beta} d\beta = e^{im\alpha} g_1(z)
\]
is non-zero and belongs to $V_f$. Let $N$ denote the translation-invariant and rotation-invariant subspace of $C(\mathbb{R}^2)$ generated by $g_1$.

By (1) the functions $e^{im\alpha} g_1(e^{i\theta} z + w)$ belongs to $V_f$ for every $\theta \in \mathbb{R}$ and $w \in \mathbb{C}$. That is, $V_f$ contains all functions $e^{im\alpha} \widetilde{g}(z)$ where $\widetilde{g} \in N$. In [1] it was proved that every closed, translation-invariant and rotation-invariant subspace of $C(\mathbb{R}^2)$ is spanned by the polynomial-exponential functions it contains. In particular, the subspace $N$ contains therefore an exponential function $e^{i(\lambda_1, \lambda_2)}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and the function $h(e^{i\alpha}, z) = e^{im\alpha} e^{i(\lambda_1, \lambda_2)}$ belongs to $V_f$. If $\lambda_1^2 + \lambda_2^2 = 0$ then the subspace $N$ contains the constant functions and $V_f$ contains therefore the character $e^{im\alpha}$. Suppose that $\lambda_1^2 + \lambda_2^2 \neq 0$.

Let $h_1 \in \mathfrak{s}_0(\mathbb{R}^2)$ of the form $h_1(w) = h_2(r) e^{-i\theta m}$ where $w = re^{i\theta}$, and $h_2 \in \mathfrak{s}_0(\mathbb{R})$ such that $h_1(\lambda_1, \lambda_2) \neq 0$. 

Then the function:

\[
\int_{\mathbb{R}^2} h(e^{i\alpha}, z - e^{i\alpha}w) h_1(w) \, dw = \hat{h}_1(\lambda_1, \lambda_2) e^{i(\lambda, z)}
\]

\[ (2) \]

(here \(dw\) denotes Lebesgue measure on \(\mathbb{R}^2\)) is non-zero and belongs to \(V_f\). It follows, by (1) and the analyticity of the elements of \(\mathcal{G}_0(r)(\mathbb{R}^2)\) that \(V_f\) contains all functions \(e^{i(\mu, z)}\) where \(\mu = (\mu_1, \mu_2) \in \mathbb{C}^2\) such that \(\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_2^2\). To prove the second part of the theorem, let \(g(z) = e^{i(\lambda, z)}\) where \(\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2\), \(\lambda_1^2 + \lambda_2^2 \neq 0\). Firstly, we will show that \(V_g\) contains no character of \(\text{M}(2)\).

Suppose that \(e^{ima} \in V_g\) for some \(m \in \mathbb{Z}\). Let \(\mu \in \text{C}(\text{M}(2))\), \(\mu(e^{i\alpha}, z) = e^{-ima} \mu_1(z)\) where \(\mu_1 \in \mathcal{G}_0(\mathbb{R}^2)\) such that \(\hat{\mu}_1(\lambda_1, \lambda_2) = 0\) and \(\hat{\mu}_1(0,0) \neq 0\). We have

\[
\int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z) \, dz = 0
\]

for every \(\theta \in \mathbb{R}\). Consequently, we deduce

\[
\int_{\text{M}(2)} e^{i(\lambda, e^{i\theta} z + we^{i\alpha})} e^{-ima} \mu_1(z) \, d\alpha \, dz = \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z)} \mu_1(z) \, dz \right] e^{i[(\lambda, we^{i\alpha}) - ma]} \, d\alpha = 0
\]

for every \(\theta \in \mathbb{R}\) and \(w \in \mathbb{C}\). Namely, \(\mu\) annihilates the subspace \(V_g\). On the other hand, we have

\[
\int_{\text{M}(2)} e^{ima} \mu(e^{i\alpha}, z) \, d\alpha \, dz = \hat{\mu}_1(0,0) \neq 0, \text{ a contradiction.}
\]

Suppose that \(e^{i(w, z)} \in V_g\) where \(w = (w_1, w_2) \in \mathbb{C}^2\). If \(\lambda_1^2 + \lambda_2^2 \neq w_1^2 + w_2^2\) then for \(\mu_2 \in \mathcal{G}_0(\mathbb{R}^2)\) where \(\hat{\mu}_2(\lambda_1, \lambda_2) = 0\) and \(\hat{\mu}_2(0,0) \neq 0\) we have

\[
\int_0^{2\pi} \int_{\mathbb{R}^2} e^{i(\lambda, e^{i\theta} z + ve^{i\alpha})} \mu_2(z) \, dz \, d\alpha = \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} e^{i(e^{i\theta} \lambda, z)} \mu_2(z) \, dz \right] e^{i(\lambda, ve^{i\alpha})} \, d\alpha = 0
\]

for each \(\theta \in \mathbb{R}\) and \(v \in \mathbb{C}\). However, we have

\[
\int_0^{2\pi} \int_{\mathbb{R}^2} e^{i(w, z)} \mu_2(z) \, dz \, d\alpha \neq 0
\]

which proves the irreducibility of \(V_g\). This completes the proof.

