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ON GROUP REPRESENTATIONS WHOSE C*-ALGEBRA IS AN IDEAL IN ITS VON NEUMANN ALGEBRA

by Edmond E. GRANIRER

Introduction. – Let τ be a unitary continuous representation of the locally compact group G on the Hilbert space H_τ and denote by $L(H_\tau)[LC(H_\tau)]$ the algebra of all bounded [compact] linear operators on H_τ. τ can be lifted in the usual way to a *-representation of $L^1(G)$. Denote by $C^*_\tau(G) = C^*_\tau$ the norm closure of $\tau[L^1(G)]$ in $L(H_\tau)$ (with operator norm) and by $VN_\tau(G) = VN_\tau$ the W*-algebra generated by $\tau[L^1(G)]$ in $L(H_\tau)$. Let

$$M_\tau(C^*_\tau) = \{\varphi \in VN_\tau ;\ \varphi C^*_\tau + C^*_\tau \varphi \subset C^*_\tau\}$$

i.e. the two sided multipliers of C^*_τ *in* VN_τ (*not* in the bidual $(C^*_\tau)''$ of C^*_τ).

The representation τ is said to be CCR if $C^*_\tau \subset LC(H_\tau)$. Furthermore, supp τ will denote the closed subset of $\hat{G}$ of all π in $\hat{G}$ which are weakly contained (à la Fell) in τ (see the notations that follow).

One of the main results in this paper (in slightly shortened fashion) is :

THEOREM 1. – *Let* G *be* σ*-compact and* τ *a unitary continuous representation of* G *such that* $M_\tau(C^*_\tau) = VN_\tau$. *Then* supp τ *is a (closed) discrete subset of* $\hat{G}$ *and each* π *in* supp τ *is* CCR (i.e. $C^*_\tau = LC(H_\tau)$).

A result of I. Kaplanski will hence imply that moreover C^*_τ is a dual C*-algebra (see [6] (10.10.6)).

Our main application of this theorem is to induced representations and in particular to the quasiregular representation π_H on $L^2(G/H)$, for some closed subgroup H, as detailed in what follows :

THEOREM 2 ([1]). — *Let* H *be a subgroup of the* σ*-compact group* G *and* $\nu = \nu^{\chi}$ *the representation of* G *induced by the representation* χ *of* H. *If* I_G *is weakly contained in* ν *and* $M_\nu(C_\nu^*) = VN_\nu$ *then* H *has finite covolume in* G, supp ν *is discrete and any* π *in* sup ν *is* CCR.

Note that I_G is the unit representation of G on C.

We improve somewhat theorem 2 for the case that $\nu = \pi_H$ is the quasiregular representation of G on $L^2(G/H)$ in

THEOREM 6. — *Let* G *be* σ*-compact,* H *a closed subgroup and* $\nu = \pi_H$. *If* π_H *weakly contains a nonzero finite dimensional representation and* $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ *then* H *has finite covolume in* G, sup π_H *is discrete and any* π *in* supp π_H *is* CCR.

It seems to be in the folklore that if G is arbitrary and H is a closed subgroup such that G/H is a compact coset space then $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ (see proposition 4).

It seems to us that the fact that H has finite covolume in G and sup π_H is discrete should imply, at least for σ-compact G that G/H is compact.

This would generalize a result of L. Baggett [20], A. H. Shtern and S. P. Wang [17] who have proved it for the regular representation (i.e. $H = \{e\}$) of a σ-compact group G. We pose the above as an open question.

The assumptions of theorem 6 still imply that G/H is compact at least in the following cases (1) H is compact. (2) H is a semidirect summand. (3) H is open in G. (4) G is connected and $H = Z_G(A)$ for some set A of automorphisms of G. (5) G is a connected Lie group and H is a connected subgroup. (6) G is a solvable Lie group and H is any subgroup (both (5) and (6) using some deep theorems of G.D. Mostow). (7) G is *any* Lie group and $H = Z_G(A)$ (A as above) etc... See corollaries 1, 2, 3 after theorem 6.

In case π_H does not weakly contain any finite dimensional nonzero representation we still have the

THEOREM 5. — *Let* G *be* σ*-compact* H *a closed subgroup. If* $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ *then* supp π_H *is discrete and any* $\pi \in$ supp π_H *is* CCR.

([1]) For second countable G thm. 8.2 of Mackey in [22] p. 120 implies that I_G can be replaced by any finite dimensional representation. Thanks are due to L. Baggett for an inspiring conversation connected with this fact.

It seems to us that the fact that supp π_H is discrete, each π in supp π_H is CCR and G is σ-compact should imply that G/H is compact.

A proof for this statement would provide an answer to the above open question. We commission hereby a proof of this statement from people in the know. It would imply, together with our theorem 5 the following statement :

« Let G be σ-compact H a closed subgroup. Then $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ if and only if G/H is a compact coset space. »

In case $H = e$ we have a slightly better result than the above, namely :

THEOREM 3. — *Let* G *be any locally compact group and* ρ *the left regular representation. If* $M_\rho(C^*_\rho) = VN_\rho$ *then* G *is compact (and conversely).*

This result improves a result of oursin [10] where it was assumed in addition that G is amenable. M. A. Barnes has informed us that the has also obtained this improvement of our result in [10] using different methods.

Notations. — Most of the notations in this paper are consistent with Dixmier [6] and Eymard [7] and [8].

Let A be a C*-algebra. Let A″ be the bidual (or the enveloping) C*-algebra of A as in [6] 12.1.4. Denote by $M(A) = \{\varphi \in A'' ; \varphi A + \varphi A \subset A\}$ the multipliers of A in A″ (or the idealizer of A in A″). If H is a Hilbert space, L(H) [LC(H)] will denote the algebra of all [compact] bounded linear operators on H.

