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FRACTIONAL CARTESIAN PRODUCTS OF SETS
by Ron C. BLEI (*)

N-fold sums of « independent » sets serve in harmonic analysis as prototypical examples of $2N/(N + 1)$-Sidon sets, and $\Lambda(q)$ sets whose $\Lambda(q)$ constants' growth is $O(q^{N/2})$. Moreover, these features are exact: N-fold sums of independent sets are not $(2N/(N + 1) - \varepsilon)$-Sidon and to not have $\Lambda(q)$ constants' growth asymptotic to $q^{(N/2)}$, for any $\varepsilon > 0$ (see [4], [6] and [2]).

In this paper, given any number $p \in (1, 2)$, we display a set that is $p$-Sidon but not $(p - \varepsilon)$-Sidon for any $\varepsilon > 0$. The same pool of examples contains, for any number $a \in [1/2, \infty)$, a set whose $\Lambda(q)$ constants' growth is $O(q^a)$ but not $O(q^{a-\varepsilon})$ for any $\varepsilon > 0$. This answers questions raised in [4] and [6], and a question that is implicit in [2]. The type of sets displayed here exhibits « combinatorial » and « analytic » properties that one would expect « fractional » cartesian products (sums) of sets to possess, and hence the title of the paper. This class of sets naturally arises in the study of multidimensional extensions of Grothendieck's inequality ([1]); it is that study that led to the present work.

1. Definitions and main results.

We employ basic notation and facts of commutative harmonic analysis as presented and followed in [10]. $\Gamma$ will be a countable discrete abelian group and $G = \Gamma$ will denote its compact dual group. Throughout, group operations in $\Gamma$ and $G$ will be designated by multiplicative notation.

We now define the type of sets that is the object of the present study. Let $J \geq K > 0$ be arbitrary integers, and

$$\mathcal{F} = \{1, \ldots, J\}.$$  

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For the sake of typographical convenience here and throughout the paper we let \( N = \binom{J}{K} \). Let

\[ \{S_1, \ldots, S_N\} \]

be the collection of all \( K \)-subsets of \( \mathcal{F} \) (sets containing \( K \) elements of \( \mathcal{F} \)), where each \( S_a \subset \mathcal{F} \) is enumerated as

\[ S_a = (\alpha_1, \ldots, \alpha_K). \]

Let \( \{P_1, \ldots, P_N\} \) be the collection of projections from \( (\mathbb{Z}^+)^J \) into \( (\mathbb{Z}^+)^K \) defined as follows: For \( 1 \leq \alpha \leq N \), and \( j = (j_1, \ldots, j_J) \in (\mathbb{Z}^+)^J \)

\[ P_a(j) = (j_{\alpha_1}, \ldots, j_{\alpha_K}). \]

Next, let \( F \subset \Gamma \) and

\[ F = \{\gamma_1\}_{\in (\mathbb{Z}^+)^K} \]

be a \( K \)-fold enumeration of \( F \). Finally, define

\begin{align*}
(1.1) \quad F_{j,K} &= \{(\gamma_{P_1(j)}, \ldots, \gamma_{P_N(j)})\}_{j \in (\mathbb{Z}^+)^J} \subset F^N \subset \Gamma^N.
\end{align*}

Throughout this paper, a set that is subscripted by \( J, K \) will denote the set defined by (1.1), for some fixed \( K \)-fold enumeration of \( F \).

**Definition 1.1.** Let \( M > 0 \) be a fixed integer. \( F = \{\gamma_j\}_{j=1}^\infty \subset \Gamma \) is \( M \)-independent if for any \( L, L' > 0 \) the relation

\[ \prod_{j=1}^L \gamma_j^\prime = \prod_{j=1}^{L'} \gamma_j^\prime, \]

where the \( \lambda_j \) 's and \( v_j \) 's are integers in \( [-M,M] \), implies that \( L = L' \) and \( \lambda_j = v_j \) for \( j = 1, \ldots, L \). If \( F \) is \( M \)-independent for every \( M \), then \( F \) is said to be independent. \( 1 \)-independent sets are referred to as dissociate sets.

**Definition 1.2** (1.6.2 and 1.6.3 in [9]). Let \( F = \{\gamma_j\}_{j=1}^\infty \subset \Gamma, s \in \mathbb{Z}^+ \) and \( \gamma \in \Gamma \). Writing (formally) the Fourier series \( h \sim \left( \sum_{j=1}^\infty \gamma_j \right) \), we define

\[ r_s(F, \gamma) = \hat{h}(\gamma). \]

Equivalently, \( r_s(F, \gamma) \) is the number of ways to write \( \gamma \) in the form of

\begin{align*}
(1.2) \quad \gamma = \gamma_{i_1} \cdots \gamma_{i_s},
\end{align*}

where \( \gamma_{i_1}, \ldots, \gamma_{i_s} \) are (not necessarily distinct) \( s \) elements in \( F \), and where different permutations on the right hand side of (1.2) are counted as different representations.
For example, it is easy to see that if $F \subset \Gamma$ is independent then, for all $s > 0$, the $J$-fold cartesian product of $F$ satisfies

\begin{equation}
\sup_{\gamma \in F^J} r_s(F_{J,1}, \gamma) = (s !)^J, \quad \text{for all } s \in \mathbb{Z}^+.
\end{equation}

The following is an extension to (1.3) and is evidence that $F_{J,K}$ could be viewed as a $J/K$-fold cartesian product of $F$.

**Theorem 1.3.** — Let $F \subset \Gamma$ be an independent set. For all $J \geq K > 0$ and $s > 0$,

\begin{equation}
\left(\frac{k}{J/K}\right)^s \sup_{\gamma \in F^J} r_s(F_{J,K}, \gamma) \leq 2^J (s !)^J.
\end{equation}

We now list the « analytic » results that are based on the above « fractional cartesian products ». For $F \subset \Gamma$, $C_F(G)$ and $L^p(F)$, $1 \leq p \leq \infty$, will be the spaces of functions in $C(G)$ and $L^p(G)$, respectively, whose spectra lie in $F$. Recall that for $2 < q < \infty$ $F$ is a $(q)$ set if there is $A > 0$ so that for all $f \in L^q(G)$

\begin{equation}
A \|f\|_2 \geq \|f\|_q.
\end{equation}

The « smallest » $A$ for which (1.5) holds is the $(q)$ constant of $F$ and is denoted by $A(q,F)$. Of particular interest are sets $F \subset \Gamma$ for which $A(q,F)$ is $O(q^a)$ for some $a \geq 1/2$. In fact, this growth condition can be neatly understood as follows : $A(q,F)$ is $O(q^a)$ if and only if every $f \in L^q(G)$ also satisfies

\begin{equation}
\exp \left(\frac{\lambda \|f\|^{1/q}}{A(q,F)}\right) < \infty, \quad \text{for all } \lambda > 0
\end{equation}

(see Remarque on p. 350 of [2]).

**Definition 1.4.** — Let $\beta \in [1, \infty)$. $F \subset \Gamma$ is a $\Lambda^\beta$ set if $A(q,F)$ is $O(q^{\beta/2})$. $F$ is said to be exactly $\Lambda^\beta$ when $F$ is $\Lambda^a$ if and only if $a \in [\beta, \infty)$. $F$ is said to be exactly non-$\Lambda^\beta$ when $F$ is $\Lambda^a$ if and only if $a \in (\beta, \infty)$.

**Definition 1.5.** — Let $p \in [1,2)$. $F \subset \Gamma$ is a $p$-Sidon set if there is $D > 0$ so that for all $f \in C_F(G)$

\begin{equation}
D \|f\|_\infty \geq \|f\|_p.
\end{equation}

The « smallest » $D$ for which (1.8) holds is the $p$-Sidon constant of $F$ and is denoted by $D(p,F)$. $F$ is exactly $p$-Sidon when $F$ is $r$-Sidon if and only if $r \in [p,2)$. $F$ is exactly non-$p$-Sidon when $F$ is $r$-Sidon if and only if $r \in (p,2)$. \]
J-fold cartesian products of dissociate sets are the classical examples of sets that are exactly $A^J$. These and other similar constructions of sets which are exactly $A^J$ are studied extensively in [2]. The same J-fold products are also the simplest examples of sets which are exactly $2J/(J+1)$-Sidon ([7], [4] and [6]). The gaps left open between the J and $(J+1)$-fold products of dissociate sets are filled by the « fractional cartesian products » that were defined at the outset.