Schwartz's Theorem in spectral synthesis is described in the following:
THEOREM 2. — Every closed, two-sided invariant subspace of $C(M(2))$ is spanned by the functions as

$$g(e^{i\alpha}, z) = e^{i m \alpha} Q(Re z, Im z) e^{i (\lambda, z)}$$

that it contains. ($\lambda \in \mathbb{C}^2$ and $Q$ is polynomial).

Proof. — For $f \in C(M(2))$, $f \neq 0$ let $V$ denote the closed subspace generated by the two-sided translates of $f$. Obviously, $f$ is contained in the closed subspace generated by the functions:

$$e^{i m \alpha} P_m(z) = \int_0^{2\pi} f(e^{i(\alpha + \beta)}, z) e^{-i m \beta} d\beta = e^{i m \alpha} \int_0^{2\pi} f(e^{i\beta}, z) e^{-i m \beta} d\beta$$

where $m \in \mathbb{Z}$.

By [1], each function $e^{i m \alpha} P_m(z)$ is contained in the closed subspace spanned by the functions $e^{i m \alpha} Q(Re z, Im z) e^{i (\lambda, z)}$ where $Q(Re z, Im z) e^{i (\lambda, z)}$ is contained in the rotation-invariant and translation-invariant subspace of $C(\mathbb{R}^2)$ generated by $P_m(z)$, and hence in the two-sided invariant subspace generated by $P_m(z)$ which completes the proof of the theorem.

4. One-sided spectral analysis.

One-sided spectral analysis of bounded functions on $M(2)$ was studied in [6].

Notation. — Let $\Gamma_w$, $w \in \mathbb{C}$, denote the closed subspace of $C(\mathbb{R}^2)$ spanned by the functions $e^{i (\lambda_1 x + \lambda_2 y)}$ (of $(x, y) \in \mathbb{R}^2$) where $\lambda_1^2 + \lambda_2^2 = w^2$. For the characterization of right-invariant subspaces of $C(M(2))$ we state the following:

THEOREM 3. — Every closed, right-invariant subspace of $C(M(2))$ contains a function as

$$g(e^{i\alpha}, z) = e^{i m \alpha} g_1(z), \ m \in \mathbb{Z}, \ g_1 \neq 0.$$ 

Moreover, if $g_1 \notin \Gamma_0$, then the closed right-invariant subspace generated by $g$ contains a function as $h(e^{i\alpha}, z) = g_2(z)$.

For $g_2 \in \Gamma_w$ and $g_1 \in \Gamma_0$ the closed right-invariant subspaces generated by $g_2$ and by $e^{i m \alpha} g_1(z)$ are irreducible.
Proof. - Let \( f \in V, f \neq 0 \), where \( V \) is a closed right-invariant subspace of \( C(M(2)) \). Then \( V \) contains all functions \( f^* \) such that 
\[
 f^*(e^{i\alpha}, z) = f(e^{i(\alpha+\beta)}, z - e^{i\alpha}w) \quad \text{where} \quad \beta \in \mathbb{R} \text{ and } w \in \mathbb{C}.
\]
Hence, for a suitable \( m \in \mathbb{Z} \) the function
\[
\int_0^{2\pi} f(e^{i(\alpha+\beta)}, z) e^{-im\beta} d\beta = e^{ima} \int_0^{2\pi} f(e^{i\beta}, z) e^{-im\beta} d\beta = e^{ima} g_1(z)
\]
is non-zero and belongs to \( V \). Suppose that \( g_1 \notin \Gamma_0 \). Then if \( g_1 \) is a polynomial (in \( \text{Re } z \) and \( \text{Im } z \)) which is harmonic on \( \mathbb{R}^2 \) there exists a function \( h \in \mathcal{S}_0(\mathbb{R}^2), h(w) = \mu(r) e^{im\theta}, \mu \in \mathcal{S}_0(r)(\mathbb{R}^2), w = re^{i\theta}, \) such that \( g_1 * h \neq 0 \).

Hence the function
\[
e^{ima} \int_{\mathbb{R}^2} g_1(z - e^{i\alpha}w) h(w) dw = \int_{\mathbb{R}^2} g_1(z - w) h(w) dw = g_2(z) (3)
\]
is non-zero and belongs to \( V \).

Otherwise, the closed rotation-invariant and translation-invariant subspace generated by \( g_1 \) contains a function \( e^{il_{(\lambda,z)}} \) where \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_1^2 + \lambda_2^2 \neq 0 \) [1]. Let \( h_1 \in \mathcal{S}_0(\mathbb{R}^2), h_1(w) = \mu_1(r) e^{im\theta} \) where
\[
\mu_1 \in \mathcal{S}_0(r)(\mathbb{R}^2), w = re^{i\theta}, \text{ such that } h_1(\lambda_1, \lambda_2) \neq 0.
\]
There exists \( \beta \in \mathbb{R} \) such that \( h_{1,\beta} * g_1 \neq 0 \), where
\[
h_{1,\beta}(w) = h_1(e^{i\beta}w) = e^{im\beta} h_1(w).
\]
Hence, \( h_1 * g_1 \neq 0 \) and proceeding as in (3) we complete the proof of the first part of the theorem.

