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HARMONIC INTERPOLATING SEQUENCES, L^p AND BMO

by John B. GARNETT

Let (z_ν) be a sequence in the upper half plane. If $1 < p \leq \infty$ and if $y_\nu^{1/p} f(z_\nu) = a_\nu$, $\nu = 1, 2, \dots$ has solution $f(z)$ in the class of Poisson integrals of L^p functions for any sequence $(a_\nu) \in l^p$, then we show that (z_ν) is an interpolating sequence for H^∞ . If $f(z_\nu) = a_\nu$, $\nu = 1, 2, \dots$ has solution in the class of Poisson integrals of BMO functions whenever $(a_\nu) \in l^\infty$, then (z_ν) is again an interpolating sequence for H^∞ . A somewhat more general theorem is also proved and a counterexample for the case $p \leq 1$ is described.

1. Let $z_\nu = x_\nu + iy_\nu$, $y_\nu > 0$ be a sequence in the upper half plane U , and let

$$P_\nu(t) = \frac{1}{\pi} \frac{y_\nu}{(t - x_\nu)^2 + y_\nu^2}$$

be the Poisson kernel for z_ν . When

$$\int f(t)(1 + t^2)^{-1} dt < \infty$$

we write $f(z_\nu) = \int f(t)P_\nu(t) dt$ and when $1 \leq p \leq \infty$ we write

$$T_p f(\nu) = y_\nu^{1/p} f(z_\nu).$$

The operator T_p maps L^p into the space l^∞ of bounded sequences, because $\|P_\nu\|_q \leq cy_\nu^{-1/p}$, $q = p/(p-1)$. If for every l^p sequence (a_ν) the interpolation

$$(1.1) \quad y_\nu^{1/p} f(z_\nu) = a_\nu, \quad \nu = 1, 2, \dots$$

has solution within the class of harmonic functions $f(z)$ on U representable as Poisson integrals of L^p functions, then for brevity we write $T_p(L^p) \supset l^p$. Similarly, $T_p(H^p) \supset l^p$ means that (1.1) has solution $f(z) \in H^p$. By a theorem of Carleson [3], [7], $T_p(H^p) = l^p$ if and only if the points z_ν satisfy

$$(1.2) \quad \inf_{\nu} \prod_{\mu, \mu \neq \nu} \left| \frac{z_\nu - z_\mu}{z_\nu - \bar{z}_\mu} \right| = \delta > 0.$$

Consequently a sequence satisfying (1.2) is called an *interpolating sequence*.

In [9] it was proved that $\{z_\nu\}$ is an interpolating sequence if and only if $T_\infty(L^\infty) = l^\infty$, and this result was refined in [4] and [13]. Here we extend the work of those papers to obtain (1.2) when $T_p(L^p) \supset l^p$, $1 < p$, or when $T_\infty(\text{BMO}) \supset l^\infty$.

Condition (1.2) holds if and only if the following two geometric conditions both hold

$$(S) \quad |z_\nu - z_\mu| \geq \alpha y_\nu, \quad \mu \neq \nu,$$

$$(C) \quad \sum_{z_\nu \in Q} y_\nu \leq Bl(Q),$$

for all squares $Q = \{a < x < a + l(Q), 0 < y < l(Q)\}$. See [10] or [9] for a proof of this well-known equivalence. Because of generalizations mentioned below we state our two theorems in terms of (S) and (C).

THEOREM 1. — *If $1 < p < \infty$ and if*

$$(1.3) \quad T_p(L^p) \supset l^p,$$

or if $p = \infty$ and if

$$(1.4) \quad T_\infty(\text{BMO}) \supset l^\infty$$

then (S) and (C) hold.

COROLLARY. — *The sequence (z_ν) is an interpolating sequence if and only if (1.3) or (1.4) holds.*

The other theorem draws the same conclusion from a weaker hypothesis, which is a version of (1.2) for harmonic functions from L^p or BMO.