**Theorem 1.6.** - Let $F \subset \Gamma$ be an independent set.

a) $F_{i,K} \subset \Gamma^N$ is exactly $A^i/K$. Moreover, there is $\eta_{i,K} > 0$ so that for all $q > 2$

$$\eta_{i,K} q^{1/k} \leq A(q,F_{i,K}) \leq q^{1/2k}. \quad (1.9)$$

b) $F_{i,K}$ is exactly $2J/(K+J)$-Sidon. Moreover,

$$D\left(\frac{2J}{(K+J)}, F_{i,K}\right) \leq 2^{i/k}. \quad (1.10)$$

(Recall that $N = \binom{J}{K}$.)

Constructions analogous to (1.1) can be carried out within $\Gamma$ by replacing the cartesian product operation with the group operation in $\Gamma$. Given (an infinite set) $F \subset \Gamma$, let $\mathcal{P}$ be a N-partition of $F$:

$$\mathcal{P} = \{F_1, \ldots, F_N\}$$

(that is, $F_1, \ldots, F_N \subset \Gamma$ are mutually disjoint sets whose union is $F$), where each $F_\alpha$ is infinite. For each $1 \leq \alpha \leq N$, endow $F_\alpha$ with a K-fold enumeration $F_\alpha = \{\gamma_\alpha^{(i)}\}_{i \in \{Z^+\}^K}$.

Define

$$F_{\mathcal{P}}^{i,K} = \{\gamma_{\mathcal{P}(0)}^{(i)} \ldots \gamma_{\mathcal{P}(N)}^{(i)}\}_{i \in \{Z^+\}^K} \subset \Gamma. \quad (1.11)$$

In this work, a set that is superscripted by $J$, $K$ and subscripted by $\mathcal{P}$ will be the set defined in (1.11), where each member of $\mathcal{P}$ is understood to have a K-fold enumeration. When $\mathcal{P}$ is fixed and understood from the context, we write $F_{\mathcal{P}}^{J,K}$ for $F_{\mathcal{P}}^{J,K}$. Observe that, for $F \subset \Gamma$, letting

$$F_\alpha = (1, \ldots, 1, F, 1, \ldots) \subset \Gamma^N$$

$\alpha$-th coordinate
and

\[ F = \bigcup_{a=1}^{N} F_a, \]

we have

\[ F_{j,K} = F_{j,K}^{1,K} \subseteq \Gamma^N \]

where

\[ \mathcal{P} = \{F_1, \ldots, F_N\}. \]

**Corollary 1.7.** Let \( F \subseteq \Gamma \) be a dissociate set and \( \mathcal{P} \) be any \( \mathbb{N} \)-partition of \( F \).

a) \( F_{j,K}^{1,K} \subseteq \Gamma \) is exactly \( \Lambda^{1/K} \). Moreover, there are constants \( \eta_{j,K}, \xi_{j,K} > 0 \) so that for all \( q > 2 \)

\[ (1.12) \quad \eta_{j,K} q^{1/2K} \leq A(q, F_{j,K}^{1,K}) \leq \xi_{j,K} q^{1/2K}. \]

b) \( F_{j,K}^{1,K} \) is exactly \( 2J/(K+J) \)-Sidon.

c) Suppose that \( \Gamma \) contains element with arbitrarily large order. Then, for every \( \beta \in [1,\infty) \) there are \( F_1, F_2 \subseteq \Gamma \) so that \( F_1 \) is exactly \( \Lambda^\beta \) and \( F_2 \) is exactly non-\( \Lambda^\beta \).

d) Let \( \Gamma \) be any discrete abelian group. Then, for every \( p \in [1,2) \) there are \( F_1, F_2 \subseteq \Gamma \) so that \( F_1 \) is exactly \( p \)-Sidon and \( F_2 \) is exactly non-\( p \)-Sidon.

The organization of the paper is as follows. In section 2, we prove the right hand inequality of (1.4) in Theorem 1.3. In section 3, fitted for \( F_{j,K}^{1,K} \), appropriate Riesz products are developed for use in later sections. The \( \Lambda^\beta \) property is treated in section 4 where Theorem 1.6 (a) and Corollary 1.7 (a), (c) are proved. The left hand inequality of (1.9) in Theorem 1.6 is then used to establish the left hand inequality of (1.4) in Theorem 1.3. \( p \)-Sidonicity is treated in section 5, where the remaining parts of Theorem 1.6 and Corollary 1.7 are proved. We conclude in section 6 with some problems.

**2. A combinatorial property of \( F_{j,K} \).**

Let \( F \subseteq \Gamma \) be an independent set. We prove here the right hand inequality of (1.4) : For all \( s \in \mathbb{Z}^+ \) and \( \gamma \in \Gamma^N \)

\[ (2.1) \quad r_s(F_{j,K}; \gamma) \leq s^{1/K}. \]
We shall use an extension of Hölder's inequality which, to facilitate referencing here and in section 5, we state below.

2.1. \textit{M-Hölder's inequality}. — Let $X$ be a measure space, $M > 1$ be an arbitrary integer and $1 < p_1 < \cdots < p_M < \infty$ be so that

$$
\sum_{i=1}^{M} \frac{1}{p_i} = 1.
$$

Then, for any $f_1, \ldots, f_M$, measurable functions on $X$,

$$
\left| \int_X f_1, \ldots, f_M \right| \leq \|f_1\|_{p_1}, \ldots, \|f_M\|_{p_M}.
$$

For typographical convenience, let

$$
N = \binom{J}{K} \quad \text{and} \quad N_1 = \binom{J-1}{K-1}.
$$

As usual, $l^{n_1}((\mathbb{Z}^+)^K)$ denotes all functions $x$ on $(\mathbb{Z}^+)^K$ so that

$$
\|x\|_{N_1} = \left( \sum_{i_1, \ldots, i_K} |x(i_1, \ldots, i_K)|^{n_1} \right)^{1/n_1} < \infty.
$$

\textbf{Lemma 2.2.} — Let $x_1, \ldots, x_N \in l^{n_1}((\mathbb{Z}^+)^K)$. Then,

$$
\left| \sum_{j \in (\mathbb{Z}^+)^J} x_1(P_1(j)) \ldots x_N(P_N(j)) \right| \leq \|x_1\|_{N_1} \ldots \|x_N\|_{N_1}.
$$

\textit{Sketch of proof.} — The key observation is that each $k \in \{1, \ldots, J\}$ appears in precisely $N_1 = \binom{J-1}{K-1}$ distinct $K$-subsets of $\{1, \ldots, J\}$. $J$ successive applications of the $N$-Hölder inequality with $p_1 = \cdots = p_{N_1} = 1/N_1$ yield the desired inequality.

\textit{Remark.} — Another form of Lemma 2.2 was used in [1], where it served as a starting point for the study of the so-called « projectively bounded » multilinear forms on a Hilbert space (Lemma 1.2 of [1]).

\textit{Proof of (2.1).} — First, by virtue of the canonical correspondence between $(\mathbb{Z}^+)^J$ and $F_{j,K}$, for notational simplicity we designate elements of $F_{j,K}$ as follows:

$$(\mathbb{Z}^+)^J \ni j \leftrightarrow \gamma(j) \in F_{j,K},$$
where
\[ \gamma(j) = (\gamma_{P_1(j)}, \ldots, \gamma_{P_N(j)}) \].

Let \( \gamma(j_1), \ldots, \gamma(j_s) \) be arbitrary elements of \( F_{J,K} \) and write
\[ \gamma = \gamma(j_1) \ldots \gamma(j_s) \]
that is,
\[ \gamma = \left( \prod_{k=1}^s \gamma_{P_k(j)}, \ldots, \prod_{k=1}^s \gamma_{P_N(j)} \right) \]
(Recall that \( j_1, \ldots, j_s \) need not be distinct.)

Let
\[ L_1 = \{P_1(j_1), \ldots, P_1(j_s)\} \subset (Z^+)^K \]
\[ \vdots \]
\[ L_N = \{P_N(j_1), \ldots, P_N(j_s)\} \subset (Z^+)^K. \]

Next, define
\[ V = \{j \in (Z^+)^J : P_\alpha(j) \in L_\alpha \text{ for all } 1 \leq \alpha \leq N \}. \]

By the independence of \( F \), the only way that \( \gamma \) can be obtained as a product of \( s \) elements from \( F_{J,K} \) is for these elements to have in their 1st, \ldots, \( N^{th} \) coordinates the members of \( F \) that appear in the 1st, \ldots, \( N^{th} \) coordinates of (2.2'), respectively. That is, if \( j_1, \ldots, j_s \in (Z^+)^J \) are so that
\[ \gamma = \gamma(j_1), \ldots, \gamma(j_s), \]
then \( j_1, \ldots, j_s \in V. \) Therefore, the game plan is to estimate \( |V| \), the «volume» of \( V \), and exploit the fact that
\[ r_s(F_{J,K}, \gamma) \leq |V|^s. \]

Let \( \chi_1, \ldots, \chi_N \) be the characteristic functions of \( L_1, \ldots, L_N \) in \((Z^+)^K\). Clearly,
\[ |V| = \sum_{j \in Z^+} \chi_1(P_1(j)) \ldots \chi_N(P_N(j)). \]

By Lemma 2.2,
\[ |V| \leq \|\chi_1\|_{N_1} \ldots \|\chi_N\|_{N_1} \leq s^{N/N_1} = s^{1/K}. \]

Combining (2.4) and (2.3), we obtain (2.1).
3. Riesz products for $F_{j,k}$.