Let \( V_1 \) be the closed right-invariant subspace generated by \( g_2(z) \) where \( g_2 \in \Gamma_{w_0} \) for some \( w_0 \in \mathbb{C}, w_0 \neq 0 \). We may show, as in the proof of Theorem 1, that \( V_1 \) contains no functions as \( e^{ima} g_1(z) \) where \( g_1 \in \Gamma_0 \). Suppose now that \( g_3 \in V_1 \) where \( g_3 \in \Gamma_{w_1}, w_1 \in \mathbb{C} \). To derive the irreducibility of \( V_1 \) we will show that \( g_3 = C g_2 \) for some \( C \in \mathbb{C} \). Let \( \{ \Phi_n \} \) be a sequence in \( \mathcal{S}_0(\mathbb{R}^2) \) such that
\[
\int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n(w) dw \xrightarrow{C(M(2))} g_3(z).
\]
Then we have
\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n(w) dw \right] d\alpha \xrightarrow{C(M(2))} g_3(z)
\]
\[
and
\[
\int_{\mathbb{R}^2} g_2(z - e^{i\alpha}w) \Phi_n^*(|w|)dw \xrightarrow{\text{C(M(2))}} g_3(z)
\]  
(5)

where
\[
\Phi_n^*(|w|) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{-i\alpha}w)d\alpha, \quad \Phi_n^* \in \mathcal{G}_0^{(r)}, \quad n = 1, 2, \ldots
\]  
(6)

But for every \( n \) we have
\[
\int_{\mathbb{R}^2} g_2(z - w) \Phi_n^*(|w|)dw = \hat{\Phi}_n^*(w_0) g_2(z).
\]
Consequently, \( g_3 = C g_2 \), as required. Similarly, we verify the irreducible of the closed right-invariant subspace generated by \( g_1(z) e^{ima} \) where \( g_1 \in \Gamma_0 \).

Remark 1. — We don't know whether Theorem 3 characterizes all the irreducible right invariant subspaces as it is not known whether the exponentials are the only functions of \( C(\mathbb{R}^n) \), \( n > 1 \) which generate irreducible translation-invariant subspaces. Whether every translation-invariant subspace of \( C(\mathbb{R}^n) \), \( n > 1 \) contains an irreducible subspace seems to be an open question.

Remark 2. — In view of Theorem 3 the right-sided analogue of Schwartz's Theorem in spectral analysis of continuous functions may be formulated as the following question; does every closed, right-invariant subspace of \( C(\text{M(2)}) \) contain either a function as \( e^{ima} g_1(z) \) where \( g_1 \in \Gamma_0 \), \( g_1 \neq 0 \), \( m \in \mathbb{Z} \), or \( g_2(z) \) where \( g_2 \in \Gamma_w \), \( g_2 \neq 0 \), for some \( w \in \mathbb{C} \)?

Notation. — Let \( \mu_R, R \geq 0 \), denote the normalized Lebesgue measure of the circle \( \{z: |z| = R\} \). For \( f \in C(\mathbb{R}^2) \) let \( N_f^{(r)} \) denote the closed subspace spanned by \( \{f * \mu_R: R \geq 0\} \) and \( \tau(f) \) the closed translation-invariant subspace generated by \( f \).

We deduce an equivalent form of the right-sided analogue of Schwartz's Theorem (as formulated in Remark 2).

It is described in

Theorem 4. — The following statements are equivalent:

(i) The right-sided analogue of Schwartz's Theorem holds for \( \text{M(2)} \).
(ii) Let $f \in C(\mathbb{R}^2)$, $f \neq 0$. Then: (a) If $\tau(f) \cap \Gamma_0 = \{0\}$ then there exists $w \in C$ such that $N_f^{(r)} \cap \Gamma_w \neq \{0\}$. (b) If $\tau(f) \cap \Gamma_0 \neq \{0\}$ then, either $N_f^{(r)} \cap \Gamma_w \neq \{0\}$ for some $w \in C$, or, there exist $m \in \mathbb{Z}$, $g \in \Gamma_0$, $g \neq 0$ and a sequence $\psi_n \in \mathcal{S}_0^{(r)}(\mathbb{R}^2)$ such that

$$f * \psi_n \xrightarrow{C(\mathbb{R}^2)} g$$

where $\psi_n(r, \theta) = \psi_n(r) e^{-im\theta}$, $n = 1, 2, \ldots$. (Here $(r, \theta)$ are the polar coordinates in $\mathbb{R}^2$).

Proof. Suppose that the right-sided analogue of Schwartz’s Theorem holds for $M(2)$. Let $f \in C(M(2))$ where $f(e^{ia}, z) = f(z)$. Suppose that $\tau(f) \cap \Gamma_0 = \{0\}$. The closed right-invariant subspace $W_f$ generated by $f$ contains no function as $e^{im\theta} g(z) \neq 0$ where $g \in \Gamma_0$ and $m \in \mathbb{Z}$. Since, otherwise

$$\int_{\mathbb{R}^2} f(z - e^{ia}w) \mu_n(w) \, dw \xrightarrow{C(M(2))} e^{im\theta} g(z)$$

implies for $\alpha = 0$ that: $f * \mu_n \xrightarrow{C(\mathbb{R}^2)} g$, a contradiction. Hence, $W_f$ contains a function $g_1(z)$ where $g_1 \in \Gamma_w$, $g_1 \neq 0$. In other words, there exist $\Phi_n \in \mathcal{S}_0^{(r)}(\mathbb{R}^2)$, $n = 1, 2, \ldots$, such that

$$\int_{\mathbb{R}^2} f(z - e^{ia}w) \Phi_n(w) \, dw \xrightarrow{C(M(2))} g_1(z).$$

Hence, by (5) we have:

$$\int_{\mathbb{R}^2} f(z - w) \Phi_n(|w|) \, dw \xrightarrow{C(M(2))} g_1(z)$$

where $\Phi_n^*$ are defined in (6). That is, $g_1 \in N_f^{(r)}$ which yields (ii) (a).