G will always stand for a locally compact group with a given left Haar measure. We say that π is a representation of G if π is a unitary continuous representation of G as in [6] 13.1.1. H_π will denote a Hilbert space on which the operators $\{\pi(x) ; x \in G\}$ act. π can be lifted as usual to a *-representation of $L^1(G)$ [in fact of M(G)] in the usual way. We denote by $C^*_\pi(G)$ or C^*_π, when G is obvious, the C*-algebra which is the operator norm closure of $\pi[L^1(G)]$ in $L(H_\pi)$. $VN_\pi(G) = VN_\pi$ will denote the W*-algebra generated by $\pi[L^1(G)]$ in $L(H_\pi)$.

The following notation is important : if π is a representation of G on H_π then

$$M_\pi(C^*_\pi) = M_\pi(C^*_\pi(G)) = \{\varphi \in VN_\pi ; \varphi C^*_\pi + C^*_\pi \varphi \subset C^*_\pi\}$$

i.e. the idealizer of C^*_π *in* VN_π. Note that $M_\pi(C^*_\pi)$ does not usually coincide with $M(C^*_\pi)$ which is the idealizer of C^*_π in its bidual $(C^*_\pi)''$. Proposition

12.1.15 of [6] is important in this respect and it shows that the bidual of C_π^* is universal in a certain sense not shared usually by VN_π.

ρ will denote the left regular representation of G on $L^2(G)$ and VN will denote VN_ρ. $C^*(G)$ will denote the full C^*-algebra of G as in [6] 13.9. The set P(G) of all positive definite continuous functions on G is identified with the positive linear functionals on $C^*(G)$. $P_\pi(G) = P_\pi$ will be the set of those u in P(G) which are weakly associated with π (this set is canonically identified with the positive linear functionals on C_π^*). $B_\pi(G) = B_\pi[B(G)]$ will be the complex linear span of $P_\pi[P(G)]$ and is identified canonically with the dual of $C_\pi^*[C^*(G)]$ see Eymard [7], pp. 189-191.

B(G) is equipped with the dual norm on $C^*(G)$ and $B_\pi(G) \subset B(G)$ with the subspace norm. $A_\pi(G) = A_\pi$ will denote the norm closure in B_π of the set of all coefficient functions of the representation π i.e. of $\{\langle\pi(x)\zeta, \eta\rangle\, \zeta, \eta \in H_\pi\}$. $A_\pi(G)$ is canonically identified with the predual of the W*-algebra VN_π. Many results on $A_\pi(G)$ are contained in G. Arsac's elegant thesis [3] which unfortunately was never published. We were unable to find them in such an elegant and suitable form anywhere else and will quote hence reference [3].

If π is a representation of G, it will be identified with its unitary equivalence class. $\hat{G}$ will as usual denote the set (of equivalence classes) of irreducible representations of G with the usual weak containment topology of Fell, i.e. that one transported from the hull-kernel topology of $C^*(G)^\wedge$ as in Dixmier [6] 18.1, p. 314. Let π_1, π_2 be representations of G. We say that π_1 is weakly contained (à la Fell) in π_2 if $B_{\pi_1} \subset B_{\pi_2}$ (see [7], p. 189).

We denote the support of π, supp π as the set of τ in $\hat{G}$ which are weakly contained in π (à la Fell) see [6] 18.1. supp π is a closed subset of G. I_G will denote the unit representation of G on **C** (the complex numbers) i.e. $\langle I_G(x)\,\alpha, \beta\rangle = \alpha\bar{\beta}$ for each x in G and α, β in **C**.

A representation π of G is CCR if $C_\pi^* \subset LC(H_\pi)$.

Let H be a closed subgroup of G. $G/H = \{\dot{x} = xH\,; \in G\}$ is the space of left cosets of G with the quotient topology. G/H admits a quasi invariant measure (see S. Gaal [9], V.3). If $\Delta_G(\Delta_H)$ are the modular functions of G(H) and if $\Delta_G = \Delta_H$ on H then G/H admits a (not necessarily finite) invariant measure [9] V. 3, p. 266. G/H may be compact and G/H need not admit an invariant measure. It happens in numerous important cases that G/H admits a *finite* invariant measure (in other terminology that H has finite covolume in G) and still G/H is not compact (see for ex G. D. Mostow [11], [12]).

LEMMA 0. — *Let* σ *be a continuous unitary representation of the locally compact group* G *such that* $M_\sigma(C^*_\sigma) = VN_\sigma$. *If* $v_n \in A_\sigma \cap P_\sigma(G)$ *is a sequence such that* $v_n \to v_0$ *uniformly on compacta then* $v_0 \in A_\sigma \cap P_\sigma(G)$ *and* $v_n \to v_0$ *weakly in the Banach space* A_σ (*i.e. in* $w(A_\sigma, VN_\sigma)$).