THEOREM 2. — *If $1 < p < \infty$ and if there are $f_\nu \in L^p$, $\nu = 1, 2, \dots$ such that $\|f_\nu\|_p \leq 1$ and*

$$(1.5) \quad \begin{aligned} T_p f_\nu(\mu) &\leq 0, & \mu \neq \nu \\ \inf_\nu T_p f_\nu(\nu) &= \delta > 0, \end{aligned}$$

then (S) and (C) hold. If there are $f_\nu \in \text{BMO}$, $\nu = 1, 2, \dots$ such that $\|f_\nu\|_{\text{BMO}} \leq 1$ and

$$(1.6) \quad \begin{aligned} T_\infty f_\nu(\mu) &\leq 0, & \mu \neq \nu \\ \inf_\nu T_\infty f_\nu(\nu) &= \delta > 0 \end{aligned}$$

then (S) and (C) hold.

Conditions (S) and (C) have analogues in the upper half space \mathbf{R}_+^{n+1} , [4], and the two theorems stated here are true in \mathbf{R}_+^{n+1} even when P_ν is replaced by

$$K_\nu(t) = \frac{1}{y_\nu^n} K\left(\frac{x_\nu - t}{y_\nu}\right)$$

where $K \geq 0$, $K \in L^1 \cap L^\infty$, $|\nabla K(t)| \leq C(1 + |t|)^{n+1}$, and $\int K dt = 1$. It is very likely that the proofs are valid in certain spaces of homogeneous type ([5], [6]), such as the unit ball of \mathbf{C}^n with T_p defined using the Poisson-Szegő kernel ([10], [11]). See [1], in which a converse of Theorem 1 is proved in that generality. For $n > 1$ it is not known if (S) and (C) imply interpolation of l^∞ by bounded harmonic functions on \mathbf{R}_+^{n+1} , and we do not claim that the corollary to Theorem 1 generalizes to \mathbf{R}^n or \mathbf{C}^n . To keep things simple we only prove the theorems for Poisson kernels on \mathbf{R}^1 .

The methods here are all real analysis; the principle tool is the lemma from § 4 of [4].

In Section 2 we obtain the inequality needed to prove Theorem 1, we show that Theorem 1 is a corollary of Theorem 2, and we verify condition (S). We also include a proof, due to Varopoulos, of Theorem 1 for $p > 2$.

Theorem 2 is proved in Section 3. In Section 4 we show by example that (C) can fail when $T_1(L^1) = l_1$ or when

$$T_p(\text{Re } H^p) = \text{Re } l^p, \quad 1/2 < p < 1,$$

for $p < 1$, T_p must be defined by (1.1) with $f = \text{Re } F$

$F \in H^p$). I suspect that $T_1(\text{Re } H^1) = \text{Re } l^1$ implies (C) but I have no proof.

I thank Eric Amar and Nicholas Varopoulos for useful correspondence and conversation.

The letters c and C stand for universal undetermined constants, the same letter denoting several constants.

2. In Theorem 1 it is not assumed that T_p is a bounded operator from L^p to l^p , or even that $T_p(L^p) \subset l^p$ (which is the same by the closed graph theorem). Indeed, if T_p were bounded then condition (C) would follow by the theorem on Carleson measures ([7] p. 193). Then, as noted in [9], $T_p(L^p) = l^p$ would trivially imply (S) and (C). However, there is an adequate substitute for boundedness.

LEMMA 2.1. — *If $1 \leq p < \infty$ and if (1.3) holds, then there is a constant M such that whenever $\sum |a_\nu|^p \leq 1$, the interpolation $T_p f(\nu) = a_\nu$ has solution with $\|f\|_p^p \leq M$. If (1.4) holds, there is a constant M such that whenever $(a_\nu) \in l^\infty$, the interpolation $T_\infty f(\nu) = a_\nu$ has solution with $\|f\|_{\text{BMO}} \leq M \sup |a_\nu|$.*

Proof. — For $1 \leq p < \infty$, the set

$$E_N = l^p \cap T_p(\{f: \|f\|_p \leq N\})$$

is closed in l^p . With (1.3) category shows that some E_N has interior in l^p , so that some E_N then contains the unit ball of l^p .