Suppose now that $\tau(f) \cap \Gamma_0 \neq \{0\}$. If $W_f \cap \Gamma_v \neq \{0\}$ for some $v \in C$ then, as proved above, $N_f^{(r)} \cap \Gamma_v \neq \{0\}$ (here, the functions of $\Gamma_v$ are looked upon as function on $M(2)$). Otherwise, the subspace $W_f$ must contain a function as $e^{im\alpha} g_2(z)$ where $g_2 \in \Gamma_0$, $g_2 \neq 0$, and $m \in \mathbb{Z}$. Namely, there exists $\phi_n \in \mathcal{S}_0(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} f(z - e^{ia}w) \phi_n(w) \, dw \xrightarrow{C(M(2))} e^{im\alpha} g_2(z).$$

Hence we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left[ \int_{\mathbb{R}^2} f(z - \xi) \phi_n(e^{-ia} \xi) \, d\xi \right] e^{-im\alpha} \, d\alpha \longrightarrow g_2(z).$$
which yields
\[ \frac{1}{2\pi} \int_{\mathbb{R}^2} f(z - \xi) \tilde{\phi}_n(\xi) \, d\xi \rightarrow g_2(z) \]
where \( \tilde{\phi}_n(\xi) = \tilde{\psi}_n(r) e^{-im\theta}, \tilde{\psi}_n(r) = \int_0^{2\pi} \phi(e^{-in}r) e^{-im\eta} d\eta, \; \xi = re^{i\theta}, \)
and we have shown that (i) implies (ii).

Suppose now that (ii) holds. By Theorem 3 we have to show that for every \( f \in C(M(2)), f(e^{ia}, z) = f(z), f \neq 0, \) the subspace \( W_f \) contains either a function \( g(z), g \neq 0, g \in \Gamma_w, \) or, a function \( g(e^{ia}, z) = e^{ima}g_1(z) \) where \( g_1 \in \Gamma_0, g_1 \neq 0 \) and \( m \in \mathbb{Z}. \)

Let \( f \in C(\mathbb{R}^2), f \neq 0 \) and suppose that \( N_f \cap \Gamma_w \neq \{0\} \) for some \( w \in \mathbb{C}. \) Then, by definition, there exist \( \psi_n \in \mathcal{S}_0^{(r)}(\mathbb{R}^2) \) \( n = 1, 2, \ldots, \) and \( g \in \Gamma_w \) such that
\[ \int_{\mathbb{R}^2} f(z - \xi) \psi_n(\xi) \, d\xi \xrightarrow{C(\mathbb{R}^2)} g(z). \]
But we have
\[ \int_{\mathbb{R}^2} f(z - e^{ia} \xi) \psi_n(\xi) \, d\xi = \int_{\mathbb{R}^2} f(z - \xi) \psi_n(\xi) \, d\xi \quad \text{for} \quad n = 1, 2, \ldots, \]
which implies (i).

Finally, suppose that \( \tau(f) \cap \Gamma_0 \neq \{0\} \) and that \( N_f^{(r)} \cap \Gamma_w = \{0\} \) for every \( w \in \mathbb{C}. \) By (ii) (b) we have
\[ \int_{\mathbb{R}^2} f(z - e^{ia} w) \phi_n(w) \, dw = \int_{\mathbb{R}^2} f(z - \xi) \phi_n(e^{-ia} \xi) \, d\xi \]
\[ = e^{ima} \int_{\mathbb{R}^2} f(z - \xi) \phi_n(\xi) \, d\xi \]
for \( n = 1, 2, \ldots, \) which yields, by (7)
\[ \int_{\mathbb{R}^2} f(z - e^{ia} \xi) \psi_n(\xi) \, d\xi \xrightarrow{C(M(2))} e^{ima}g(z). \]
This completes the proof.

5. Invariant subspaces of \( C(\mathbb{R}^2). \)

For \( f \in C(\mathbb{R}^2) \) we say that \( w \in \text{Sp}^{T,R}(f), \ w \in \mathbb{C} \) if the translation-invariant and rotation-invariant subspace generated by \( f \) contains an exponential in \( \Gamma_w. \) Actually, the fact announced in [1] that unless \( f = 0 \) we have \( \text{Sp}^{T,R}(f) \neq \emptyset \) implies the main
results of [1] concerning the Pompeiu problem [4, 7]. By Theorem 4, the one-sided Schwartz’s Theorem for the motion group is intimately connected to the following problem:

For $f \in C(\mathbb{R}^2)$ we say that $w \in Sp^{(r)}(f), w \in \mathcal{C}, w \neq 0$, if $N_{f}^{(r)} \cap \Gamma_{w} \neq \{0\}$, and that $0 \in Sp^{(r)}(f)$ if $N_{f}^{(r)} \cap \Gamma_{0} \neq \{0\}$, where $\Gamma_{0}$ denotes the space of harmonic functions on $\mathbb{R}^2$. Suppose that $f \neq 0$. Does this imply that $Sp^{(r)}(f) \neq \emptyset$?