Proof. — Clearly $\|v_n\| = v_n(e)$ is bounded and by the w^* compactness of closed bounded balls in the dual B_σ of C^*_σ a subnet of v_n will converge w^* (hence in $w(B_\sigma, L^1)$) and also uniformly on compacta to some $u_0 \in B_\sigma$. Hence $u_0(x) = v_0(x)$ for each $x \in G$ i.e. $v_0 \in P_\sigma$. By Akeman and Walter's proposition 2 of [2], p. 458 it follows that $\langle \Phi, v_n \rangle'' \to \langle \Phi, u_0 \rangle''$ for each $\Phi \in M(C^*_\sigma)$ where $\langle\ \rangle''$ ($\langle\ \rangle$) will stand for the $\langle (C^*_\sigma)'', B_\sigma \rangle$ ($\langle VN_\sigma, A_\sigma \rangle$) duality. Let $i : C^*_\sigma \to VN_\sigma$ be the identity embedding then i can be extended uniquely to a *faithful* representation also denoted by i to all of $M(C^*_\sigma)$ and $i(M(C^*_\sigma))$ is the *idealiser* of $i(C^*_\sigma)$ in its weak closure i.e. in VN_σ, by a result of Akeman Pedersen and Tomiyama [1], p. 280, Prop. 2.4 (and independently obtained by M.C. Flanders in his thesis at Tulane Univ. 1968). If $\kappa : C^*_\sigma \to (C^*_\sigma)''$ is the canonical embedding then by Dixmier [6] (12.1.5) there exist a unique ultraweakly continuous representation $\tilde{\imath}$ of $(C^*_\sigma)''$ to VN_σ such that $\tilde{\imath}\kappa(\varphi) = i(\varphi)$ for each $\varphi \in C^*_\sigma$. As remarked in the proof of [1], p. 280, prop. 2.4 $\tilde{\imath}$ restricted to $M(C^*_\sigma)$ is just i. Since we assume that $M_\sigma(C^*_\sigma) = VN_\sigma$ it follows that $\tilde{\imath}(M(C^*_\sigma)) = VN_\sigma$.

Claim 1. — If $\Phi \in (C^*_\sigma)''$ and $v \in A_\sigma \subset B_\sigma$ the predual of VN_σ then $\langle \Phi v \rangle'' = \langle \tilde{\imath}\Phi, v \rangle$. In fact let $f_\alpha \in L^1(G)$ be such that

$$\langle \kappa\sigma(f_\alpha), u \rangle'' \to \langle \Phi u \rangle''$$

for each $u \in B_\sigma$ (i.e. ultraweakly in $(C^*_\sigma)''$, see Dixmier [6] 12.1). Then $\tilde{\imath}\kappa\sigma(f_\alpha) \to \tilde{\imath}\Phi$ ultraweakly in VN_σ by the u.w. continuity of $\tilde{\imath}$. But $\tilde{\imath}\kappa\sigma(f_\alpha) = i\sigma(f_\alpha)$ and since $v \in A_\sigma$ one has

$$\langle \tilde{\imath}\kappa\sigma(f_\alpha), v \rangle = \langle i\sigma(f_\alpha), v \rangle = \int f_\alpha(x) v(x)\, dx = \langle \kappa\sigma(f_\alpha), v \rangle'' \to \langle \Phi, v \rangle''$$

by Dixmier [6] 12.13 (ii). But the left hand side converges to $\langle \tilde{\imath}\Phi, v \rangle$ which proves the present claim.

Claim 2. — v_n is a weak (i.e. $w(A_\sigma, VN_\sigma)$) Cauchy sequence in A_σ. In fact by the Akeman Pedersen Tomiyama result quoted above each element of

VN_σ can be expressed as $\tilde{\iota}\Phi$ for some $\Phi \in M(C_\sigma^*)$. Hence

$$\langle \tilde{\iota}\Phi, v_n \rangle = \langle \Phi, v_n \rangle'' \to \langle \Phi, u_0 \rangle''$$

by the Akeman Walter prop. 2.4 of [2], p. 458, which proves claim 2.

Now A_σ is weakly sequentially complete as any predual of any W^* algebra by Sakai [13]. Hence there is some $v' \in A_\sigma$ such that $v_n \to v'$ in $w(A_\sigma, VN_\sigma)$ and in particular $\int f(x) v_n(x)\, dx \to \int f(x) v(x)\, dx$ for each $f \in L^1(G)$. But the left hand side converges to $\int f(x) u_0(x)\, dx$. Thus $u_0 = v'$ almost everywhere and since u_0 and v' are continuous $u_0(x) = v(x)$ for each x. Thus $u_0 \in A_\sigma \cap P_\sigma(G)$ and $v_n \to u_0$ weakly as claimed.

COROLLARY. — *If* G *is* σ*-compact and* σ *is a representation of* G *such that* $M_\sigma(C_\sigma^*) = VN_\sigma$ *then* $A_\sigma = B_\sigma$.

Proof. — For any $v_0 \in P_\sigma$ there exists a sequence $v_n \in A_\sigma \cap P_\sigma$ such that $v_n \to v_0$ uniformly on compacta. Thus $v_0 \in A_\sigma$, Hence $P_\sigma \subset A_\sigma$ and $A_\sigma = B_\sigma$.

THEOREM 1. — *Let* G *be any* σ*-compact loc. cpt. group and* σ *a unitary continuous representation of* G *such that* $M_\sigma(C_\sigma^*) = VN_\sigma$.

Then $\operatorname{supp} \sigma$ *is a discrete (closed) subset of* $\hat{G}$, *and* $A_\nu = B_\nu$ *for each representation* ν *weakly contained* σ. *In addition :*

a) for some cardinal c, $c\sigma \simeq c\{\Sigma \oplus \pi ; \pi \in \operatorname{supp} \sigma\}$, ($\simeq$ *stands for equivalence of representations and* $\Sigma \oplus$ *for direct Hilbert sum*),

(*b*) $B_\sigma = A_\sigma = \{\Sigma \oplus A_\pi ; \pi \in \operatorname{supp} \sigma\}$ (*the* l^1 *direct sum, see Arsac* [3], p. 27 *and* 39) *and*

(*c*) $C_\pi^*(G) = LC(H_\pi)$ *for each* $\pi \in \operatorname{supp} \sigma$.

By a theorem of Kaplanski, as stated in Dixmier [6] (10.10.6) C_σ^* is *moreover*, a dual C^*-algebra (see also [6] (4.7.20) and [1] prop. 2.4 and thm. 2.8).

Remark. — If for some $\pi \in \hat{G}$, $C_\pi^* = LC(H_\pi)$ then clearly $M_\pi(C_\pi^*) = VN_\pi = L(H_\pi)$.

