

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 29, n° 4 (1979), p. 159-171

http://www.numdam.org/item?id=AIF_1979__29_4_159_0

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ZEROS OF RANDOM FUNCTIONS IN BERGMAN SPACES

by Joel H. SHAPIRO (*)

1. Introduction.

Let μ be a finite, positive, rotation invariant Borel measure on the open unit disc Δ of the complex plane, and suppose that μ gives positive mass to each annulus $r < |z| < 1$. For $0 < p < \infty$ the *weighted Bergman space* A_μ^p is the collection of functions f holomorphic in Δ with

$$\|f\|_p^p = \int |f|^p d\mu < \infty.$$

Let $A_\mu^{p+} = \bigcup_{q>p} A_\mu^q$, so $A_\mu^{p+} \subset A_\mu^p$. For f holomorphic in Δ , let $Z(f)$ denote the *zero set* of f , with each zero counted according to its multiplicity. If f belongs to some class \mathfrak{F} of holomorphic functions we frequently refer to $Z(f)$ as an \mathfrak{F} -*zero set*.

Recently we showed [7] that for each such μ and p there exists f in A_μ^p such that :

- (a) $Z(f)$ is contained in no A_μ^{p+} zero set, and
- (b) $Z(f+1) \cup Z(f-1)$ lies in no $A_\mu^{(p/2)+}$ zero set, hence in no A_μ^p zero set.

These results continued the work of Charles Horowitz [2] and Walter Rudin [4]. Horowitz considered the special measures

$$d\mu(z) = (1-|z|)^\alpha dx dy \quad (\alpha > -1),$$

(*) Research partially supported by the National science Foundation.

and used infinite products to construct the desired functions; while Rudin got similar results for Hardy spaces on the unit ball and polydisc in \mathbb{C}^n by means of an ingenious « multiplier argument ». The proof in [7] used Rudin's idea, with the desired function f constructed as a gap series.

The point of this paper is that Rudin's method (which we will describe in the next section) also works very naturally in the context of *random power series*. We show that a *Gaussian power series which almost surely lies in $A_\mu^p A_\mu^{p+}$ must almost surely have properties (a) and (b) listed above.*

More precisely, let $(\zeta_n)_0^\infty$ be a sequence of independent complex Gaussian random variables with mean zero and variance one [3; Ch. 9, sec. 3, p. 118]. Suppose $(a_n)_0^\infty$ is a sequence of complex numbers with

$$(1.1) \quad \limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq 1,$$

and consider the random power series

$$(1.2) \quad f(z) = \sum_{n=0}^{\infty} \zeta_n a_n z^n.$$

Since almost surely $|\zeta_n| = O(\sqrt{\log n})$ [3; Ch. XI, sec. 4, p. 121, Prop. 3], condition (1.1) insures that with probability one the series (1.2) converges uniformly on compact subsets of Δ to a holomorphic function. The quantity which controls the random behavior of f is its variance $\sigma_f^2(z)$, defined for $z \in \Delta$ by

$$(1.3) \quad \sigma_f^2(z) = \mathcal{E}\{|f(z)|^2\} = \sum_{n=0}^{\infty} |a_n|^2 |z|^{2n}.$$

The main result of this paper is the following.

THEOREM 1. — *Suppose f is defined by formulas (1.1) and (1.2). Then*

(a) *the following are equivalent :*

(i) $\sigma_f \in L^p(\mu)$ but $\notin L^{p+}(\mu)$.

(ii) *With probability one : $f \in A_\mu^p$ but $\notin A_\mu^{p+}$.*

(iii) *With probability one : $f \in A_\mu^p$ but $Z(f)$ is not contained in any A_μ^{p+} zero set.*

(b) If any (hence all) of the above conditions hold, then with probability one : $Z(f+1)$ and $Z(f-1)$ are A_μ^p zero sets, but their union is not even an $A_\mu^{(p/2)+}$ zero set.