**Remark 3.** — We notice that for $f \in C(\mathbb{R}^2)$ we have $Sp^{(r)}(f) \subseteq Sp^{T.R.}(f)$.

**Remark 4.** — This question is connected to the following problem of Pompeiu type:

Determine for which family $P \subset M_0(\mathbb{R}^2)$, the only continuous function $f$ on $\mathbb{R}^2$ such that $T(f \ast \mu_\Gamma) = 0$ for all $T \in P$ and $\Gamma \geq 0$, is the zero function.

Let $J_n$ denote the $n$th Bessel function of the first kind. By definition, we deduce

$$J_n(r) e^{in\theta} = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{ir\cos(\phi-\theta)} e^{in\phi} d\phi.$$

Hence we have $J_n(wr) e^{in\theta} \in \Gamma_w$, $Sp^{(r)}(J_n(wr) e^{in\theta}) = \{w\}$ for $w \in \mathcal{C}, w \neq 0$ and $N_{f}^{(r)}$ is one-dimensional where $I_n(r, \theta) = J_n(wr) e^{in\theta}$.

A partial answer to the above question is provided by the following result:

**Theorem 5.** — Let $f \in C(\mathbb{R}^2), f \neq 0$ where

$$f(r, \theta) = \sum_{m=0}^{N} g_m(r) e^{im\theta}, \ g_m \in C^{(r)}(\mathbb{R}^2) \ (m = 0, 1, \ldots, N).$$

Then $Sp^{(r)}(f) \neq \emptyset$. If $0 \notin Sp^{(r)}(f)$ there exist $\lambda, a_m \in \mathcal{C}$ $(m = 0, 1, \ldots, N), \lambda \neq 0$, where $\sum_{m=0}^{N} |a_m| > 0$ such that

$$\sum_{m=0}^{N} a_m J_m(\lambda r) e^{im\theta} \text{ belongs to } N_{f}^{(r)}. \text{ Moreover, we have }$$

$$Sp^{(r)}(f) = \bigcup_{m=0}^{N} Sp^{(r)}(g_m(r) e^{im\theta}).$$
The proof will be accomplished in several lemmas.

**Lemma 6.** — Every proper closed ideal in \( \mathcal{S}'_r(\mathbb{R}^2) \) has a common zero in \( \mathbb{C}^2 \).

**Proof.** — Let \( J \) be a proper closed ideal in \( \mathcal{S}'_r(\mathbb{R}^2) \) and suppose that the functions in \( J \) have no common zeroes. Every \( f \in J \) is a function of \( z_1^2 + z_2^2 \). That is, there exists an even entire function \( Q_f \) of one complex variable such that

\[
f(z_1, z_2) = Q_f(\sqrt{z_1^2 + z_2^2}) \quad \text{and} \quad Q_f \in \mathcal{S}'(\mathbb{R}).
\]

Let \( J^* \) be the ideal in \( \mathcal{S}'(\mathbb{R}) \) generated by \( \{Q_f : f \in J\} \).

Obviously, the functions in \( J^* \) have no common zeroes. Thus, applying Schwartz's Theorem [5] we deduce that \( J^* = \mathcal{S}'(\mathbb{R}) \). That is, there exists a sequence \( \{P_n\} \) in \( J^* \) converging to 1 in \( \mathcal{S}'(\mathbb{R}) \). Each \( P_n \) must be of the form \( \sum_{j=1}^{k} T_j(w)S_j(w) \) where each \( T_j \in \mathcal{S}'(\mathbb{R}) \) and \( S_j \in J \). But then the function

\[
\sum_{j=1}^{k} T_j(w)S_j(w) + \sum_{j=1}^{k} T_j(-w)S_j(-w) = \sum_{j=1}^{k} (T_j(w) + T_j(-w))S_j(w)
\]

belongs to \( J \) since each \( T_j(w) + T_j(-w) \) belongs to \( \mathcal{S}'_r(\mathbb{R}^2) \). Hence, \( Q_n(w) = \frac{1}{2} (P_n(w) + P_n(-w)) \) belongs to \( J \) and \( Q_n \rightarrow 1 \) in \( \mathcal{S}'_r(\mathbb{R}^2) \), a contradiction.