Let $F \subseteq \Gamma$ be a dissociate set. Let $F_{j,k}' = F_{j,k}'$ be defined by (1.11) where $\mathcal{P}$ is an arbitrary $N$-partition of $F$ (as usual, our convention is that $N = \binom{J}{K}$). For any $\gamma \in \Gamma$ and $\theta \in \mathbb{R}$, define
\[
\cos(\gamma + \theta) = \frac{e^{i\theta \gamma} + e^{-i\theta \gamma}}{2}.
\]
Next, let $\varphi_1, \ldots, \varphi_N \in L^\infty((\mathbb{Z}^+)^k)$ be so that
\[
\|\varphi_1\|_\infty, \ldots, \|\varphi_N\|_\infty \leq 1,
\]
and write, for $1 \leq \alpha \leq N$,
\[
\varphi_\alpha(k) = |\varphi_\alpha(k)|e^{i\theta_\alpha(k)}, \quad k \in (\mathbb{Z}^+)^k.
\]
We now consider the following Riesz product:
\[
(3.1) \quad \mu \sim \left[ \prod_{j \in (\mathbb{Z}^+)^k} (1 + |\varphi_1(j)|\cos(\gamma_1(j) + \theta_1(j))) \right] \ldots \left[ \prod_{j \in (\mathbb{Z}^+)^k} (1 + |\varphi_N(j)|\cos(\gamma_N(j) + \theta_N(j))) \right].
\]
As usual, $\|\mu\| = 1$ and the spectral analysis of $\mu$ yields the following.

**Lemma 3.1.** Let $\varphi_1, \ldots, \varphi_N \in L^\infty((\mathbb{Z}^+)^k)$, $\|\varphi_1\|_\infty, \ldots, \|\varphi_N\|_\infty \leq 1$. Then, there is $\mu \in M(G)$ so that
\[
(3.2) \quad \|\mu\| \leq 2^N,
\]
and
\[
(3.3) \quad \hat{\mu}(\gamma_{P_1,j}^{(1)} \ldots \gamma_{P_N,j}^{(N)}) = \varphi_1(P_1(j)) \ldots \varphi_N(P_N(j)),
\]
for all $j \in (\mathbb{Z}^+)^k$.

If a higher degree of independence is assumed for $F \subseteq \Gamma$, then the norm estimate in (3.2) can be correspondingly improved. We illustrate this in $\Gamma = \mathbb{Z}$. Let $M > 1$ and $F = \bigcup_{z=1}^N F_z \subseteq \mathbb{Z}$ be an $M$-independent set where
\[
F_z = \{ \lambda_k^{(z)} \}_{k \in (\mathbb{Z}^+)^k}
\]
(as usual, the F\textsubscript{s}'s are infinite and mutually disjoint). Let K\textsubscript{M} \in L\textsuperscript{1}(T) be the M\textsuperscript{th} Fejer kernel:

\[ K\textsubscript{M}(t) = \sum_{l=-M}^{M} \left(1 - \frac{|l|}{M+1}\right)e^{ilt}. \]

Let \( \varphi_1, \ldots, \varphi_N \) be unimodular functions in \( l^\infty((\mathbb{Z}^+)^K) \) given by

\[ \varphi_j(k) = e^{i\theta_j(k)} \]

for \( k \in (\mathbb{Z}^+)^K \). Replacing (3.1) by

\[ H - f \mathfrak{R}_M \left( ^{+6}_1, \ldots, ^{+9}_1 \right), \]

we obtain

**Lemma 3.2.** Let \( \varphi_1, \ldots, \varphi_N \in l^\infty((\mathbb{Z}^+)^K) \) be given by (3.4). Then, there is \( \mu \in M(T) \) so that

\[ ||\mu|| \leq (1 + 1/M)^N \]

and

\[ \hat{\mu}(\lambda_{F_1}^{(1)} + \cdots + \lambda_{F_N}^{(N)}) = \varphi_1(P_1(j)) \ldots \varphi_N(P_N(j)), \]

for all \( j \in (\mathbb{Z}^+)^J \).

**Definition 3.3.** \( F_1, F_2 \subset \Gamma \) are said to be harmonically separated if there is \( \mu \in M(G) \) so that

\[ \hat{\mu} = \begin{cases} 1 & \text{on } F_1 \\ 0 & \text{on } F_2 \setminus F_1 \end{cases}. \]

**Lemma 3.4.** Let \( J > K > 0, \ J' > K' > 0, \) and \( N = \binom{J}{K}, N' = \binom{J'}{K'} \). Let \( F \subset \Gamma \) be dissociate, and \( \mathcal{P}, \mathcal{P}' \) be N,N'-partitions of F, respectively. Then \( F^{JK}_{\mathcal{P}} \) and \( F^{J'K'}_{\mathcal{P}'} \) are harmonically separated.

**Proof.** If \( N \neq N' \), then an application (whose details are left to the reader) of a Riesz product such as the one given by (3.1) yields the desired conclusion. Assume that \( N = N' \), and let

\[ \mathcal{P} = \{F_1, \ldots, F_N\}. \]
be an $N$-partition for $F$, where, as usual, $F_x$ is enumerated

$$F_x = \{ \gamma_i^{(x)} \}_{i \in (Z^+)^K}.$$ 

Let $k = \binom{J-1}{K-1}$. $\otimes Z_k = \Omega(k)$ will denote the compact direct product of $Z_k$, and $\otimes Z_k = \hat{\Omega}(k)$ will be its (discrete) dual group, the direct sum of $Z_k$.

$E(k) = \{ r_n^{(k)} \}_{n=1}^{\infty}$ will be the system of $k$-Rademacher functions realized as characters in $\hat{\Omega}(k)$; The $n^{th}$ $k$-Rademacher function $r_n^{(k)}$ is defined by

$$r_n^{(k)}(\omega) = \exp\left(2\pi i \omega(n)/k\right),$$

for all $\omega \in \Omega(k) = \{(\omega(j))_{j=1}^\infty : \omega(j) \in \{0, 1, \ldots, k-1\}\}$. Observe that $E(k)$ is $(k-1)$-independent and that for all $r \in E(k)$, $r^m = 1$ iff $m \equiv 0 \pmod{k}$. In the sequel below, $k$ is fixed and, for the sake of simplicity, will be omitted from all superscripts and subscripts. As usual, for $\gamma \in \Gamma$ and $\theta \in R$, write

$$\cos(\gamma + \theta) = (e^{i\theta} + e^{-i\theta})/2.$$ 

For each $\omega = (\omega_1, \ldots, \omega_J) \in \Omega^J$ define the Riesz product

$$\mu_\omega \sim \left\{ \prod_{i \in (Z^+)^K} \left[ 1 + \cos \left( \gamma_i^{(1)} + 2\pi \sum_{m=0}^{K} \omega_1(m) \right) \right] \right\} \cdots$$

$$\cdots \left\{ \prod_{i \in (Z^+)^K} \left[ 1 + \cos \left( \gamma_i^{(N)} + 2\pi \sum_{m=1}^{K} \omega_N(m) \right) \right] \right\},$$

where $i = (i(1), \ldots, i(K)) \in (Z^+)^K$, and (see Section 1)

$$S_\alpha = (x_1, \ldots, x_K) \subset [1, \ldots, J],$$

for $1 \leq \alpha \leq N$. Next, we integrate over $\Omega^J$ the $M(G)$-valued function whose value at $\omega \in \Omega^J$ is $\mu_\omega$:

$$\mu = \int_{\Omega^J} \mu_\omega \, d\omega \in M(G).$$

The spectral analysis of $\mu$ yields the following:

$$\hat{\mu}(\gamma_1^{(1)} \ldots \gamma_N^{(N)}) = 2^{-N} \int_{\Omega^J} \left[ \prod_{m=1}^{K} r_{i_1(m)}(\omega_1(m)) \right] \cdots$$

$$\cdots \left[ \prod_{m=1}^{K} r_{i_N(m)}(\omega_N(m)) \right] d\omega_1 \ldots d\omega_J,$$

for all $i_1, \ldots, i_N \in (Z^+)^K$. 