The most important of these results are (b), and the implication (i) \rightarrow (iii) of (a) : these imply the corresponding results in [7]. For their proof we require only the most basic facts about Gaussian random variables. The other non-trivial implication in (a) is (ii) \rightarrow (i), which follows from a beautiful result of X. Fernique concerning moments of vector valued Gaussian random variables. These matters, Rudin's multiplier argument, and some other preliminaries are reviewed in section 2. Theorem 1 is proved in the third section, and the paper closes with some remarks and open problems.

I want to thank my colleague Joel Zinn of Michigan State University for several interesting discussions, and especially for pointing out Fernique's theorem to me.

2. Preliminaries.

(a) *Rudin's multiplier argument.* As exploited in both this paper and [7], Rudin's idea is this : if the zero set of $f \in A_\mu^p \setminus A_\mu^{p+}$ is contained in some A_μ^{p+} zero set, then $fh \in A_\mu^{p+}$ for some h holomorphic in Δ . Since h decreases the growth of f , it must have relatively small values where f is large. Assuming (without loss of generality) that $h(0) = 1$, we obtain from the subharmonicity of h :

$$(2.1) \quad 0 = \log |h(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| d\theta$$

for $0 \leq r < 1$, which forces h on the circle $|z| = r$ to balance out any small values with appropriate large ones. Therefore if $f \in A_\mu^p$ does not get into A_μ^{p+} because it has large values on substantial portions of certain circles $|z| = r_n (r_n \rightarrow 1-)$, then we should expect that no h holomorphic in Δ can multiply f into A_μ^{p+} . We will show in the next section that any Gaussian series (1.2) which almost surely lies in $A_\mu^p \setminus A_\mu^{p+}$ will almost surely be such an f . This complements the work in [7] where such f 's were constructed as gap series.

(b) *Gaussian random variables.* The reference for all of this material is [3; Ch. XI, sec. 1-4]. From now on $(\zeta_n)_0^\infty$ denotes a sequence of independent complex Gaussian random variables with mean zero and variance one, defined on a probability space $(\Omega, \mathfrak{F}, \Pr)$. In particular, (ζ_n) is an orthonor-

mal sequence in $L^2(\Omega, \mathfrak{F}, \text{Pr})$, and for each Borel subset B of the complex plane :

$$\text{Pr}\{\zeta_n \in B\} = \frac{1}{2\pi} \iint_B e^{-(x^2+y^2)/2} dx dy.$$

From this it follows quickly that for $0 \leq \lambda < \infty$,

$$\text{Pr}\{|\zeta_n| > \lambda\} = e^{-\pi\lambda^2}$$

[3; Ch. XI, sec. 4, p. 121, formula (3.1)]. A crucial property of the sequence (ζ_n) is that if (a_n) is a complex sequence with $\|a\|_2^2 = \sum |a_n|^2 < \infty$, and if $Z = \sum a_n \zeta_n$, then the random variable $Z/\|a\|_2$ has the same distribution as ζ_n . In particular :

$$(2.2) \quad \text{Pr}\{|Z| > \lambda \|a\|_2\} = e^{-\pi\lambda^2},$$

and for $0 < p < \infty$:

$$(2.3) \quad \mathcal{E}\{|Z|^p\} = C_p^p (\mathcal{E}\{|Z|^2\})^{p/2} = C_p^p \|a\|_2^p,$$

where C_p is independent of (a_n) , and \mathcal{E} denotes integration with respect to Pr . These are the only properties of (ζ_n) that we require for the main part of the proof of Theorem 1.

We remark in passing that the statement « f has property Q with probability one » (or « almost surely ») means that there exists $E \in \mathfrak{F}$ with $\text{Pr}\{E\} = 1$ such that f has property Q for every $\omega \in E$. We do not require $\{\omega \in \Omega : f \text{ has property } Q\}$ to belong to \mathfrak{F} . Similar remarks apply to statements like « with probability $\geq \delta$, f has property Q ».

(c) *Interchanging measure and probability.* Some form of the next result occurs frequently in applications of probability to analysis.