**Lemma 7.** — Let \( f \in C(\mathbb{R}^2) \) where \( f(r, \theta) = g(r)e^{im\theta} \), \( g \in C^{(r)}(\mathbb{R}^2) \), \( g \neq 0 \), \( m \in \mathbb{Z} \). Then \( Sp^{(r)}(f) \neq \emptyset \). If \( 0 \notin Sp^{(r)}(f) \) there exists \( \lambda \in \mathbb{C} \), \( \lambda \neq 0 \), such that \( H \in N^{(r)}_f \) where

\[
H(r, \theta) = J_m(\lambda r)e^{im\theta}.
\]

**Proof.** — We may assume that \( f \in \mathcal{S}(\mathbb{R}^2) \). Let \( M_f^{(r)} \) denote the closed subspace of \( \mathcal{S}(\mathbb{R}^2) \) spanned by \( \{f * \mu_r : R \geq 0\} \). For \( m \in \mathbb{Z} \) let \( \mathcal{S}_m(\mathbb{R}^2) \) denote the closed subspace of functions \( s \in \mathcal{S}(\mathbb{R}^2) \) such that \( s(r, \theta) = h(r)e^{im\theta} \). We have \( M_f^{(r)} \subseteq \mathcal{S}_m(\mathbb{R}^2) \).

Let \( \mathcal{S}_m'(\mathbb{R}^2) \subseteq \mathcal{S}'(\mathbb{R}^2) \) denote the dual of \( \mathcal{S}_m(\mathbb{R}^2) \).

Let \( M_f^{(r)\perp} = \{T \in \mathcal{S}_m'(\mathbb{R}^2) : T(f) = 0, f \in M_f^{(r)}\} \).
Every element of $S_m(R^2)$ is of the form $p(r)e^{im\theta}$ (as a function on $R^2$). Let $P = \{p : \hat{T}(r, \theta) = p(r)e^{im\theta}, T \in M_f^{(r)}\}$.

We notice that all functions of $P$ are even or odd depending on $m$.

Let $k$ be the larger integer such that 0 is a zero of order $k$ for each $p \in P$. It follows that $\frac{p(w)}{w^k}, p \in P$, is an even entire function of $w$ and by complexification of $\frac{p(r)}{r^k}$

$$p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}$$

is an entire function on $C^2$. The space

$$J^* = \left\{ p^* : p^*(z_1, z_2) = \frac{p(\sqrt{z_1^2 + z_2^2})}{(z_1^2 + z_2^2)^{k/2}}, p \in P \right\}$$

is therefore a closed ideal in $S_{(r)}(R^2)$. If $0 \notin Sp^{(r)}(f) J^*$ is a proper ideal.

Hence, by Lemma 6, there exists

$$\lambda^* = (\lambda_1, \lambda_2) \in C^2, \lambda_1^2 + \lambda_2^2 = \lambda^2 \neq 0$$

which is a common zero of $J^*$. Consequently, for each $T \in M_f^{(r)}$ we have $\hat{T}(w) = 0$ where $w = (w_1, w_2) \in C^2, w_1^2 + w_2^2 = \lambda^2$.

It follows that $T(Q) = 0$ for $T \in M_f^{(r)}$ where

$$Q(x, y) = \frac{1}{2\pi i m} \int_0^{2\pi} e^{i\lambda_1(x \cos \phi + y \sin \phi)} e^{im\phi} d\phi.$$ 

But we have

$$Q(r, \theta) = \frac{1}{2\pi i m} \int_0^{2\pi} e^{i\lambda r \cos (\phi - \theta)} e^{im\phi} d\phi = J_m(wr) e^{im\theta}.$$ 

Consequently, $Q \in M_f^{(r)} \cap \Gamma_\lambda$ which completes the proof.

**Notation.** – Let $C(R^2, C^N)$ denote the space of all continuous functions on $R^2$ which take values in $C^N$, with the usual topology. Let $M_0(R^2, C^N)$ be the dual of $C(R^2, C^N)$, the space of vector-valued measures having compact support. For $f \in C(R^2, C^N)$, (resp. $\mu \in M_0(R^2, C^N)$) let $(f)_n$ (resp. $(\mu)_n$) denote the $n$th coordinate of $f$ (resp. $\mu$). For $m = (m_1, m_2, \ldots, m_N) \in Z^N$ let $B_{(m)}$ denote
the closed subspace of $C(R^2, C^N)$ which consists of all functions $f$ where

$$(f)_n(r, \theta) = h_n(r) e^{im_n\theta} \quad n = 1, 2, \ldots, N.$$  

Let $B'_m$ be the dual of $B(m)$, the space of all $\eta \in M_0(R^2, C^N)$ such that $(\eta)_n = \mu_n e^{-im_n\theta}$ where $\mu_n \in M_0(r)(R^2)$, $n = 1, 2, \ldots, N$. We will use the following equality:

$$(J_k(w^r) e^{ik\theta}) * (\mu(r) e^{im\theta}) (r, \theta) = \phi(w) J_{k+m}(wr) e^{i(k+m)\theta} \quad (8)$$

where $\mu \in M_0(r)(R^2)$, $w \in C$, and $\mu(r) e^{im\theta}(r, \theta) = \phi(r) e^{im\theta}$. Finally, we notice that $M_0(r)(R^2)$ acts on $B(m)$ by convolution. Namely, $f \in B(m)$ and $\mu \in M_0(r)(R^2)$ imply that $f * \mu \in B(m)$.