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Since every \( I \in \{1, \ldots, J\} \) appears in precisely \( k = \binom{J-1}{K-1} \) \( S_z \)'s, and
\[
\left\{ \begin{array}{ll}
r^m = 1 & \text{if } m \equiv 0 \pmod{k} \\
0 & \text{otherwise}
\end{array} \right., \quad r \in E^{(k)}
\]
we obtain from (3.4.2) that
\[
\hat{\mu}(\gamma^{(1)}_n \ldots \gamma^{(N)}_n) = \left\{ \begin{array}{ll}
2^{-N} & \text{if there is } j \in (Z^+)^J \text{ so that } P_a(j) = i_z \text{ for all } 1 \leq \alpha \leq N; \\
0 & \text{otherwise}.
\end{array} \right.
\]

4. The \( \Lambda^\theta \) property.

Proof of Theorem 1.6, part (a). — The right hand inequality in (1.9) follows immediately from (2.1) via the following.

Lemma 4.1. (Théorème 3 in [2]). — Let \( s \) be a positive integer. Then, for \( F \subset \Gamma \),
\[
A(2s,F) \leq \left[ \sup_{\gamma \in F} r_\gamma(F,\gamma) \right]^{1/2s}.
\]
We now prove that there is \( \eta_{l,K} > 0 \) so that
\[
\eta_{l,K} q^{1/2k} \leq A(q,F_{l,K}),
\]
where \( F \subset \Gamma \) is independent (the idea for the argument that follows originates — as far as we can determine — in [5]). Let \( n > 0 \) be arbitrary, and denote
\[
V_n = \{ j = (j_1, \ldots, j_J) \in (Z^+)^J : 1 \leq j_1, \ldots, j_J \leq n \}.
\]
Let \( g \) be the trigonometric polynomial defined by
\[
g = \sum_{j \in V} (Y_{P,j_1} \ldots Y_{P,j_J})
\]
\[
\left( \text{as always, } \ N = \binom{J}{K} \right).
\]
Clearly,
\[
(4.1) \quad ||g||_2 = n^{1/2}.
\]
Next, let \( h \) be the Riesz product defined by
\[
h(t_1, \ldots, t_N) = \prod_{i \in U_n} (1 + \cos \gamma_i(t_i)) \prod_{i \in U_n} (1 + \cos \gamma_i(t_n)),
\]
where \( U_n = \{i = (i_1, \ldots, i_K) \in (\mathbb{Z}^*)^K : 1 \leq i_1, \ldots, i_K \leq n\} \),
\( \cos \gamma = (\gamma + \bar{\gamma})/2 \), and \( t_1, \ldots, t_N \in G \). We observe that

\[
(4.2) \quad ||h||_1 = 1,
\]
and

\[
(4.3) \quad ||h||_2 \leq ||h||_\infty \leq 2^{N^n}.
\]
Combining (4.2) and (4.3), we obtain for any \( 1 < R < 2 \)

\[
(4.4) \quad ||h||_R \leq 2^{N^n_{R+1}}.
\]
\( (1/R + 1/R' = 1) \).

Also, note that

\[
(4.5) \quad \hat{h} = 1/2^N \quad \text{on} \quad \{(\gamma_{F_{\mu}(\nu)}, \ldots, \gamma_{F_{\mu}(\nu)})\}_{\nu \in V_n}.
\]

Letting \( R' = n^k \) and applying (4.4) and (4.5), we deduce

\[
2^{-N}n^\delta = g \ast h(0) \leq ||g||_{n^k} ||h||_{n^k},
\]
\[
\leq ||g||_{n^k} \cdot 2^{N}.
\]
Therefore (from (4.1)),

\[
(4.6) \quad 2^{-2N} ||g||_2 n^{\delta/2} \leq ||g||_{n^k}.
\]
\( n \) was arbitrary, and the left hand inequality of (1.9) follows. \( \square \)

Completion of the proof of Theorem 1.3. — The left hand inequality in (1.4) follows from Lemma 4.1 and (4.6). \( \square \)

Exploiting (1.9) of Theorem 1.6 via an « averaging » procedure, we now prove part (a) of Corollary 1.7:

Let \( F \subset \Gamma \) be a dissociate set, and \( F_{\mu}^{k} = F^{k} \) be defined by (1.11). Let \( f \in L^{2}_{F_{\mu}^{k}}(G) \) be given by

\[
f \sim \sum_{\mu \in \mathbb{Z}^{*}} a_{\gamma}^{(1)}(u_{0}) \ldots \gamma_{F_{\mu}(\nu)}^{(N)}.
\]
We select \( E = \{e_{i}\}_{i \in \mathbb{Z}^{*}} \), an infinite independent set in some \( \Gamma_0 (= \hat{G}_0) \),
and «randomize» the Fourier coefficients of $f$ as follows. Let $t = (t_1, \ldots, t_N) \in G_0^N$, and

$$f_t \sim \sum_{j \in \mathbb{Z}^N} a e_{P,j}(t_1) \cdots e_{P,j}(t_N) \gamma_{P,j}^{(1)} \cdots \gamma_{P,j}^{(N)}.$$ 

For each $t = (t_1, \ldots, t_N) \in G_0^N$, let $\mu_t \in M(G)$ be so that

$$\hat{\mu}(\gamma_{P,j}^{(1)} \cdots \gamma_{P,j}^{(N)}) = e_{P,j}(t_1) \cdots e_{P,j}(t_N),$$

and

$$(4.7) \quad ||\mu_t|| \leq 2^N$$

(Lemma 3.1).

Observe that

$$f = f_t \ast \mu_t.$$ 

Therefore, for all $t \in G_0^N$ and $q > 2$,

$$(4.8) \quad ||f||_q^q \leq ||f_t||_q^q ||\mu_t||^q \leq 2^{Nq} ||f_t||_q^q.$$ 

It follows from (4.8) that

$$||f||_q^q \leq 2^{Nq} \int_{G_0^N} \left( \int_G |f_t|^q \right) dt.$$ 

Interchanging the order of integration, and applying the right hand inequality of (1.9), we deduce that

$$(4.9) \quad ||f||_q \leq 2^{Nq} A(q, E_{j,k}) ||f||_2 \leq 2^{Nq^{1/2k}} ||f||_2.$$ 

(4.9) proves the right hand inequality in (1.12). To show the left hand inequality, we follow a computation identical to the one used in showing the left hand inequality of (1.9). The proof of 1.7 (a) is complete. $\square$

We assume now that $\Gamma$ is such that for every $M \geq 1$, $\Gamma$ contains an infinite $M$-independent set. In fact, for concreteness’ sake, assume $\Gamma = \mathbb{Z}$.

**Lemma 4.3.** Let $J > K > 0$ be arbitrary. There exists $F \subset \mathbb{Z}$ so that for any $\mathcal{P}$, an $N$-partition of $F$, $\eta_{j,k} q^{1/2k} \leq A(q, F_{\mathcal{P}}^{j,k}) \leq 4q^{1/2k}$, for some $\eta_{j,k} > 0$ and all $q > 2$.

**Proof.** Let $M > 0$ be so that

$$[(M+1)/M]^N \leq 4$$
\( (N = \left( \begin{array}{c} J \\ K \end{array} \right) ) \), and let \( F \subset Z \) be an \( M \)-independent set. Run through the argument leading to (4.9), where an application of Lemma 3.2 replaces that of 3.1. The result is an improved estimate (over (4.9)):

\[
A(q, F)^{J^K} \leq 4q^{J^K}.
\]

**Proof of Corollary 1.7, part (c).** — Again, without loss of generality, we shall work in \( Z \). Let \( \beta \in [1, \infty) \) be arbitrary, and \( J_n > K_n > 0 \), \( n = 1, \ldots, \) be so that \( (J_n/K_n)^{\infty}_{n=1} \) is a monotonically increasing sequence converging to \( \beta \). Predictably, the strategy is to let \( E_n \subset Z \) be the « fractional » sum corresponding to \( J_n > K_n > 0 \) given by Lemma 4.3, and then select finite sets \( F_n \subset E_n \) with the following properties:

(i) \( A\left( q, \bigcup_{k=1}^{\infty} l_k F_k \right) \leq 8q^{\beta/2} \),

where \( l_{k+1} \gg l_k > 0 \) are chosen appropriately

\[
(l_nF_n = \{ l_n \lambda_j^{(n)} : \lambda_j^{(n)} \in F_n \});
\]

(ii) the \( F_n \)'s are sufficiently « thick » so that \( \bigcup_{k=1}^{\infty} l_k F_k \) is not \( \Lambda^{\beta-\epsilon} \) for any \( \epsilon > 0 \).