LEMMA A [3; Ch. V, sec. 4, p. 42]. — Suppose $(\Omega, \mathfrak{F}, P)$ and (T, \mathfrak{B}, m) are probability spaces, and $E \in \mathfrak{F} \otimes \mathfrak{B}$ (product sigma-algebra). Define the usual cross-sections;

$$\begin{aligned} E^\omega &= \{t \in T : (\omega, t) \in E\} \quad (\omega \in \Omega) \\ E_t &= \{\omega \in \Omega : (\omega, t) \in E\} \quad (t \in T), \end{aligned}$$

and suppose $0 \leq \theta, \eta \leq 1$. If $P\{E_t\} \geq \eta$ for $[m]$ a.e. t in T , then

$$P\{\omega \in \Omega : m(E^\omega) \geq \theta\eta\} \geq \frac{(1-\theta)\eta}{1-\theta\eta}.$$

Proof. — Let $A = \{\omega \in \Omega : m(E^\omega) \geq \theta\eta\}$. Then by Fubini's theorem :

$$\begin{aligned} \eta &\leq \int_{\Gamma} P\{E_t\} dm(t) \\ &= \int_{\Omega} m(E^\omega) dP(\omega) \\ &= \int_A + \int_{\Omega \setminus A} m(E^\omega) dP(\omega) \\ &\leq P(A) + \theta\eta[1 - P(A)], \end{aligned}$$

and the result follows upon solving for $P(A)$.

(d) *Fernique's Theorem.* For $f \in A_\mu^p$ let $\|f\| = \|f\|_p$ if $p \geq 1$, and $\|f\|_p^p$ if $0 < p < 1$. Then $\|\cdot\|$ is a norm on A_μ^p if $p \geq 1$, and a « p -norm » if $0 < p < 1$ (that is, $\|af\| = |a|^p\|f\|$ when $0 < p < 1$). It is not difficult to use the subharmonicity of $|f|^p$ to check that for each $z \in \Delta$ the linear functional of « evaluation at z »

$$f \rightarrow f(z) \quad (f \in A_\mu^p)$$

is continuous on A_μ^p ($0 < p < \infty$). From this it follows that A_μ^p , in the metric induced by $\|\cdot\|$, is complete, i.e., it is a Banach space when $p \geq 1$ and a « p -Banach space » when $0 < p < 1$. Even when $0 < p < 1$ there are enough continuous linear functionals to separate points (the point evaluations, for example), and the Borel structure induced on A_μ^p by the « norm » topology coincides with the one induced by the topology of uniform convergence on compact subsets of Δ (since the closed unit ball of A_μ^p is closed in this weaker topology).

From these considerations it follows routinely that if (u_n) is a sequence in A_μ^p for which the Gaussian series $Z = \sum \zeta_n u_n$ converges almost surely in A_μ^p , then, even when $0 < p < 1$, Z is an A_μ^p -valued Gaussian random variable in the following sense : if Z' and Z'' are independent and similar to Z , then so are $(Z' + Z'')/\sqrt{2}$ and $(Z' - Z'')/\sqrt{2}$.

Thus X. Fernique's Theorem [1] (or more precisely when $0 < p < 1$, its proof) applies to Z , and shows that the tail distribution $\Pr\{\|Z\| > \lambda\}$ decays exponentially as $\lambda \rightarrow \infty$. In particular,

$$(2.4) \quad \mathcal{E}\{\|Z\|_p^p\} < \infty,$$

which yields the following characterization of Gaussian Taylor series which a.s. belong to A_μ^p .

LEMMA B. — Suppose f and σ_f are given by (1.1)-(1.3). Then $f \in A_\mu^p$ almost surely if and only if $\sigma_f \in L^p(\mu)$.

Proof. — For any f given by (1.1) and (1.2), we have from (2.3) and Fubini's Theorem :

$$\int \sigma_f^p d\mu = C_p^{-p} \int \mathcal{E}\{|f|^p\} d\mu = C_p^{-p} \mathcal{E}\{\|f\|_p^p\}.$$