Proof. — Let ν be a representation weakly contained in σ. Let $v_0 \in P_\nu \subset P_\sigma$ and $v_n \in P_\nu \cap A_\nu \subset P_\sigma = P_\sigma \cap A_\sigma$ (by the above corollary) be such that $v_n \to v_0$ uniformly on compacta. We apply lemma 0 to the sequence $v_n \in A_\sigma$, $v_0 \in B_\sigma$. It implies that $v_0 \in A_\sigma$ and $v_n \to v_0$ in $w(A_\sigma, VN_\sigma)$. Thus a net of convex combinations of the $v_n' s$ will converge in the norm of A_σ to v_0. Since $v_n \in A_\nu$ and A_ν is norm closed it follows that $v_0 \in A_\nu$ i.e. $A_\nu = B_\nu$.

Let $\pi_0 \in \operatorname{supp} \sigma$. We show that $\{\pi_0\}$ is open in $\operatorname{supp} \sigma$. Let

$$\nu = \{\Sigma \oplus \pi\,;\, \pi \in \operatorname{supp} \sigma, \pi \neq \pi_0\}.$$

If π_0 is not open in $\operatorname{supp} \sigma$ then $\pi_0 \in \operatorname{supp} \nu$. Thus $A_{\pi_0} = B_{\pi_0} \subset B_\nu = A_\nu$ by above. Thus π_0 is quasi-equivalent to a subrepresentation ν_0 of ν by Arsac [3] Cor. 3.14, p. 40. Thus ν_0 is equivalent to $k\pi_0$ for some cardinal k (see Dixmier [6], p. 105 (iii) $\Rightarrow$ (iv)). But then π_0 which is irreducible, is a subrepresentation of ν_0 and hence of ν, which cannot be, by the definition of ν. Thus $\{\pi_0\}$ is open in $\operatorname{supp} \sigma$ which has hence, to be discrete.

Let now $\tau = \{\Sigma \oplus \pi\,;\, \pi \in \operatorname{supp} \sigma\}$. Then, by Arsac [3], p. 39, Cor. 1, A_τ is the l^1 direct sum of all $A_\pi = B_\pi \subset B_\sigma = A_\sigma$ with $\pi \in \operatorname{supp} \sigma$. Hence $A_\tau \subset A_\sigma$ and by Arsac [3], p. 43, thm. 3.18 $A_\sigma = A_\tau \oplus A_{\tau'}$ where τ' is the linear span of the coefficients of the representations ν of G *contained* in σ and *disjoint* from τ (see also Dixmier [6], 5.2. p. 101).

Let $\pi_1 \in \operatorname{supp} \tau' \subset \operatorname{supp} \sigma$. Then $\pi_1 \leqslant \tau$. Thus $A_{\pi_1} \subset A_\tau \cap A_{\tau'} = \{0\}$ by Arsac [3], p. 37, (3.12). This shows that $A_\sigma = A_\tau$ which implies that for some cardinal c, $c\sigma \simeq c\tau$ (see Dixmier [6], p. 105).

Now $A_\sigma = A_{c\sigma} = A_{c\tau} = A_\tau$ by Arsac [3], p. 29 and $A_\tau = \{\Sigma \oplus A_\pi; \pi \in \operatorname{supp} \sigma\}$ (the l_1 direct sum) by [3], p. 39, Cor. 1. To complete the proof one still has to show that $C^*_\pi(G) = LC(H_\pi)$ for each $\pi \in \operatorname{supp} \sigma$. Fix now such a π. It is enough to show that $C^*_\pi(G)$ is a norm separable C*-algebra, since then using the fact that $A_\pi = B_\pi$ we get by Arsac [3], p. 47 that $C^*_\pi(G) = LC(H_\pi)$. (This result is stated in [3] only for separable groups G. However, only the separability of $C^*_\pi(G)$ is used in the proof). Let $a \in H_\pi$, $a \neq 0$. Then $\{\pi(x)a\,;\, x \in G\}$ spans a dense linear subspace of H_π. If $K \subset G$ is compact then $\{\pi(x)a\,;\, x \in K\}$ is a norm compact hence separable subset of H_π. Since G is σ-compact, it follows that $\{\pi(x)a\,;\, x \in G\}$ is separable hence so is H_π. Thus $C^*_\pi(G)$ is a C*-algebra acting on the separable H_π, whose dual is the singleton $\{\pi\}$ (if π' is irreducible and weakly contained in π then, since π is closed in $\hat{G}$, π' is equivalent to π). We apply now lemma 1.5 of S. P. Wang [17], p. 21 and get that $C^*_\pi(G)$ is separable. This finishes the proof.

Remark. — If $A_\sigma = B_\sigma$ for some representation σ, it does not follow that $M_\sigma(C^*_\sigma) = VN_\sigma$, even if G is abelian. In fact let G be locally compact abelian and $\{\gamma_n\}_0^\infty = S$ be such that $\gamma_n \to \gamma_0$ uniformly on compact subsets of G. (Each γ_n is a continuous character on G). Then S is a compact subset of $\hat{G}$.