We start with

\[
E_n = \{ \lambda_{P(0)}^{(1,n)} + \cdots + \lambda_{P(0)}^{(N_n,n)} \}_{j \in Z^{+} \setminus \{0\}},
\]

a « fractional » sum corresponding to \( J_n > K_n > 0 \) as in Lemma 4.3

\( \left( N_n = \left( \begin{array}{c} J_n \\ K_n \end{array} \right) \right) \). Next, fix a positive sequence \( (\epsilon_n)^{\infty}_{n=1} \) that is monotonically converging to 0. For each \( n > 0 \) select \( L_n \in Z^+ \) so that

\[
(4.10) \quad L_n^{c, K_n} \geq n16^{N_n}.
\]

Let

\[
F_n = \{ \lambda_{P(0)}^{(1,n)} + \cdots + \lambda_{P(0)}^{(N_n,n)} \}_{j \in V_{L_n}} \subset E_n,
\]

where

\[
V_{L_n} = \{ j = (j_1, \ldots, j_n) \in (Z^{+})^n : 1 \leq j_2 \leq L_n \}.
\]

Observe that by running through an argument identical to the one that led to (4.6), we obtain

\[
g_n \in L_{L_n}^2(T)
\]
so that

\[(4.11) \quad \|g_n\|_{L^p_nK_n} \geq 4^{-N_n(L_n)^{1/2}}\|g_n\|_2.\]

We now determine by induction that « dilation » factors of the $F_n$'s ($l_n$ in (i) above): Let $l_1 = 1$, and suppose that $l_1, \ldots, l_k$ were determined so that

\[(4.12) \quad A\left(q, \bigcup_{n=1}^k l_nF_n\right) \leq 8q^{1/2M_k},\]

for all $2 < q < \infty$. Observe that the cardinality of $\bigcup_{n=1}^{k+1} (L_n)^{1/p}$, and, therefore, however we choose $l_{k+1}$, we will have

\[(4.13) \quad A\left(q, \bigcup_{n=1}^{k+1} l_nF_n\right) \leq \left[\sum_{n=1}^{k+1} (L_n)^{1/p}\right]^{1/2},\]

for all $2 < q < \infty$. Guided by (4.13), we choose $M_k$ so that

\[(4.14) \quad \left[\sum_{n=1}^{k+1} (L_n)^{1/p}\right]^{1/2} \leq 8M_k^{1/2M_{k+1}}.\]

Finally, select $l_{k+1}$ so that

\[(4.15) \quad (\lambda_1 \pm \cdots \pm \lambda_r) + (v_1 \pm \cdots \pm v_r) \neq 0\]

for all $\{\lambda_i\}_{i=1}^r \subseteq \bigcup_{n=1}^k l_nF_n$ and $\{v_j\}_{j=1}^{r'} \subseteq l_{k+1}F_{k+1}$, where $r, r' \leq 2M_k$ (we allow repetitions in $\{\lambda_i\}_{i=1}^r$ and $\{v_j\}_{j=1}^{r'}$).

Claim. $A\left(q, \bigcup_{n=1}^{k+1} l_nF_n\right) \leq 8q^{k+1/2M_{k+1}}$, for all $2 < q < \infty$.

Let $f$ be a trigonometric polynomial with spectrum in $\bigcup_{n=1}^{k+1} l_nF_n$ and write $f = f_1 + f_2$ where

\[
\text{spectrum} (f_1) \subseteq \bigcup_{n=1}^k l_nF_n, \\
\text{spectrum} (f_2) \subseteq l_{k+1}F_{k+1}.
\]

It follows from (4.15) that

\[(4.16) \quad \|f_1 + f_2\|_{L^m}^2 = \|f_1\|^2 + \|f_2\|^2,\]
for all \( m \leq M_n^2 \). Therefore, it follows from (4.16) that
\[
\|f_1 + f_2\|_{2m}^2 \leq \|f_1\|_{2m}^2 + \|f_2\|_{2m}^2.
\]
Applying the induction hypothesis, and the monotonicity of \((J_n/K_n)^{\infty}_{n=1}\), we obtain
\[
\|f_1 + f_2\|_{2m} \leq 8(2m^{k+1/2}K_{k+1})\|f_1 + f_2\|_{2}.
\]
Combining (4.14) and (4.17), we obtain the claim. Combining the « claim », (4.11) and (4.10), we obtain that \( \bigcup_{n=1}^{\infty} I_n F_n \) is exactly \( \Lambda^B \).

By choosing \( J_n > K_n > 0 \), where \((J_n/K_n)^{\infty}_{n=1}\) is a monotonically decreasing sequence converging to \( \beta \), and following a procedure similar to the above, we obtain a set in \( \mathbb{Z} \) which is exactly non-\( \Lambda^B \). The details are left to the reader.

\[ \square \]

5. The \( p \)-Sidonicity property.

Proof of Theorem 1.6, part (b). — First, for the sake of economy in notation, we adopt the following conventions: Let
\[
U = \{i_1, \ldots, i_M\} \subseteq \{1, \ldots, J\} = F, \quad \text{and} \quad P_U : (\mathbb{Z}^+)^J \rightarrow (\mathbb{Z}^+)^M
\]
be the projection defined by
\[
P_U((j_1, \ldots, j_J)) = (j_{i_1}, \ldots, j_{i_M}).
\]
\( \sum_U \) will denote summation over \( P_U((\mathbb{Z}^+)^J) \). For example,
\[
\sum_U a_{i_1, \ldots, i_M} = \sum_{j_{i_1}, \ldots, j_{i_M}} a_{j_1, \ldots, j_J}.
\]
Also, in what follows \( \gamma_{S_j} \) will mean \( \gamma_{P_j(0)} \) whenever these occur in summands. For example,
\[
\sum_{j=0}^{J_{\infty}} \gamma_{P_{S_j}} = \sum_{j_1, \ldots, j_J} \gamma_{S_{j_1}} \cdots \gamma_{S_{j_J}}.
\]
Let \( E \subseteq \Gamma \) be an independent set, and \( E_{1,K} \subseteq \Gamma^N \) be as in (1.1) (as usual, \( N = \binom{J}{K} \)).
LEMMA 5.1. — Let $f$ be a trigonometric polynomial in $C_{E_{j,k}}(G^N)$ given by

$$f = \sum_{j_1 \ldots j_N} a_{j_1, \ldots, j_N} \gamma_{s_1} \ldots \gamma_{s_N}.$$  

Then,

$$2^{1/k} \|f\|_\infty \geq \sum_{s_N} \left( \sum_{s_1, \ldots, s_{N-1}} |a_{j_1, \ldots, j_N}| \right)^{1/2}$$

for all $\alpha = 1, \ldots, N$.

**Proof.** — Let $1 \leq \alpha \leq N$ be arbitrary. Since $E$ is a 1-Sidon set with Sidon constant $= 1$, it is easy to see that by a proper choice of $t_s \in G$ we obtain

$$(5.1.1) \quad \|f\|_\infty \geq \sup_{t_1, \ldots, t_N} \left| \sum_{s_N} \sum_{s_1, \ldots, s_{N-1}} a_{j_1, \ldots, j_N} \gamma_s(t_1) \ldots \gamma_s(t_N) \right|.$$  

Since the sup-norm dominates the $L^1$-norm, it follows from (5.1.1) that

$$\|f\|_\infty \geq \sum_{s_N} \left( \sum_{s_1, \ldots, s_{N-1}} |a_{j_1, \ldots, j_N}| \gamma_s(t_1) \ldots \gamma_s(t_N) \right)$$

(we make the obvious modification for $\alpha = 1$ and $\alpha = N$). Since $E_{j,k}$ is a $\Lambda(2)$ set whose $\Lambda(2)$ constant is bounded by $2^{1/k}$ (this follows from part (a) of Theorem 1.6), we obtain

$$2^{1/k} \|f\|_\infty \geq \sum_{s_N} \left( \sum_{s_1, \ldots, s_{N-1}} |a_{j_1, \ldots, j_N}| \right)^{1/2} \quad \square$$

In a previous version of this manuscript, by following a multidimensional version of Littlewood's rearrangement argument (see p. 168 of [7] and Lemma 3 of [6]) we proved that

$$\sum_{\alpha=1}^N \sum_{s_N} \left( \sum_{s_1, \ldots, s_{N-1}} |a_{j_1, \ldots, j_N}| \right)^{1/2} \geq \frac{1}{K!} \left( \sum_{j_1, \ldots, j_N} |a_{j_1, \ldots, j_N}| \right)^{(K+J)/2J}.$$  

Then, combining Lemma 5.1 and (5.1), we deduce that $E_{j,k}$ is a $2J/(K+J)$-Sidon and that

$$D\left( \frac{2J}{(K+J)} E_{j,k} \right) \leq 2^{1/k} \binom{J}{K} K!.$$  

The estimate in (5.2), however, is not as tight as we would like it to be. In order to obtain the existence of sets that are exactly $p$-Sidon, for any $p \in (1,2)$, we
require the sharper Lemma 5.3 below. We are grateful to S. Kaijser at Uppsala University for pointing out to us that Littlewood’s classical inequality (the case $J = 2$ and $K = 1$ in (5.1)) can be proved without Littlewood’s rearrangement argument. Indeed, this is a key observation in the demonstration below. In the course of the proof of 5.3 we greatly benefited also from stimulating conversations with M. Benedicks at the Institut Mittag-Leffler.