Thus $\sigma_f \in L^p(\mu)$ implies $\mathcal{E}\{\|f\|_p^p\} < \infty$, hence $\|f\|_p < \infty$ a.s. Conversely, suppose $\|f\|_p < \infty$ a.s. We claim f is an A_μ^p -valued Gaussian random variable. Indeed, the fact that the integral means

$$(2.5) \quad M_p^p(f; r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

increase with r [6; Theorem 17.6, p. 363], along with the monotone convergence theorem, show that the Taylor series (1.2) of f is a.s. Abel summable to f in A_μ^p . Thus by [3; Theorem 1, Ch. II, p. 11] (whose proof works even when $0 < p < 1$), the series converges a.s. in A_μ^p to f ; hence f is Gaussian. From (2.4) we see $\mathcal{E}\{\|f\|_p^p\} < \infty$, hence by the calculations at the beginning of this proof, $\sigma_f \in L^p(\mu)$. This completes the proof.

(e) *Two technical lemmas.* We close this section with two lemmas needed to deduce Theorem 1 from the essential probabilistic arguments, which will be isolated in Proposition 2 of the next section.

LEMMA C. — Given μ as usual, and $0 < p < \infty$, there exists a finite positive rotation invariant Borel measure ν whose closed support is $\{|z| \leq 1\}$, and such that $A_\mu^p = A_\nu^p$.

Proof. — For f holomorphic in Δ the integral mean $M_p^p(f; r)$ defined by (2.5) increases with r , so if $f \in A_\mu^p$ then

$$(2.6) \quad 2\pi M_p^p(f; r) \leq \|f\|_p^p \mu(r)^{-1}$$

where

$$\mu(r) = \mu\{z : r \leq |z| < 1\}.$$

Our standing hypotheses on the measure μ insure that $\mu(r) > 0$ for each $0 < r < 1$, and $\mu(r) \downarrow 0$ as $r \uparrow 1$. In particular μ is a bounded, strictly positive, measurable function on $[0,1)$, hence the measure

$$d\nu(z) = d\mu(z) + \mu(|z|) dx dy/\pi$$

has closed support equal to $\{|z| \leq 1\}$, and dominates μ . Since (2.6) insures that

$$\int_{\Delta} |f|^p dv \leq 2\|f\|_p^p$$

we see that $A_{\mu}^p = A_{\nu}^p$, as desired.

LEMMA D. — Suppose γ is a finite positive Borel measure on the interval $[0,1]$ which is either (i) purely atomic, or (ii) continuous with closed support equal to $[0,1]$. Then for $0 < \alpha < 1$:

$$(2.7) \quad \int_0^1 \gamma[r,1]^{-\alpha} d\gamma(r) < \infty.$$

Proof. — (i) Suppose γ is purely atomic, say with mass γ_n at r_n ($n = 1, 2, \dots$), and no mass anywhere else. Let $\rho_n = \sum_{k \geq n} \gamma_k$. Then the integral in (2.7) is just the series $\sum \rho_n^{-\alpha} \gamma_n$, whose convergence for $0 < \alpha < 1$ is a standard exercise in advanced calculus (see [5; Ch. 3, pp. 79-80, problem 12(b)] for the special case $\alpha = 1/2$).

(ii) If γ is continuous with closed support = $[0,1]$, then the function

$$\gamma(r) = \gamma([r,1]) \quad (0 \leq r < 1)$$

is continuous and strictly decreasing on $[0,1]$. The integral in (2.7) can then be interpreted as the Riemann-Stieltjes integral

$$- \int_0^1 v(r)^{-\alpha} dv(r)$$

which, after making the change of variable $x = v^{-1}(r)$ (composition inverse), and paying due respect to the singularity at $r = 1$, becomes [5; Theorem 6.19, p. 132]

$$\int_0^1 x^{-\alpha} dx < \infty.$$

This completes the proof.

3. Proof of the Main Theorem.

We isolate the essential part of Theorem 1 in the following proposition, which we state in somewhat more generality than actually required. The

following notations help the exposition. As in the proof of Lemma C, let

$$\mu(r) = \mu\{z \in \Delta : r \leq |z| < 1\}.$$

For b holomorphic in Δ and $0 \leq r < 1$, let

$$M_\infty(b; r) = \max\{|b(z)| : |z| = r\}$$

and write

$$b_r(e^{i\theta}) = b(re^{i\theta}).$$

From now on, f always represents a Gaussian power series as given by (1.1) and (1.2), with σ_f given by (1.3). We also assume that the measure μ has total mass 1, so $0 < \mu(r) \leq 1$.