Let κ be the counting measure on S and σ the representation of G on $L^2(S,\kappa) = l^2(S)$ given by $(\sigma(x)f)(\gamma) = \langle \gamma, x\rangle f(\gamma)$ for $\gamma \in S$, $x \in G$ and $f \in l^2(S)$. If $\mu \in M(G)$ then for $f_1, f_2 \in l^2(S)$ one has

$$\langle \sigma(\mu)f_1, f_2\rangle = \int_S \int_G \gamma(x)\, d\mu(x) f_1(\gamma)\overline{f_2}(\gamma)\, d\kappa$$

$$= \int_S \hat{\mu}(\gamma) f_1(\gamma)\overline{f_2}(\gamma)\, d\kappa = \langle \hat{\mu} f_1, f_2\rangle$$

i.e. $\sigma(\mu)f_1(\gamma) = \hat{\mu}(\gamma)f_1(\gamma)$ (pointwise multiplication of functions) and the operator norm of $\sigma(\mu)$ on $l_2(S)$ is just $\sup\{|\mu(\gamma)|; \gamma \in S\}$. Thus $C_\sigma^* = \{$sup norm closure of $L^1(G)^\wedge$ restricted to $S\} = C(S)$ the continuous complex functions on S. Furthermore $VN_\sigma = l^\infty(S)$ hence

$$M_\sigma(C_\sigma^*) = \{f \in l^\infty(S); fg \in C(S) \text{ for each } g \in C(S)\} = C(S) \neq l^\infty(S) = VN_\sigma$$

since

$$C(S) = \{f \in l^\infty(S);\ \lim_{n\to\infty} f(\gamma_n) = f(\gamma_0)\}.$$

(Note that multiplication of operators in VN_σ is just pointwise multiplication of functions.) However $A_\sigma = B_\sigma$ since B_σ (the dual of C(S)) is just $M(S) = l^1(S)$, since S is countable and the predual A_σ of $l^\infty(S)$ is just $l^1(S)$. In fact if $\gamma_0 \in S$ and $h \in L^1(G)$ then

$$\langle \delta_{\gamma_0}, \hat{h}\rangle = \langle \check{\delta}_{\gamma_0}, h\rangle = \int \check{\delta}_{\gamma_0}(x)h(x)\, dx = \hat{h}(\gamma_0).$$

Thus $A_\sigma = B_\sigma$ but $M_\sigma(C_\sigma^*) \neq VN_\sigma$.

Denote by I_G the trivial unit representation of G on $H = \mathbf{C}$. For definitions related to induced representations we follow S. Gaal [9]. Applying the previous theorem to induced representations we get.

THEOREM 2. – *Let* H *be a closed subgroup of the* σ*-compact group* G *and* $\nu = \nu^\chi$ *the representation induced on* G *by the representation* χ *of* H. *If* I_G *is weakly contained in* ν *and* $M_\nu(C_\nu^*) = VN_\nu$ *then* H *has finite covolume in* G (*i.e.* G/H *admits a finite invariant measure*) *and* supp ν *is discrete. Thus* ([19] *Gaal*, p. 407), $n(1_G, \nu^\chi) = n(1_H, \chi)$. Any π in supp ν is CCR.

Remark. – In particular, if H is open in G i.e. G/H is discrete, then G/H is finite. See also footnote (1) in introduction.

Proof. – I_G is weakly contained in ν which by our theorem 1 necessarily contains I_G as a direct summand. Hence, by proposition 3 in S. Gaal [9], p. 406, G/H admits a finite invariant measure. The result on the multiplicities n of $1_G(1_H)$ in $\nu^{\chi}(\chi)$ follows from Gaal [9], p. 407.

We use next our main theorem to the regular representation ρ of any locally compact group. $VN_\rho(G)$ is denoted by $VN(G)$ in Eymard [7].

THEOREM 3. – *Let* G *be any locally group such that* $M_\rho(C_\rho^*(G)) = VN_\rho(G)$, *where* ρ *denotes the left regular representation. Then* G *is compact.*

Proof. – Let G_1 be any open σ-compact subgroup of G. If $u \in B_{\rho_1}(G_1)$ (ρ_1 is the left regular representation of G_1) let $\mathring{u}$ extend u to all of G by $\mathring{u}(x) = 0$ if $x \in G \sim G_1$. Then $\mathring{u} \in B_\rho(G)$ (see Eymard [7], 2.31, p. 205). Also, $P_\rho(G)|_{G_1} = \{u|_{G_1}\,;\, u \in P_\rho(G)\} = P_{\rho_1}(G_1)$. ([7], 2.31, p. 205).

We claim that $(G_1)^\wedge_{\rho_1}$ the reduced dual of G_1 is discrete. It is enough to prove that $M_{\rho_1}(C^*_{\rho_1}(G_1)) = VN(G_1)$. By [7], (3.21) 2°, p. 215, the map $v \to v|_{G_1}$ from A(G) onto $A(G_1)$ has as its transpose a map $T \to \mathring{T}$ from $VN(G_1)$ to VN(G) which is an isomorphism of W*-algebras onto VN_{G_1} (the w^*-subalgebra of VN(G) generated by $\{\rho(y)\,; y \in G_1\}$) which maps $C_{\rho_1}(G_1)$ into $C^*_\rho(G)$. Let $T \in VN(G_1)$, $\Phi \in C^*_{\rho_1}(G_1)$. Then $(T\Phi)^\circ = \mathring{T}\mathring{\Phi} \in C^*_\rho(G)$ by the assumption of our theorem. Note that $T \to \mathring{T}$ is one to one since $u \to u|_{G_1}$ is onto. We show that $T\Phi \in C^*_{\rho_1}(G_1)$. If $\Lambda \in VN(G)$ let $\Lambda|_{G_1} \in VN(G_1)$ be defined for $v \in A(G_1)$ by $<\Lambda|_{G_1}, v\rangle = \langle \Lambda, v\rangle$, as in [7] 3.21,1°. If $\psi \in VN(G_1)$ then $\mathring{\psi}|_{G_1} = \psi$. Since if $u \in A(G_1)$ then

$$\langle \mathring{\psi}|_{G_1}, u\rangle = \langle \mathring{\psi}, \mathring{u}\rangle = \langle \psi, \mathring{u}|_{G_1}\rangle = \langle \psi, u\rangle .$$

It follows that $T\Phi = [(T\Phi)^\circ]|_{G_1}$ where $(T\Phi)^\circ \in C_\rho(G)$. But by [7], 3.21, 1° $(T\Phi)^\circ|_{G_1} \in C^*_{\rho_1}(G_1)$. Thus $T\Phi \in C^*_{\rho_1}(G_1)$ for each $T \in VN(G_1)$.