We require two basic inequalities: Minkowski’s inequality which, to facilitate referencing, we state below, and the M-Hölder inequality that is given in section 2 ((2.1)).

(5.2) Minkowski’s inequality. — Let $X$ and $Y$ be measure spaces and $g$ be a measurable function on $X \times Y$. For any $1 < r < \infty$,

$$\left( \int_X \left( \int_Y |g(x,y)|^r \right)^{1/r} \right) \leq \int_Y \left( \int_X |g(x,y)|^r \right)^{1/r}.$$  (5.3)

**Lemma 5.3.** — Let $J > K > 0$.

$$\left( \sum_{i_1, \ldots, i_J} |a_{i_1, \ldots, i_J}|^{2J/(K+J)} \right)^{(K+J)/2J} \leq \left( \frac{N}{\binom{J}{K}} \right)^{1/N} \left( \sum_{i_1} \left( \sum_{j} |a_{i_1, \ldots, i_J}|^2 \right)^{1/2} \right)^{1/N}$$  (5.4)

**Proof.** — We prove (5.4) by induction on $J$. We start with the case $J = 2$ and $K = 1$. Write

$$\sum_{i,j} |a_{ij}|^{4/3} = \sum_{i,j} |a_{ij}|^{2/3} |a_{ij}|^{2/3}.\quad (5.3.1)$$

By applying 2-Hölder’s inequality to the sum over $j$ with $p_1 = 3/2$ and $p_2 = 3$ for the first and second factors, respectively, in the summand of the right side of (5.3.1), we obtain

$$\sum_{i,j} |a_{ij}|^{4/3} \leq \sum_i \left( \sum_j |a_{ij}| \right)^{2/3} \left( \sum_j |a_{ij}|^2 \right)^{1/3}.\quad (5.3.2)$$

Next, applying 2-Hölder’s inequality to the sum over $i$ with $p_1$ and $p_2$ for the second and first factors, respectively, in the summand of the right side of
(5.3.2.), we obtain

\[
(5.3.3) \quad \sum_{i,j} |a_{ij}|^{4/3} \leq \left[ \sum_i \left( \sum_j |a_{ij}|^2 \right) \right]^{1/3} \left[ \sum_j \left( \sum_i |a_{ij}|^2 \right) \right]^{2/3}.
\]

From Minkowski's inequality, we have

\[
\left( \sum_i \left( \sum_j |a_{ij}|^2 \right) \right)^{1/2} \leq \sum_j \left( \sum_i |a_{ij}|^2 \right)^{1/2},
\]

and, therefore (from (5.3.3)), we deduce

\[
(5.3.4) \quad \left( \sum_{i,j} |a_{ij}|^{4/3} \right)^{3/4} \leq \left[ \sum_i \left( \sum_j |a_{ij}|^2 \right) \right]^{1/2} \left[ \sum_j \left( \sum_i |a_{ij}|^2 \right) \right]^{1/2}.
\]

(5.3.4) starts the inductive proof of (5.4), and we now assume that \( J > 2 \) and that (1.4) holds for all \( J_1 < J \). We write

\[
(5.3.5) \quad \sum_{j_1, \ldots, j_J} |a_{j_1, \ldots, j_j}|^{\frac{2j}{K+J}} = \sum_{j_1, \ldots, j_J} |a_{j_1, \ldots, j_j}|^{\frac{2}{K+1}} \ldots |a_{j_1, \ldots, j_j}|^{\frac{2}{K+J}}.
\]

For notational reasons, we shall write

\[
f_n(j_1, \ldots, j_j) = a_{j_1, \ldots, j_j}
\]

for the \( n \)-th factor in the summand of the right hand side of (5.3.5) (even though \( f_1 \equiv \cdots \equiv f_j \)), and whenever there is no confusion we shall write merely \( f_n \):

\[
(5.3.5') \quad \sum_{j_1, \ldots, j_J} |a_{j_1, \ldots, j_j}|^{\frac{2j}{K+J}} = \sum_{j_1, \ldots, j_J} |f_{j_1}^{2(K+J)} \cdots |f_{j_J}^{2(K+J)}|.
\]

Now, apply J-Hölder's inequality to the sum over \( j_j \) with

\[
p_1 \equiv p = (K+J)/2
\]

for \( f_1 \) and \( p_2 = \cdots = p_1 \equiv q = \frac{(J-1)(K+J)}{(K+J-2)} \) for \( f_2, \ldots, f_j \) and obtain that the right side of (5.3.5') is majorized by

\[
(5.3.6) \quad \sum_{j_1, \ldots, j_{J-1}} \left( \sum_j |f_{j_1}|^{\frac{2j}{K+J}} \prod_{n=2}^J \left( \sum_j |f_n|^{\frac{2(j-1)}{(J-1)(K+J)}} \right) \right)^{\frac{K+J-2}{J-1}(K+J)}.
\]
Next, apply J-Hölder’s inequality to the sum over \( j_{j-1} \) with \( p \) for the factor containing \( f_2 \) and \( q \) for the remaining factors in the summand of (5.3.6), and obtain that (5.3.6) is majorized by

\[
(5.3.7) \quad \sum_{j_{j-1}=1}^{n-1} \left\{ \left[ \sum_{j_{j-1}} \left( \sum_{j_{j} \setminus j_{j-1}} |f_1| \right)^{2(p-1)} \right]^{(K+J-2)(p-1)(K+J)} \cdot \left[ \sum_{j_{j-1}} \left( \sum_{j_{j} \setminus j_{j-1}} |f_2| \right)^{2(q-1)} \right]^{(K+J-2)(q-1)(K+J)} \cdot \prod_{n=3}^{J} \left[ \sum_{j_{j-1}=1}^{n} |f_n|^{2(q-1)} \right]^{(K+J-2)(q-1)(K+J)} \right\}.
\]

We continue in this fashion: At then \( n \)-th step, \( n > 2 \), we apply the J-Hölder inequality to the sum over \( j_{j-n+1} \) with \( p \) for the factor that contains \( f_n \) and with \( q \) for the remaining factors. After \( J \) such operations we obtain that (5.3.5') is majorized by

\[
(5.3.8) \quad \left[ \sum_{j_{j-1}, \ldots, j_{j-1}} \left( \sum_{j_{j}} |f_1| \right)^{2(p-1)} \right]^{(K+J-2)(p-1)(K+J)} \cdot \left[ \sum_{j_{j-1}, \ldots, j_{j-n+1}} \left( \sum_{j_{j-n+2} \cdots j_{j}} |f_n|^{2(q-1)} \right) \right]^{(K+J-2)(q-1)(K+J)} \cdots \left[ \sum_{j_{j-n+1} \cdots j_{j}} |f_J|^{2(q-1)} \right]^{(K+J-2)(q-1)(K+J)}.
\]

Now, apply Minkowski’s inequality to each of the first \((J-1)\) factors in (5.3.8) as follows: To the first factor, apply Minkowski’s inequality (as stated above) with \(|g| = |f_1|\),

\[
\int_X = \sum_{j_{j-1}=1}^{n-1}, \quad \int_Y = \sum_{j_{j}} \quad \text{and} \quad r = \frac{2(J-1)}{(K+J-2)}.
\]