PROPOSITION 2. — *Suppose that*

$$(3.1) \quad \limsup_{r \rightarrow 1^-} \frac{\sigma_f(r)\mu(r)^{1/p}}{-\log \mu(r)} > 0.$$

Then the following holds with probability one : for each positive integer N , every b holomorphic in Δ with

$$\limsup_{r \rightarrow 1^-} M_\infty(b; r)\mu(r)^{N/p} < 1,$$

and every h holomorphic in Δ , we have

$$(f^N + b)h \notin A_\mu^{p/N}.$$

Remark. — For part (a) of Theorem 1 we need only the case $N = 1$, $b \equiv 0$, while for part (b) we require $N = 2$, $b \equiv -1$. However, these special cases are no easier to prove than the general proposition, which gives some further information regarding remark (i), section 5 of [7].

Proof. — Let \mathbf{T} denote the unit circle $\{|z| = 1\}$, m normalized Lebesgue measure on \mathbf{T} , and μ_1 the unique finite positive Borel measure on $[0,1)$ such that

$$\int_{\Delta} g \, d\mu = \int_{[0,1)} \left\{ \int_{\mathbf{T}} g(rt) \, dm(t) \right\} d\mu_1(r)$$

for each $g \in C_0([0,1))$.

Fix $k > 0$. We are going to show that the desired result holds with probability at least $k/(k+1)$; hence with probability one, since k is

arbitrary. In this regard the reader should note that although the set $\{\omega : f^N \notin A_\mu^{p/N}\}$ is a tail event, the set we are interested in :

$$\{\omega : (f^N + b)h \notin A_\mu^{p/N} \text{ for all } h, b \text{ as in the Proposition}\}$$

is *not* (in fact it is not even clear that it is an event), so the zero-one law does not apply.

According to the hypothesis (3.1) there is a positive number δ and a positive sequence $r_n \rightarrow 1 -$ such that

$$\sigma_f(r_n)\mu(r_n)^{1/p} \geq -\delta \log \mu(r_n).$$

Let $\lambda_n^{-1} = \sigma_f(r_n)\mu(r_n)^{1/p}$, so

$$(3.2) \quad 0 < \lambda_n \leq \frac{1}{-\delta \log \mu(r_n)} \rightarrow 0.$$

For each positive integer n , (2.2) insures that for all $t \in T$ we have

$$(3.3) \quad \Pr\{|f(r_n t)| > \lambda_n \sigma_f(r_n)\} = e^{-\pi \lambda_n} = \eta_n,$$

where $\eta_n \rightarrow 1$ because $\lambda_n \rightarrow 0$. Let

$$E(n) = \{(\omega, t) \in \Omega \times T : |f(r_n t)| > \lambda_n \sigma_f(r_n)\}.$$

Then using the notation of Lemma A, equation (3.3) asserts that $\Pr\{E_t(n)\} = \eta_n$ for every t in T ; hence Lemma A, with $\theta = \eta_n^k$, shows that with probability at least

$$\beta_n = \frac{\eta_n(1 - \eta_n^k)}{1 - \eta_n^{k+1}}$$

we have $m\{E^\omega(n)\} \geq \eta_n^{k+1}$ ($n = 1, 2, \dots$). Since $\beta_n \rightarrow k/(k+1)$ as $n \rightarrow \infty$, it follows that with probability $\geq k/(k+1)$:

$$(3.4) \quad m\{E^\omega(n)\} \geq \eta_n^{k+1} \text{ for infinitely many } n.$$

Let $F = \{\omega \in \Omega : (3.4) \text{ holds}\}$; then $\Pr\{F\} \geq k/(k+1)$. We are going to show that for each $\omega \in F$;

$$(f^N + b)h \notin A_\mu^{p/N}$$

whenever h, b, N are as in the hypothesis of the proposition. This will complete the proof.