Hence $VN(G_1) = M_{\rho_1}(C^*_{\rho_1}(G_1))$. Our previous theorem implies that supp ρ_1 is discrete. We use now thm. 7.6, p. 33 of S. P. Wang [17] and get that G_1 is compact.

Since any σ-compact subgroup G_1 of G is compact it follows that G is compact.

Applications to the quasiregular representation.

Notations. – H will always be a closed subgroup of the locally compact group G. We denote by π_H the quasiregular representation of G on $L^2(G/H,\lambda)$ where λ is a quasi invariant measure on G/H (for definition see Eymard [8], p. 27) and $G/H = \{\dot{x} = xH ; x \in G\}$.

PROPOSITION 4. – Let H *be a closed subgroup of* G *such that the coset space* G/H *is compact. Then* $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ *(In fact* $C^*_{\pi_H}$ *is even an ideal in its second dual).*

Proof. – As known [8], p. 27, π_H is the representation of G induced by I_H the trivial one dimensional representation of H on **C** (which is certainly CCR). Hence, if G is second countable, then π_H is CCR by I Schochetman [14] thm. 4.1, p. 482. i.e. $\pi_H(f)$ is compact for all f in $L^1(G)$. Schochetman's result has been generalized to arbitrary G by J.M.G. Fell in [21], p. 57. Consequently $C^*_{\pi_H}$ contains only compact operators for arbitrary G.

Let $A = C^*_{\pi_H}$. Then by Berglund thm. 5.5, p. 25, A is an ideal in its second dual A″. Let π_0 be the identity representation of A, $\pi_0(a) = a$ for all $a \in A$ where a is an operator on $L^2(G/H)$. Then π_0 is faithful and hence lifts to a faithful representation again denoted by π_0 of

$$A'' = M(A) = \{b \in A'' ;\ bA + Ab \subset A\}$$

into VN_{π_H} (see Ackeman Pedersen Tomiyama [1] prop. 2.4, attributed there to M. C. Flanders in his Tulane Univ. 1968 thesis) and furthermore $\pi_0 M(A)$ is the idealiser of A in VN_{π_H} i.e. exactly our $M_{\pi_H}(C^*_{\pi_H})$. Note also that by the same prop. 2.4, π_0 lifted to M(A) is just the restriction of the normal extension from A to A″ as in Dixmier [6] 12.1.5. Hence $\pi_0 A'' = VN_{\pi_H}$ i.e.

$$M_{\pi_H}(C^*_{\pi_H}) = \pi_0 M(A) = \pi_0(A'') = VN_{\pi_H}$$

which finishes the proof.

Remark. – The following know fact has been shown above : If τ is a representation of the dual C*-algebra A (i.e. A is an ideal in A″) on the Hilbert space H_τ and VN_τ is the W*-algebra generated by $\tau(A)$ then $M_\tau[\tau(A)] = VN_\tau$.

Our main goal in what follows is to get some converse to the above proposition.

Our main theorem yields immediately the following.

THEOREM 5. — *Let* G *be* σ*-compact,* $H \subset G$ *a closed subgroup and* π_H *the quasiregular representation. If* $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ *then* (1) $\operatorname{supp} \pi_H$ *is a discrete closed subset of* $\hat{G}$ (*and* $A_\nu = B_\nu$ *for any* ν *weakly contained in* π_H) (2) *for some cardinal* c, $c\pi_H$ *is equivalent to* $c\{\Sigma \oplus \pi ; \pi \in \operatorname{supp} \sigma\}$ *and* (3) *any* $\nu \in \operatorname{supp} \pi_H$ *is* CCR.

Remarks. — We hoped to find in the literature a result stating the following :

(*) Let G be σ-compact $H \subset G$ a closed subgroup. If $\operatorname{supp} \pi_H$ is discrete then the coset space G/H is compact.

(*) Would generalize the well known result of L. Baggett [20], A. I. Shtern and S. P. Wang [17], p. 33, that if G is-σ-compact $H = \{e\}$ (thus $\pi_H = \rho$ is the regular representation) and supp ρ is discrete then G is compact. We could neither find such a result in the literature, nor prove it ourselves. Clearly, our main theorem together with (*) would imply that if H is a closed subgroup of the σ-compact G and $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ then G/H is compact.

We still will be able to prove this assertion in many cases.

G/H is said to be an *amenable* coset space, if the identity representation I_G is weakly contained in the quasiregular representation π_H. We follow Eymard [8] in notations and results regarding π_H. Many equivalent conditions as well as many examples of amenable coset spaces G/H are given in Eymard [8]. In particular, if G is an amenable group then G/H is a amenable coset space for any H. In quite a few interesting cases, G and $H \subset G$ are not amenable while G/H is an amenable coset space. For example if $G = SL(2R)$, $H = SL(2, Z)$ then G/H is an amenable coset space even though none of G or H are amenable groups. Furthermore, if G/H is compact then it is not necessarily an amenable coset space.

THEOREM 6. — *Let* G *be* σ*-compact and* $H \subset G$ *a closed subgroup such that the quasiregular representation weakly contains a finite dimensional nonzero representation.*

If $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ *then* G/H *admits a finite invariant measure and the support of* π_H *is discrete. In addition any* π in $\operatorname{supp} \pi_H$ is CCR.