**Lemma 8.** -- Every closed non-trivial subspace of $B(m)$, invariant under $M_0(r)(R^2)$ contains an invariant one-dimensional subspace. Moreover, if $f \in B(m)$ such that $\lambda \in S^r((f)_n)$, $\lambda \neq 0$, for some $n$, $1 \leq n \leq N$, then the closed subspace spanned by $\{f * \mu_R : R \geq 0\}$ contains a function $g \neq 0$, such that

$$(g)_n(r, \theta) = a_n J_{m_n}(\lambda r) e^{im_n\theta} \quad n = 1, 2, \ldots, N.$$  

**Proof.** -- By induction on $N$ where the case $N = 1$ is provided by Lemma 7. Let $f \in B(m)$ and suppose that $0 \neq \lambda \in S^r((f)_1)$. Let $V_f$ denote the closed subspace of $B(m)$ spanned by $\{f * \mu_R : R \geq 0\}$ and $V_f^\perp = \{\eta \in B'_m : \eta(g) = 0, g \in V_f\}$. We notice that for $\eta \in V_f^\perp$ we have:

$$\sum_{n=1}^{N} (g_n(r) e^{im_n\theta}) * (\mu_n e^{-im_n\theta}) = 0 \quad (9)$$

where $(\eta)_n = \mu_n e^{-im_n\theta}$ and $(f)_n = g_n(r) e^{im_n\theta}$, $n = 1, 2, \ldots, N$.

Thus we may assume that there exists $\eta \in V_f^\perp$ such that

$$(J_{m_N}(\lambda r) e^{im_N\theta}) * (\mu_N e^{-im_N\theta}) \neq 0. \quad (10)$$

Otherwise, the subspace $V_f$ contains a function $g^*$ such that $(g^*)_n = 0$ for $n = 1, 2, \ldots, N - 1$, and $(g^*)_N = J_{m_N}(\lambda r) e^{im_N\theta}$ which completes the proof. To this end, let $h \in B_{(m')}^\perp$ where $(h)_n = (f)_n$ for $n = 1, 2, \ldots, N - 1$, $m' = (m_1, m_2, \ldots, m_{N-1})$ and $B_{(m')} \subset C(R^2, C^{N-1})$. By the induction hypothesis the subspace $V_h$ contains a function $h^* \neq 0$ such that
(h^n)_n = b_n J_{m_n}(\lambda r) e^{i n \theta} \quad \text{for} \quad n = 1, 2, \ldots, N - 1.

That is, there exists a sequence \( \{\phi_k\} \), \( \phi_k \in M^{(r)}_0(\mathbb{R}^2) \), such that

\[
(g_n(r') e^{i n \theta} * \phi_k)(r, \theta) \xrightarrow[k \to \infty]{} b_n J_{m_n}(\lambda r) e^{i n \theta}
\]

for \( n = 1, 2, \ldots, N - 1 \), where \( \sum_{n=1}^{N-1} |b_n| > 0 \). Let \( \psi_k \in M^{(r)}_0(\mathbb{R}^2) \) where

\[
\psi_k = \phi_k * \mu_N e^{-i n \theta} * \mu_N e^{i n \theta} \quad k = 1, 2, \ldots.
\]

Then by (8), (10) and (11) we obtain:

\[
g_n(r') e^{i n \theta} * \psi_k \xrightarrow[k \to \infty]{} b_n J_{m_n}(\lambda r) e^{i n \theta} \]

for \( n = 1, 2, \ldots, N - 1 \) where \( C_1 \in \mathbb{C}, C_1 \neq 0 \).

For \( n = N \) we have by (9) and (8):

\[
g_N(r) e^{i n \theta} * \psi_k = g_N(r) e^{i n \theta} * \mu_N e^{-i n \theta} * \phi_k * \mu_N e^{i n \theta} \]

\[
= - \left[ \sum_{n=1}^{N-1} g_n(r) e^{i n \theta} * \mu_n e^{-i n \theta} \right] * \phi_k * \mu_N e^{i n \theta}.
\]

Hence we obtain

\[
g_n(r) e^{i n \theta} * \psi_k \xrightarrow[k \to \infty]{} \left[ \sum_{n=1}^{N-1} b_n J_{m_n}(\lambda r) e^{i n \theta} * \mu_n e^{-i n \theta} \right] * \mu_N e^{i n \theta}
\]

\[
= C J_0(\lambda r) * \mu_N e^{i n \theta} = C' J_N(\lambda r) e^{i n \theta}.
\]

Similarly, we may prove that if \( 0 \in Sp^{(r)}((f)_n) \) for some \( n \), \( 1 \leq n \leq N \), then \( V_f \) contains a function \( g \neq 0 \) such that:

\[
(g)_n(r, \theta) = a_n r^{m_n} e^{i n \theta} \quad n = 1, 2, \ldots, N.
\]

**Proof of Theorem 5.** – Let \( h \in B_{(m)} \), \( B_{(m)} \subset C(\mathbb{R}^2, \mathbb{C}^{N+1}) \) where \( m = (0, 1, \ldots, N) \) and \( (h)_n(r, \theta) = g_{n-1}(r) e^{i(n-1) \theta} \), \( n = 1, 2, \ldots, N + 1 \), and suppose that \( \lambda \in Sp^{(r)}((h)_{k_0}) \), \( \lambda \neq 0 \), for some \( k_0 \), \( 1 \leq k_0 \leq N + 1 \). Then by Lemma 8, there exists a sequence \( \{\phi_k\} \), \( \phi_k \in M^{(r)}_0(\mathbb{R}^2) \) \( k = 1, 2, \ldots, \) such that

\[
(g_{n-1}(r') e^{i(n-1) \theta} * \phi_k)(r, \theta) \xrightarrow[k \to \infty]{} a_{n-1} J_{n-1}(\lambda r) e^{i(n-1) \theta}
\]
for \( n = 1, 2, \ldots, N + 1 \) where \( \sum_{n=0}^{N} |a_n| > 0 \). Hence, we have

\[
\left[ \sum_{n=0}^{N} g_n(r') e^{i n \theta} \right] \phi_k (r, \theta) \xrightarrow{C(R^2)} \sum_{n=0}^{N+1} a_n J_n(\lambda r) e^{in \theta}.
\]