To this end, fix $\omega \in F$ and b, N , and h . Suppose, as we may, that $h(0) = 1$, and choose $0 < \varepsilon < 1$ so that

$$\limsup_{r \rightarrow 1^-} M_\omega(b; r)\mu(r)^{N/p} < \varepsilon.$$

Letting $R_n = \{r_n \leq |z| < 1\}$ we have :

$$\begin{aligned} \int_{R_n} |(f^N + b)h|^{p/N} d\mu &= \int_{[r_n, 1)} \left\{ \int_T |(f_r^N + b_r)h_r|^{p/N} dm \right\} d\mu_1(r) \\ &\geq \int_{[r_n, 1)} \exp \left\{ (p/N) \int_T \log |(f_r^N + b_r)h_r| dm \right\} d\mu_1(r) \end{aligned}$$

by the arithmetic-geometric mean inequality. Let $I(r)$ denote the integral inside the braces. Then using (2.1) and the fact that $\int \log |g_r| dm$ increases with r for any holomorphic function g on Δ [6; Theorems 17.3 and 17.5, pp. 362-363] we obtain for $r_n \leq r < 1$:

$$\begin{aligned} I(r) &\geq \int_T \log |f_r^N + b_r| dm \\ &\geq \int_T \log |f_{r_n}^N + b_{r_n}| dm \\ &\geq \int_{E^{\omega(n)}} \log \|f_{r_n}^N - |b_n|\| dm. \end{aligned}$$

Since $\omega \in F$, this yields for infinitely many n :

$$\begin{aligned} I(r) &\geq \int_{E^{\omega(n)}} \log \|[\lambda_n \sigma_f(r_n)]^N - M_\infty(b; r_n)\| dm \\ &\geq m\{E^{\omega(n)}\} \log [\mu(r_n)^{-N/p} - \varepsilon \mu(r_n)^{-N/p}] \\ &\geq \eta_n^{k+1} \log [(1 - \varepsilon)\mu(r_n)^{-N/p}] \quad (\text{by (3.4)}) \end{aligned}$$

whenever $r_n \leq r < 1$. Thus, infinitely often :

$$\begin{aligned} \int_{R_n} |(f^N + b)h|^{p/N} d\mu &\geq \int_{[r_n, 1)} \exp \{(p/N)I(r)\} d\mu_1(r) \\ &\geq (1 - \varepsilon)^{p/N} \mu(r_n)^{1 - \eta_n^{k+1}}. \end{aligned}$$

Recalling the definition of η_n :

$$1 - \eta_n^{k+1} = 1 - e^{-(k+1)\pi\lambda_n} \leq (k+1)\pi\lambda_n,$$

so

$$\begin{aligned} \mu(r_n)^{1 - \eta_n^{k+1}} &\geq [\mu(r_n)^{\lambda_n}]^{(k+1)\pi} \\ &\geq \{\mu(r_n)^{1/\log \mu(r_n)}\}^{-\delta(k+1)\pi} \quad (\text{by 3.2}) \\ &= e^{-\delta(k+1)\pi}. \end{aligned}$$

Thus for each $\omega \in F$:

$$\limsup_{n \rightarrow \infty} \int_{R_n} |(f^N + b)h|^{p/N} d\mu \geq (1 - \varepsilon)^{p/N} e^{-\delta(k+1)\pi} > 0,$$

hence $(f^N + b)h \notin A_\mu^{p/N}$. This completes the proof.

Deduction of Theorem 1. — In part (a) the equivalence (i) \rightarrow (ii) is immediate from Lemma B (section 2), and the implication (iii) \rightarrow (ii) is trivial. So it remains to show that (i) implies both (iii) and (b). In view of Lemma B it is enough to show that if $\sigma_f \notin L^{p+}(\mu)$, then with probability one : $Z(f)$ is not contained in any A_μ^{p+} zero set, and $Z(f + 1) \cup Z(f - 1)$ is not contained in any $A_\mu^{(p/2)+}$ zero set.