Proof. — Assume that α is a non zero representation of G on the Hilbert space H_α with $\dim H_\alpha = n$, $0 < n < \infty$. Let $v(x) = \langle \alpha(x)\xi, \eta > \xi$, $\eta \in H_\alpha$ be such that $v(x_0) \neq 0$ at some x_0. Then $v(x)$ is a bounded continuous almost periodic function on G (AP(G)) and $M(|v|) > 0$ where M is the unique invariant mean on the continuous weakly almost periodic functions on G (denoted WAP(G). Note that $|v|$ is defined by $|v|(x) = |v(x)|$ for all x). This is due to the direct sum decomposition $WAP(G) = AP(G) \oplus W_0(G)$ where $W_0(G) = \{f \in WAP(G);\ M(|f|) = 0\}$ *see* Burckel [5], pp. 29-30. Clearly $v \in B_\alpha \subset B_{\pi_H} = A_{\pi_H}$ (by our theorem 1). Apply now the majorisation principle of Hertz as given in Arsac [3], p. 54 (take $\omega = I_H$). Then there is some $u \in A_{\pi_H}$ such that $\|u\| \leqslant \|v\|$ (in B(G) norm) but $|v(x)| \leqslant u(x)$ for each x in G. Hence $M(u) = d \neq 0$ and by Dixmier [6], 13.11.6, p. 274 the constant $d = M(u)$ function belongs to the *uniform* closure of $\mathrm{Co}\{l_x u; x \in G\}$ where $(l_x u)(y) = u(xy)$ and Co denotes convex hull. Clearly, by M. Walter [15], p. 33, $l_x u \in B_{\pi_H}$. Also, for any $w \in \mathrm{Co}\{l_x u; x \in G\}$ $\|w\| \leqslant \|v\|$ see [7], p. 186. Let $w_\beta \in \mathrm{Co}\{l_x u; x \in G\}$ be a net such that $w_\beta(x) \to d > 0$ *uniformly* in $x \in G$. Then a subnet w_{β_γ} will converge w^* and a fortiori $\sigma(B_{\pi_H}, L^1(G))$ to some $w_0 \in B_{\pi_H}$ (which is the dual of $C^*_{\pi_H}$). If $f \in C_{00}(G)$ then $\int w_{\beta_\nu} f\,dx \to \int w_0 f\,dx$. Thus $\int (w_0 - d) f\,dx = 0$ for each $f \in C_{00}(G)$. This shows that $w_0 = d$ a.e. (G is σ-compact). Since w_0 is continuous $w_0 = d > 0$ everywhere. Thus $1 \in B_{\pi_H}$ and I_G is weakly contained in π_H. Apply now theorem 2 and the fact that π_H is in fact I_H induced to G. Thus H has finite covolume in G and $\mathrm{supp}\,\pi_H$ is discrete.

COROLLARY 1. — *Let* G *be* σ*-compact* H *a closed subgroup be such that* π_H *weakly contains a finite dimensional* $\neq 0$ *representation. If* $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$. *Then* G/H *is compact and admits a finite invariant measure at least in the following cases :*

1. H *is compact.*

2. H *is a semidirect summand.*

3. H *is open in* G (*i.e.* G/H *is discrete*) [2]. *In particular if* G *is discrete.*

4. G *is connected and* $H = Z_G(A) = \{x \in G; \alpha x = x$ *for all* $\alpha \in A\}$ *where* A *is any set of continuous automorphisms of* G.

[2] Any π in $\mathrm{supp}\,\pi_H$ is CCR and since G is discrete π is finite dimensional. The requirement that π_H weakly contains a finite dimensional rep. is superflous.

4'. *It is enough in* 4 *that* G *is only* σ-*compact* $H = Z_G(A)$ *and* $F(G/G_0)$ *is open in* G/G_0 (*where* G_0 *is the connected component of* e *and* $F(K) = \{x \in K\,;\ x$ *has relatively compact conjuacy class in* $K\}$.

Proof. – 1. $A_{\pi_H} = B_{\pi_H}$ by our theorem 1 and $1 \in B_{\pi_H}$ by theorem 6. Thus $1 \in A_{\pi_H}$. However $A_{\pi_H} \subset A(G)$ by Arsac [3], p. 83, proposition. Since $1 \in A(G) \subset C_0(G)$ it follows that G is compact.

2. $G = HN$ and $H \times N \to HN$ defined by $(h, n) \to hn$ is a homeomorphism. Also $H \cap N = \{e\}$ and N is normal in G. Thus $N = N/N \cap H \to G/H$ and $H = H/N \cap N \to G/N$ (canonical maps) are homeomorphisms. By prop. 4.4 of S. P. Wang [18], p. 413, H has finite invariant covolume in G (which holds by our theorem 6 above) iff $\{e\} = N \cap H$ has finite covolume in N. This just means that the *Haar* measure λ on the group N satisfies $\lambda(N) < \infty$. Thus N has to be a compact group (as well known) and since N and G/H are homeomorphic, G/H is compact.

3. G/H is discrete and admits a finite invariant measure by our theorem. Thus G/H is finite.

4 and 4'. Our theorem 6 implies that $H = Z_G(A)$ has finite invariant covolume in G. We apply now Corollary 5.7, p. 416 of S. P. Wang [18] and get that G/H is compact. Note that a slightly better result than 4 can be obtained, by using thm. 5.6., p. 416 of S. P. Wang [18].

COROLLARY 2. – *Let* G *be* σ-*compact* H *a closed subgroup be such that* π_H *weakly contains* $a \neq 0$ *finite dimensional representation. If* $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$, *then* G/H *is compact and admits a finite invariant measure at least in the following cases :*

1. G *is a connected Lie group and* H *is a connected subgroup.*

2. G *is a solvable Lie group and* H *any closed subgroup.*

3. G *is any Lie group and* $H = Z_G(A)$ *where* A *is any set of continuous automorphisms of* G.

4. G *is a Lie group all whose connected semisimple subgroups are compact and* H *any closed subgroup.*

Proof. – In any case H has finite covolume in G by our theorem 6. We use now some deep results of G. D. Mostow which state that if G is a connected [or solvable] Lie group and H is any connected [arbitrary] closed subgroup such that H has finite covolume in G then G/H is compact (see [19], p. 317 and [16], p. 392 or Mostow [11]). To prove (3) use S. P. Wang's thm. 3.6 on p. 319 of [19] which states that $H = Z_G(A)$ has finite covolume in G if and only if G/H is compact.