For $p = \infty$, we use the fact that BMO is the dual of the real Banach space $\text{Re } H^1$ [8], although a more elementary argument can be given in a few more words. Since

$$P_\nu - P_1 \in \text{Re } H^1,$$

the set

$$E_N = \{(a_\nu) \in l^\infty: f(z_\nu) - f(z_1) = a_\nu - a_1, \\ \nu = 1, 2, \dots, \|f\|_{\text{BMO}} \leq N\}$$

is closed in l^∞ . Since constant functions have zero BMO norm, (1.4) and category as above show interpolation is possible with $\|f\|_{\text{BMO}} \leq M \sup_\nu |a_\nu|$.

Because of the lemma, Theorem 2 clearly implies Theorem 1.

In Theorem 2 (or Theorem 1), condition (S) is easy to verify. For $p < \infty$ there is $f \in L^p$, $\|f\|_p \leq 1$ such that $f(z_\mu) \leq 0$, $f(z_\nu) \geq \delta y_\nu^{-1/p}$. The harmonic function $f(z)$ satisfies $|\nabla f(z)| \leq c y^{-(1+1/p)} \|f\|_p \leq c y^{-(1+1/p)}$, so that

$$\delta y_\nu^{-1/p} \leq |f(z_\nu) - f(z_\mu)| \leq c y_\nu^{-(1+1/p)} |z_\nu - z_\mu|$$

if

$$|z_\nu - z_\mu| < y_\nu/2.$$

Hence

$$\frac{|z_\nu - z_\mu|}{y_\nu} \geq \text{Max} \left(\frac{1}{2}, \frac{\delta}{c} \right),$$

and we have verified (S). When $p = \infty$ there is $f \in \text{BMO}$, $\|f\|_{\text{BMO}} \leq 1$ such that $f(z_\mu) \leq 0$, $f(z_\nu) \geq \delta$. The elementary estimate $y|\nabla f(z)| \leq c\|f\|_{\text{BMO}}$, then yields (S) just as in the case $p < \infty$ above.

N. Varopoulos has a simple proof of Theorem 1 for $p > 2$ which we now present. By the lemma, (1.3) has the dual formulation

$$(2.1) \quad \Sigma |\lambda_\nu|^q \leq M \|\Sigma \lambda_\nu y_\nu^{1/p} P_\nu\|_q^q$$

for all finite sequences (λ_ν) , where $q = p/(p - 1)$. To prove (C), fix a square Q with base I and let \tilde{I} be the interval concentric with I having length $|\tilde{I}| = 3|I|$. Let z_1, z_2, \dots, z_N be finitely many points from our sequence lying in Q . Let $\lambda_\nu = \pm y_\nu^{1/q}$, $\nu = 1, 2, \dots, N$ with random \pm sign. Taking expectations in (2.1) gives

$$\sum_1^N y_\nu \leq M \int |\Sigma y_\nu^2 P_\nu^2|^{q/2} dt = M \int_{\tilde{I}} + M \int_{\mathbb{R} \setminus \tilde{I}}.$$

Since $q/2 < 1$, Hölder's inequality gives

$$\begin{aligned} \int_{\tilde{I}} |\Sigma y_\nu^2 P_\nu^2|^{q/2} dt &\leq |\tilde{I}|^{1-\frac{q}{2}} \left(\int_{\tilde{I}} |\Sigma y_\nu^2 P_\nu^2| dt \right)^{q/2} \\ &\leq 3^{1-\frac{q}{2}} |I|^{1-\frac{q}{2}} \left(\sum_1^N y_\nu \right)^{q/2}. \end{aligned}$$

Fixing $x_0 \in I$, we have $P_\nu^2(t) \leq c y_\nu^2 / (t - x_0)^q$ if $z_\nu \in Q$ and

$t \notin \tilde{I}$, so that

$$\begin{aligned} \int_{\mathbb{R} \setminus \tilde{I}} \left| \sum_1^N y_\nu^2 P_\nu^2 \right|^{q/2} dt &\leq c \left(\sum_1^N y_\nu^4 \right)^{q/2} \int_{(\mathbb{R} \setminus \tilde{I})} \frac{dt}{|t - x_0|^{2q}} \\ &\leq c \left(\sum_1^N y_\nu \right)^{q/2} \cdot |\mathbb{I}|^{3q/2} \cdot \int_{|\mathbb{I}|}^\infty \frac{ds}{s^{2q}} \\ &\leq C_q |\mathbb{I}|^{1 - \frac{q}{2}} \left(\sum_1^N y_\nu \right)^{q/2}. \end{aligned}$$

Hence

$$\left(\sum_1^N y_\nu \right)^{1 - \frac{q}{2}} \leq C |\mathbb{I}|^{1 - \frac{q}{2}}$$

and condition (C) holds.