To the \( n \)-th factor, \( 1 < n < J \), apply Minkowski’s inequality with

\[
|g| = \sum_{j_{j-n+2} \cdots j_{j}} |f_n|^{\frac{2(J-1)}{(K+J-2)}},
\]

\[
\int_X = \sum_{j_{j-1}=1}^{n-1}, \quad \int_Y = \sum_{j_{j-n+1}} \quad \text{and again with} \quad r = \frac{2(J-1)}{(K+J-2)}.
\]
We therefore obtain
\[
(5.3.9) \left( \sum_{j_1, \ldots, j_J} |a_{j_1, \ldots, j_J}|^{2j/(K+J)} \right)^{(K+J)/(2J)} \leq \prod_{n=1}^J \left( \sum_{a_n} \left( \sum_{j_1, \ldots, j_J} |a_{j_1, \ldots, j_J}|^{2(j-1)/(K+J-2)} \right)^{(K+J-2)/(2j-2)} \right)^{1/2} \]

(\sum_{a_n} \text{denotes summation with respect to indices in } \{j_1, \ldots, j_J\} \setminus \{j_n\}). \text{ If } K = 1, (5.3.9) \text{ reduces to (5.4) and the lemma is proved. We now assume } K > 1 \text{ and apply the induction hypothesis that (5.4) holds for } J - 1 \text{ and } K - 1 \text{ for each of the } J \text{ factors on the right hand side of (5.3.9). Namely, for each } 1 \leq n \leq J \text{ we have}
\[
(5.3.10) \sum_{a_n} \left( \sum_{j_1, \ldots, j_J} |a_{j_1, \ldots, j_J}|^{2(j-1)/(K+J-2)} \right)^{(K+J-2)/(2j-2)} \leq \sum_{a_n} \prod_{a_s} \left( \sum_{j_1, \ldots, j_J} |a_{j_1, \ldots, j_J}|^{2(J-1)} \right)^{1/2} \right)^1/N_1
\]
[Recall that \(N_1 \equiv \binom{J-1}{K-1}\). \{S_1, \ldots, S_{N_1}\} \text{ denotes the collection of all } (K-1)-\text{subsets of } \{1, \ldots, J\} \setminus \{n\}; \sum_{S_a} \text{ denotes summation with respect to the indices } j_{a_1}, \ldots, j_{a_{K-1}}, \text{ where } S_a = (a_1, \ldots, a_{K-1}); \sum_{S_a} \text{ denotes summation with respect to the remaining indices (except } j_n).]
\text{ We now apply } N_1\text{-Hölder inequality, on the right hand side of (5.3.10), to the sum over } j_n \text{ with}
\[p_1 = \cdots = p_{N_1} = 1/N_1 \text{ for each of the } N_1 \equiv \binom{J-1}{K-1} \text{ factors in the summand. Combining this application with (5.3.9), we obtain that the left hand side of (5.3.9) is majorized by}
\[
(5.3.11) \prod_{n=1}^J \prod_{a=1}^{N_1} \left[ \sum_{j_1, \ldots, j_J} \left( \sum_{s_a} |a_{j_1, \ldots, j_J}|^2 \right)^{1/2} \right]^{1/N_1}
\]
\text{ Finally, observe that}
\[
(5.3.12) \{\{1\} \cup S_1^1, \{1\} \cup S_2^1, \ldots, \{1\} \cup S_n^1, \ldots, \{n\} \cup S_1^1, \ldots, \{n\} \cup S_n^1, \ldots, \{J\} \cup S_1^1, \ldots, \{J\} \cup S_n^1\} = \{S_1, \ldots, S_n\}.
\]
Also, notice that each set in the enumeration of the collection on the left hand side of (5.3.12) occurs precisely $K$ times. Therefore, the expression in (5.3.11) equals

$$\prod_{\alpha=1}^{N_i} \left( \sum_{\sim S_{\alpha}} |a_{j_\alpha \cdots j_\beta}|^2 \right)^{1/2} \prod_{\alpha=1}^{K} \left( \sum_{\sim S_{\alpha}} |a_{j_\alpha \cdots j_\beta}|^2 \right)^{1/2} \prod_{\alpha=1}^{N} \left( \sum_{\sim S_{\alpha}} |a_{j_\alpha \cdots j_\beta}|^2 \right)^{1/2} \prod_{\alpha=1}^{N_i} \left( \sum_{\sim S_{\alpha}} |a_{j_\alpha \cdots j_\beta}|^2 \right)^{1/2},$$

and the proof of the Lemma is complete. \hfill \Box

Combining Lemmas 5.1 and 5.3, we deduce that

$$D(2J/(K+J), E_{j,k}) \leq 2^{3/2}.$$

We now proceed to show that

$$D((2J/(K+J)) - \varepsilon, E_{j,k}) = \infty$$

for all $\varepsilon > 0$. The argument that follows is similar to the one used in showing that $E_{j,k}$ is not $A^{1/2}$ for any $\varepsilon > 0$ (and is an adaptation of a proof used in 2.7 of [4]). We use the fact that if $F \subset \Gamma$ is $p$-Sidon, then there is $B > 0$ so that for all $f \in L_p^1(G)$

$$||f||_b \leq B \sqrt{b} ||f||_a$$

for all $1 < b < \infty$, where

$$a = 2p/(3p - 2)$$

(see (9) in [3]). Let $n > 0$ be arbitrary, and let

$$V_n = \{j = (j_1, \ldots, j_d) \in (\mathbb{Z}^+)^d : 1 \leq j_1, \ldots, j_d \leq n\}.$$

Let

$$g = \sum_{j \in V_n} (\gamma_{P_{1j}, \ldots, P_{nj}}).$$

Clearly,

$$||g||_a = n^{1/a}.$$

Next, let $U_n = \{i = (i_1, \ldots, i_k) \in (\mathbb{Z}^+)^k : 1 \leq i_1, \ldots, i_k \leq n\}$ and define

$$h(t_1, \ldots, t_N) = \left[ \prod_{i \in U_n} (1 + \cos \gamma_i(t_1)) \right] \ldots \left[ \prod_{i \in U_n} (1 + \cos \gamma_i(t_N)) \right].$$

As in section 4, we conclude (see (4.2)-(4.4)) that for any $1 < R < 2$

$$||h||_R \leq 2^{N/2} R^N,$$

and

$$\hat{h} = 1/2^N \quad \text{on} \quad \{(\gamma_{P_{1j}, \ldots, P_{nj}})_{j \in V_n}.}$$
Therefore, as in section 4,
\begin{equation}
2^{-N}n^J = |g \ast h(0)| \leq \|g\|_{a(K)} \|h\|_{b(K)},
\end{equation}
\begin{equation}

\leq \|g\|_{a(K)} \cdot 2^N.
\end{equation}

If $E_{j,k}$ is $p$-Sidon, it follows from (5.5), (5.6) and (5.7) that
\begin{equation}
2^{-2N}n^J \leq Bn^{\left(\frac{K}{2}\right) + \left(\frac{J}{a}\right)}.
\end{equation}

But, (5.8) holds for all $n > 0$ only if 
\begin{equation}
p \geq \frac{2J}{(K + J)}.
\end{equation}

The proof of part b) of Theorem 1.6 is complete. 

\textit{Proof of Corollary 1.7, part (b).} – Let $F \subset \Gamma$ be dissociate and $F^{j,k} = F^{j,k}_\gamma$ be given by (1.11), for an arbitrary $\gamma$. Let $f$ be a trigonometric polynomial in $C_{\gamma,k}(\Gamma)$ given by
\begin{equation}
f = \sum_{j \in (Z^+)^J} a_j \gamma_{P(1)} \cdots \gamma_{P(N)}^{(N)}.
\end{equation}

Select $E = \{e_i\}_{i \in (Z^+)^K}$, an infinite independent set in some $\Gamma_0 = \hat{\Gamma}_0$. Let $t = (t_1, \ldots, t_N) \in \Gamma_0^N$ be arbitrary, and $\mu_\gamma \in M(\Gamma)$ be so that
\begin{equation}
\hat{\mu}_\gamma(\gamma_{P(1)} \cdots \gamma_{P(N)}^{(N)}) = e_{P(1)}(t_1) \cdots e_{P(N)}(t_N)
\end{equation}
for all $j \in (Z^+)^J$, and
\begin{equation}
||\mu_\gamma|| \leq 2^N
\end{equation}
(Lemma 3.1).

Therefore,
\begin{equation}
\sup_{t \in \Gamma_0^N} \left| \sum_{j \in (Z^+)^J} a_j e_{P(1)}(t_1) \cdots e_{P(N)}(t_N) \right| = \sup_{t \in \Gamma_0^N} |\langle f, \mu_\gamma \rangle| \leq 2^N \|f\|_\infty.
\end{equation}

By Theorem 1.6, part (b), we have
\begin{equation}
\left( \sum_{j \in (Z^+)^J} |a_j|^{2J/(K + J)} \right)^{(K + J)/2J} \leq 2^{J} \cdot 2^N \|f\|_\infty.
\end{equation}

This proves that $F^{j,k}$ is $\frac{2J}{(K + J)}$ – Sidon. To show that $F^{j,k}$ is exactly $\frac{2J}{(K + J)}$ – Sidon we follow the same route that was used to prove the corresponding fact for $E_{j,k}$. 