To prove 4 note that by theorem 2.1 of S. P. Wang [19], p. 315 any closed subgroup H of such G for which G/H has finite invariant volume necessarily satisfies that G/H is compact.

COROLLARY 3. – *Let* G *be* σ-*compact* H *a closed subgroup be such that* π_H *weakly contains* $a \neq 0$ *finite dimensional representation and* $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$.

If G *is a Lie group* H *any closed subgroup then* G/H_1 *is compact and has a finite invariant measure, where* $H_1 = Z_G(Z_G(H))$.

Note. – $Z_G(K) = \{x \in G; xk = kx$ for all k in $K\}$.

Proof. – Our theorem 6 implies that G/H has a finite invariant measure Thus by Wang [19] Cor. 3.7 G/H_1 is compact. But $H \subset H_1$. Thus by Mostow [11] lemma 2.5 G/H_1 admits a finite invariant measure.

The reader will notice that in all above corollaries 1, 2 and 3, G and H were so chosen that the assertion G/H admits a finite invariant measure (i.e. I_G is an isolated point of supp π_H) implied that G/H is in fact compact.

Our theorem 6 implies however more that this fact, namely it implies that supp π_H is discrete and G/H admits a finite invariant measure, and any π in supp π_H is CCR.

We expect in fact that the following statement (weaker than the statement (*) after theorem 5) to be true :

(**) Let G be σ-compact, H a closed subgroup be such that G/H admits a finite invariant measure. If supp π_H is discrete then G/H is compact. The truth of (**) would imply an improved version of theorem 6 namely.

Assertion 6'. – Let G be σ-compact H a closed subgroup such that π_H weakly contains a finite dimensional nonzero representation.

If $M_{\pi_H}(C^*_{\pi_H}) = VN_{\pi_H}$ then G/H is compact and admits a finite invariant measure. (Theorem 6 only asserts that supp π_H is discrete and G/H admits a finite invariant measure). One notes that Assertion 6′ would make corollaries 1, 2, 3 superfluous.

We pose hereby, statements (*) or at least (**) as open questions.

BIBLIOGRAPHY

[1] C. A. AKEMAN, G. K. PEDERSEN, J. TOMIYAMA, Multipliers of C*-Algebras, *J. of Functional Analysis*, 13 (1973), 277-301.

[2] C. A. AKEMAN, M. E. WALTER, Nonabelian Pontriagin duality, *Duke Math. J.*, 39 (1972), 451-463.

[3] G. ARSAC, Sur l'espace de Banach engendré par les coefficients d'une représentation unitaire, Thèse, Université Claude-Bernard, Lyon I, 1973.

[4] M. C. F. BERGLUND, Ideal C*-algebras, *Duke Math. J.*, 40 (1973), 241-257.

[5] R. B. BURCKEL, Weakly almost periodic functions on semigroups, Gordon and Breach, 1970.

[6] J. DIXMIER, Les C*-algèbres et leurs représentations, Gauthier-Villars, Paris, 2e edition, 1969.

[7] P. EYMARD, l'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France*, 92 (1964), 181-236.

[8] P. EYMARD, Moyennes Invariantes et Représentations Unitaires, *Lecture notes in mathematics*, Springer Verlag, 1972, n° 300.

[9] Steven A. GAAL, Linear Analysis and Representation Theory, Springer Verlag, 1973.

[10] E. GRANIRER, Density theorems for some linear subspaces and some C*-subalgebras of VN(G), *Proc. of Symp. on Harmonic Analysis and Function Spaces*, Inst. Nazionale di Alta Matematica Rome March 1976, pp. 61-70.

[11] G. D. MOSTOW, Homogeneous spaces with finite invariant measure, *Annals of Math.*, 75 (1962), 17-37.

[12] G. D. MOSTOW, Factor spaces of solvable groups, *Ann. of Math.*, 60 (1954), 1-27.

[13] S. SAKAI, On topological properties of W*-algebras, *Proc. Japan Acad.*, 33 (1957), 439-444.

[14] I. SCHOCHETMAN, Topology and the duals of certain locally compact groups, *Trans. AMS*, 150 (1970), 477-489.

[15] M. E. WALTER, W*-algebras and nonabelian harmonic analysis, *J. of Functional Analysis*, 11 (1972), 17-38.

[16] S. P. WANG, On S-subgroups of solvable Lie groups, *Amer. J. Math.*, 92 (1970), 389-397.

[17] S. P. WANG, On isolated points in the dual spaces of locally compact groups, *Math. Annalen*, 218 (1975), 19-34.

[18] S. P. WANG, Compactness properties of topological groups III, *Trans AMS*, 209 (1975), 399-418.

[19] S. P. WANG, Homogeneous spaces with invariant measure, *Amer. J. Math.*, 98 (1976), 311-324.

[20] L. BAGGETT, A separable group having a discrete dual space is compact, *J. Functional Analysis*, 10 (1972), 131-148.

[21] J. M. G. FELL, A new look at Mackey's imprimitivity theorem, Proc. Conference on Harmonic Analysis Maryland 1971, *Springer Lecture Notes*, Vol. 266, 42-58.

[22] G.W. MACKEY, Induced representations of locally compact groups I, *Annals of Math.*, 55 (1952), 101-139.

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