Varopoulos' argument can be modified to give the BMO case of Theorem 1 in this way. It is enough to verify (C) for a square Q whose upper half contains a point z_0 from the sequence. Let z_1, \dots, z_N be finitely many other points from the sequence and in Q . By the lemma and by duality, (1.4) gives

$$\begin{aligned} \sum_{j=1}^N |\lambda_j| &\leq M \sup \left\{ \left| \sum_{j=1}^N \lambda_j f(z_j) \right| : \|f\|_{\text{BMO}} \leq 1, f(z_0) = 0 \right\} \\ &\leq cM \left\| \sum_{j=1}^N \lambda_j (P_j - P_0) \right\|_{\mathbb{H}^1} \\ &= cM \left\| \sum_{j=1}^N \lambda_j \left(\frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right) \right\|_{\mathbb{L}^1}. \end{aligned}$$

We again set $\lambda_j = \pm y_j$ and take the expectation, getting

$$\sum_1^N y_j \leq cM \int_{\mathbb{R}} \left\{ \sum y_j^2 \left| \frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right|^2 \right\}^{1/2} dt.$$

Now

$$\begin{aligned} &\int_{\tilde{I}} \left\{ \sum_1^N y_j^2 \left| \frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right|^2 \right\}^{1/2} dt \\ &\leq 3^{1/2} |\mathbb{I}|^{1/2} \left\{ 2 \sum_1^N \int_{\tilde{I}} \frac{y_j^2}{|t - \bar{z}_j|^2} dt + 2 \sum_1^N y_j^2 \int_{\tilde{I}} \frac{dt}{|t - \bar{z}_0|^2} \right\}^{1/2} \\ &\leq 3^{1/2} |\mathbb{I}|^{1/2} \left\{ 2 \sum_1^N y_j \int_{\tilde{I}} \frac{y_j}{(t - x_j)^2 + y_j^2} dt + 2c \sum y_j^2 / |\tilde{I}| \right\}^{1/2} \\ &\leq C |\mathbb{I}|^{1/2} \left(\sum_1^N y_j \right)^{1/2}. \end{aligned}$$

For $t \notin \tilde{I}$,

$$\left| \frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right|^2 \leq \frac{c|I|^2}{(t - x_0)^4},$$

so that

$$\begin{aligned} \int_{\mathbb{R} \setminus \tilde{I}} \left\{ \sum_1^N y_j^2 \left| \frac{1}{t - \bar{z}_j} - \frac{1}{t - \bar{z}_0} \right|^2 \right\}^{1/2} dt \\ \leq \left(\sum_1^N y_j \right)^{1/2} |I|^{1/2} c \int_{\mathbb{R} \setminus \tilde{I}} \frac{|I|}{(t - x_0)^2} dt \\ \leq C|I|^{1/2} (\sum y_j)^{1/2}. \end{aligned}$$

Hence $(\sum_1^N y_j)^{1/2} \leq C|I|^{1/2}$ and (C) holds.

This reasoning does not apply to the case $p \leq 2$ nor to the situation in Theorem 2.

3. In proving Theorem 2 we can now assume the points satisfy (S)

$$|z_\nu - z_\mu| \geq \alpha y_\nu, \quad \mu \neq \nu.$$

We prove (C) by contradiction. The idea is that if (C) fails with a large constant B then there are relations among the kernels P_ν which are inconsistent with (1.5) or (1.6). Our main tool is this lemma from [4].