\qed
COROLLARY 5.4. — Let $J \geq K > 0$ and $J' \geq K' > 0$ be arbitrary. Let $F \subset \Gamma$ be dissociate and $\mathcal{P}$, $\mathcal{P}'$ be $\left(\frac{J}{K}\right)$, $\left(\frac{J'}{K'}\right)$ — partitions of $F$ respectively. Then $F^{J,K}_{\mathcal{P}} \cup F^{J',K'}_{\mathcal{P}'}$ is exactly a $p$-Sidon set, where

$$p = \max \left\{ \frac{2J}{(K+J)}, \frac{2J'}{(K'+J')} \right\}.$$

Proof. — Apply Lemma 3.4.

Proof of Corollary 1.7, part (d). — We consider two cases.

I. $\Gamma$ contains elements with arbitrarily large order. As usual, we shall assume that $\Gamma = \mathbb{Z}$. First, observe that for any $J > K > 0$ the (M-independent) set $F \subset \mathbb{Z}$ given in Lemma 4.3 has the property

$$(5.10) \quad D(2J/(K+J), F^{J,K}_{\mathcal{P}}) \leq 16^{-2^J}.$$  

($5.10$) is achieved by applying, en route to ($5.9$), Lemmas 3.2 and 4.3 in place of Lemma 3.1 and Theorem 1.6 (b), respectively, whence $2^N$ is replaced by $4$ and $2^{J/K}$ by $4 \cdot 2^{J/K}$ in ($5.9$). Let $p \in [1, \infty)$, and $J_n > K_n > 0$ be so that $(J_n/K_n)_{n=1}^{\infty}$ is a monotonically increasing sequence converging to $p/(2-p)$. For each $n > 0$, let $F_n \subset \mathbb{Z}$ be the set given in Lemma 4.3 corresponding to $J_n > K_n > 0$. Select finite sets $E_n \subset F_n$ so that

$$(5.11) \quad D([2J_n/(J_n+K_n)] - 1/n, E_n) > n.$$  

Next, determine $d_n \in \mathbb{Z}^+$ with the following property: Whenever $f$ is a trigonometric polynomial with spectrum in $\bigcup_{n=1}^{\infty} d_nE_n$, then

$$(5.12) \quad 6 \|f\|_{\infty} \geq \sum_{n=1}^{\infty} \|f_n\|_{\infty},$$

where $f = \sum_{n=1}^{\infty} f_n$ and spectrum $(f_n) \subset d_nE_n$. Combining ($5.10$), ($5.11$), and ($5.12$), we conclude that $\bigcup_{n=1}^{\infty} d_nE_n$ is exactly $p$-Sidon.

To obtain a set in $\mathbb{Z}$ which is exactly non $p$-Sidon, we choose $J_n > K_n > 0$ so that $(J_n/K_n)_{n=1}^{\infty}$ is a monotonically decreasing sequence to $p/(2-p)$, and carry out a construction similar to the one above. Details are left to the reader.
II. $\Gamma$ contains $\bigoplus Z_a$ for some $a \in Z^+$. Clearly, we may assume that for any given $J > K > 0$, there is a dissociate set 

$$E = \{\gamma_i\}_{i \in (Z^+)^k} \subset \bigoplus Z_a$$

so that

$$(5.13) \quad E_{J,K} = \{(\gamma_{P(0)}, \ldots, \gamma_{P(N)})\}_{j \in (Z^+)^J} \subset \bigoplus Z_a = (\bigoplus Z_a)^N.$$ 

**Sublemma.** Let $E_{J,K} \subset \bigoplus Z_a$ be given by (5.13). Then,

$$A(q, E_{J,K}) \leq (Bq)^{[J/K]^2},$$

where $[J/K] = $ least integer equal to or greater than $J/K$, and $B$ is the 1-Sidon constant of $E$.

**Proof.** Let $g \in L^2_{E_{J,K}}(\bigotimes Z_a)$, $\|g\|_2 = 1$, be given by

$$g \sim \sum_{j \in (Z^+)^J} a_j(\gamma_{P(0)}, \ldots, \gamma_{P(N)}).$$

Observe that there are $[J/K]$ subsets of $\{1, \ldots, J\}$, $S_1, \ldots, S_{[J/K]}$, so that

$$\bigcup_{n=1}^{[J/K]} S_n = \{1, \ldots, J\}.$$ 

Therefore, we may estimate the $q^{\text{th}}$-norm of $g$ as follows:

$$\|g\|^q = \int (\omega_1, \ldots, \omega_N) \in (\bigotimes Z)^N \left| \sum_{j \in (Z^+)^J} a_j(\gamma_{P(0)}(\omega_1) \cdot \cdot \cdot \gamma_{P(N)}(\omega_N))^q \right| d\omega_1 \cdot \cdot \cdot d\omega_N$$

$$= \int (\bigotimes Z_a)^{N-J/K} \left| \sum_{j \in (Z_a)^{J/K}} a_j(\gamma_{P(0)}) \cdot \cdot \cdot \gamma_{P(N)})^q \right| \leq \int (\bigotimes Z_a)^{N-J/K} (Bq)^{[J/K]^2}$$

$$\leq (Bq)^{[J/K]^2}$$

(the appearance of $B$ above is explained by (2) of 5.7.7 in [10]).

To prove Corollary 1.7 (d), in the present context, we follow a route identical to the one followed in Case I where the use of (5.10) is replaced by a use of the Sublemma.

**6. Problems.**

The « $J/K$-fractional » product of a dissociate set $E \subset \Gamma$ was defined in this work as a subset of the $\left(\begin{array}{c} J \\ K \end{array}\right)$-fold product of $E$. Subsequently, through
the use of Riesz products, \( \binom{J}{K} \) appeared in some of the estimates. The first four problems focus on replacing \( \binom{J}{K} \) by \( J/K \) in the various computations.

**Problem 6.1.** — Improve the lower bound on \( \sup_{\gamma \in \Gamma^N} r_s(F, \gamma) \) in (1.4) of Theorem 1.3. In particular, is there \( C > 0 \) so that for all \( s \in \mathbb{Z}^+ \)

\[
C^{-J/Kg^{(J/K)b}} \leq \sup_{\gamma \in \Gamma^N} r_s(F, \gamma),
\]

where \( F \subset \Gamma \) is independent?

**Problem 6.2.** — Improve the norm estimate in (3.2) of Lemma 3.1. In particular, is there \( \mu \in M(G) \) fulfilling (3.3) and so that

\[
\|\mu\| \leq C^{j/k},
\]

for some (universal) \( C > 0 \)?

**Problem 6.3.** — Given any rational \( \beta \in [1, \infty] \), we constructed in this paper \( E \subset \Gamma \) for which there are \( \eta_{\beta}, \zeta_{\beta} > 0 \) so that for all \( q > 2 \)

\[
(6.4.1) \quad \eta_{pq}^{\beta/2} \leq A(q, E) \leq \zeta_{pq}^{\beta/2}.
\]

While (6.4.1) is a stronger statement than « \( E \) is exactly \( \Lambda^\beta \) », in Corollary 1.7 (c) we deduced for all \( \beta \in [1, \infty) \) no more than the existence of sets that are exactly \( \Lambda^\beta \). Given any \( \beta \in [1, \infty) \), can we find sets (say in \( \mathbb{Z} \)) for which (6.4.1) holds?

**Problem 6.4.** — Prove Corollary 1.7 (c) for \( \Gamma = \bigoplus \mathbb{Z}_a, \ a \in \mathbb{Z}^+ \).

**Problem 6.5.** — A classical theorem due to Rudin ([10]) states that every \( 1 \)-Sidon set is \( \Lambda^1 \). The converse was recently established by Pisier ([9]). Could it be that \( E \subset \Gamma \) is a \( \Lambda^\beta \) set if and only if \( E \) is \( 2\beta/(1+\beta) \)-Sidon?

**Problem 6.6.** — In Section 5, an essentially probabilistic method was employed to show that \( E_{J,K} \) is not \( (2J/(K+J)-\varepsilon) \)-Sidon for any \( \varepsilon > 0 \). In the case \( J = 2 \) and \( K = 1 \), an explicit construction of trigonometric polynomials (bounded bilinear forms) displaying this fact is given on p. 172 of [8]. Replace the probabilistic procedure in Section 5 by a constructive one.

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