So suppose $\sigma_f \notin L^{p+}(\mu)$. We will show in a moment that this implies

$$(3.5) \quad \limsup_{r \rightarrow 1^-} \sigma_f(r)\mu(r)^{1/q} = \infty$$

for each $q > p$, which yields :

$$\limsup_{r \rightarrow 1^-} \frac{\sigma_f(r)\mu(r)^{1/q}}{-\log \mu(r)} = \infty$$

for each $q > p$. Thus Proposition 2 (with q replacing p) guarantees that for each $q > p$ it is almost sure that

$$(f^N + b)h \notin A_\mu^{q/N}$$

for every h holomorphic in Δ , b constant, and $N = 1, 2, \dots$. Since a countable intersection of sets of probability one again has probability one, it follows upon quoting the above result for a sequence $q_n \downarrow p$ that almost surely : $(f^N + b)h \notin A_\mu^{(p/N)+}$ for all b, h, N as above.

Taking $N = 1, b \equiv 0$ we see from the discussion of section 2 (a) that a.s. $Z(f)$ is contained in no A_μ^{p+} zero set, which proves (iii). Taking $N = 2$ and $b \equiv -1$ we see that a.s.

$$Z(f^2 - 1) = Z(f + 1) \cup Z(f - 1)$$

lies in no $A_\mu^{(p/2)+}$ zero set, which proves (b).

It remains only to prove that (3.5) holds for each $q > p$. Suppose not. Then for some $q > p$:

$$(3.6) \quad \sigma_f(r) = O(\mu(r)^{-1/q}) \quad (r \rightarrow 1^-).$$

Fix $p < s < q$. We will show that $\sigma_f \in L^s(\mu)$, contrary to the hypothesis on σ_f . By Lemma C we may assume that the measure μ has $\{|z| \leq 1\}$ as its closed support, hence the closed support of μ_1 is the interval $[0,1]$. Thus $\mu_1 = \gamma_1 + \gamma_2$, where γ_1 is purely atomic and γ_2 is continuous with closed support $[0,1]$. By (3.6) we have

$$\sigma_f(r) = O(\gamma_i[r,1]^{-1/q}) \quad (r \rightarrow 1 -)$$

for $i = 1,2$; hence by Lemma D,

$$\int_0^1 \sigma_f(r)^s d\gamma_i(r) < \infty \quad (i = 1,2),$$

hence $\sigma_f \in L^s(\mu)$: a contradiction. This completes the proof of Theorem 1.

4. Concluding Remarks.

Lemma B suggests that Proposition 2 should be capable of improvement.

CONJECTURE. — *If f is not a.s. in A_μ^p (hence by the zero-one law, a.s. not in A_μ^p), then a.s. $(f^N + b)h \notin A_\mu^{pN}$ for all b, h, N as in the statement of Proposition 2.*

The arithmetic-geometric mean inequality seems to give away too much to get this result : In the case $N = 1, b \equiv 0$, Fernique's inequality might be a possibility. It is not difficult to check that if $fh \in A_\mu^p$ a.s. for some fixed holomorphic h in Δ , then fh is an A_μ^p -valued Gaussian random variable. Then Fernique's inequality, the rotational symmetry of $\sigma_f(z)$, and the monotonicity of $M_p^p(h,r)$ yield :

$$\begin{aligned} \infty &> \mathcal{E} \{ \|fh\|_p^p \} \\ &= \int \mathcal{E} \{ |fh|^p \} d\mu \\ &= \int \mathcal{E} \{ |f|^p \} |h|^p d\mu \\ &= C_p^p \int \sigma_f^p |h|^p d\mu \\ &= C_p^p \int_0^1 \sigma_f(r)^p M_p^p(h;r) d\mu_1(r) \\ &\geq C_p^p \int \sigma_f^p d\mu, \end{aligned}$$

hence $f \in A_\mu^p$ a.s. But this merely shows that :

$$f \notin A_\mu^p \text{ a.s.} \Rightarrow \forall h \text{ holomorphic in } \Delta; fh \notin A_\mu^p \text{ a.s.}$$

whereas the desired result is :

$$f \notin A_\mu^p \text{ a.s.} \Rightarrow \text{a.s. : } fh \notin A_\mu^p \forall h \text{ holomorphic in } \Delta.$$

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Manuscrit reçu le 5 juin 1979.

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