If \( 0 \in \text{Sp}^r((h)_{k_0}) \) then, similarly, \( \mathcal{N}^r \) contains \( g \in \tilde{T}_0 \), \( g \neq 0 \), where \( g(r, \theta) = \sum_{n=0}^{N} b_n r^n e^{in \theta} \). Finally, we may easily prove that \( \text{Sp}^r(f) \subseteq \bigcup_{m=0}^{N} \text{Sp}^r(g_m e^{im \theta}) \) and the result follows.

**Corollary 6.** Let \( f \in C(R^2) \), \( f \neq 0 \) where

\[
f(r, \theta) = \sum_{m=0}^{N} g_m(r) e^{im \theta}, \quad g_m \in C^r(R^2) \quad (m = 0, 1, \ldots, N).
\]

Then the translation-invariant closed subspace \( \tau(f) \) generated by \( f \) contains an exponential function.

**Proof.** If \( 0 \in \mathcal{N}^r(f) \) then \( \tau(f) \) contains a polynomial and hence \( 1 \in \tau(f) \). Otherwise, by Theorem 5, \( g \in \tau(f), \ g \neq 0 \) where:

\[
g(r, \theta) = \sum_{m=0}^{N} a_m J_m(\lambda r) e^{im \theta}
\]

for some \( \lambda, \ a_m \in \mathbb{C}, \ \lambda \neq 0, \ (m = 0, 1, \ldots, N) \).

The subspace \( \tau(f) \) contains therefore all the functions \( h \) where

\[
h(x, y) = (g \ast \mu)(x, y)
\]

\[
= C \sum_{m=0}^{N} a_m \int_{R^2} \left[ \int_{0}^{2\pi} e^{i\lambda[(x-\alpha)\cos \phi+(y-\beta)\sin \phi]} e^{im \phi} \right] d\mu(\alpha, \beta)
\]

\[
= C \sum_{m=0}^{N} a_m \int_{0}^{2\pi} \tilde{\mu}(\lambda \cos \phi, \lambda \sin \phi) e^{i\lambda(x \cos \phi+y \sin \phi)} e^{im \phi} d\phi
\]

for every \( \mu \in M_0(R^2) \) where \( C \in \mathbb{C}, \ C \neq 0 \).

Thus \( \tau(f) \) contains all the functions \( u \) where

\[
u(x, y) = \sum_{m=0}^{N} a_m \int_{0}^{2\pi} s(\phi) e^{i\lambda(x \cos \phi+y \sin \phi)} e^{im \phi} d\phi
\]

for every \( s \in C[0, 2\pi] \), \( s(0) = s(2\pi) \). For a sequence \( \{s_n\} \) converg-
ing to the Dirac mass \(\delta_{\phi_0}\) concentrated in \(\phi_0\) where \(\sum_{m=0}^{N} a_m e^{im\phi_0} \neq 0\), we obtain, by passing to the limit, that \(v \in \tau(f)\) where

\[
v(x, y) = \left( \sum_{m=0}^{N} a_m e^{im\phi_0} \right) e^{i(x\lambda \cos \phi_0 + y\lambda \sin \phi_0)}
\]

which completes the proof.

**Remark 5.**—To this end we may introduce the following proof to the fact that every translation-invariant and rotation-invariant closed subspace of \(C(\mathbb{R}^2)\) contains an exponential function [1]. Let \(R_f\) denote the closed translation-invariant and rotation invariant subspace generated by \(f \neq 0\). Then, for a suitable \(m \in \mathbb{Z}\) the function \(g\) where

\[
g(r, \theta) = \int_{0}^{2\pi} f(r, \theta + \beta) e^{-im\beta} d\beta = e^{im\theta} \int_{0}^{2\pi} f(r, \beta) e^{-im\beta} d\beta
\]

is non-zero and belongs to \(R_f\). Let \(\mu \in M_0^{(r)}(\mathbb{R}^2)\) where \(\mu(f_1) \neq 0\). Hence the function \(g_1 = g * (\mu e^{-im\theta})\) is non-zero and belongs to \(R_f \cap C(\mathbb{R}^2)\).

By Lemma 6, or by Lemma 7 for \(m = 0\), there exists \(\lambda \in \mathbb{C}\) such that \(J_0(\lambda r) \in R_f\). Arguing as in the proof of Corollary 6, we deduce that \(R_f\) contains the exponentials \(e^{i(x\lambda \cos \phi + y\lambda \sin \phi)}\) for every \(\phi \in \mathbb{R}\) and the result follows.

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