LEMMA 3.1. — For $\varepsilon > 0$ there is a constant $B(\varepsilon, \alpha)$ such that if

$$(3.1) \quad \sum_{z_\nu \in Q} y_\nu \geq B(\varepsilon, \alpha) l(Q)$$

for some square $Q = \{a < x < a + l(Q), 0 < y < l(Q)\}$, then there is a point z_ν in the sequence and there are weights λ_μ such that

$$(3.2) \quad \lambda_\mu \geq 0, \quad \sum \lambda_\mu = 1$$

$$(3.3) \quad \lambda_\nu = 0$$

$$(3.4) \quad \|P_\nu - \sum \lambda_\mu P_\mu\|_1 < \varepsilon$$

$$(3.5) \quad \sum_{z_\mu \in Q} \lambda_\mu \leq l(Q) \|P_\nu\|_\infty \leq \frac{l(Q)}{\pi y_\nu}, \quad \text{for all } Q.$$

Except for (3.5) the lemma is proved in Section 4 (and Section 2) of [4], and (3.5) is implicit in that proof because the

functions constructed there are non-negative. We refer to [4] for the details.

Suppose $1 < p < \infty$, let $\varepsilon > 0$ be determined later, and assume (3.1) holds. Write $G = P_\nu - \sum \lambda_\mu P_\mu$, where the λ_μ are given by Lemma 3.1.

LEMMA 3.2. — $\|G\|_{\text{BMO}} \leq c/y_\nu$.

Proof. — Fix an interval I with center t_0 and let

$$Q_n = \{z : y < 2^n |I|, |x - t_0| < 2^{n-1} |I|\}.$$

For $z_\mu \in Q_1$ we have trivially

$$\frac{1}{|I|} \int_I P_\mu dt \leq \frac{1}{|I|},$$

while for $z_\mu \in Q_n \setminus Q_{n-1}$, $n \geq 2$, we have

$$\begin{aligned} \frac{1}{|I|} \int_I |P_\mu - P_\mu(t_0)| dt &\leq \frac{c}{|I|} \int_I \frac{|t - t_0|}{(x_\mu - t_0)^2 + y_\mu^2} dt \\ &\leq \frac{c}{2^{2n} |I|}. \end{aligned}$$

Letting $a = \sum_{z_\mu \notin Q_1} \lambda_\mu P_\mu(t_0)$, we then have

$$\begin{aligned} \frac{1}{|I|} \int_I |G - a| dt &\leq \|P_\nu\|_\infty + \sum_{z_\mu \notin Q_1} \frac{\lambda_\mu}{|I|} + \sum_{n=2}^\infty \sum_{z_\mu \notin Q_n \setminus Q_{n-1}} \frac{c\lambda_\mu}{2^{2n} |I|} \\ &\leq c/y_\nu \end{aligned}$$

by (3.5), and the lemma is proved.

Now define $G^\#(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |G - G_I| dt$, where G_I denotes the mean of G over I . By Lemma 3.2, $\|G^\#\|_\infty \leq c/y_\nu$, and by the Hardy-Littlewood maximal theorem and (3.4), $G^\#$ has small weak L^1 norm

$$m(\lambda) = |\{x : G^\#(x) > \lambda\}| \leq \frac{c\varepsilon}{\lambda}.$$

Consequently for $q = p/(p - 1)$,

$$\|G^\#\|_q^q = q \int_0^\infty \lambda^{q-1} m(\lambda) d\lambda \leq Cq\varepsilon \int_0^{c/y_\nu} \lambda^{q-2} d\lambda$$

and

$$\|G^\#\|_q \leq C_q \varepsilon^{1/q} y_\nu^{-1/p}.$$

From Theorem 5 of [8] we conclude that $\|G\|_q \leq C'_q \varepsilon^{1/q} y_v^{-1/p}$. But then if $C'_q \varepsilon^{1/q} < \delta$, (1.5) and (3.2) give this contradiction:

$$\delta y_v^{-1/q} \leq \left| \int G f_v dt \right| \leq C'_q \varepsilon^{1/q} y_v^{-1/p}.$$

We conclude that (C) holds with constant $B\left(\frac{\delta^q}{C_q}, \alpha\right)$.

Now suppose $p = \infty$. Again if (C) fails we have a point z_ν and weights λ_μ such that (3.2), (3.3) and (3.4) hold for some $\varepsilon > 0$ to be determined. By (1.6) there is $f \in \text{BMO}$ such that $\|f\|_{\text{BMO}} \leq M = 1/\delta$, and

$$(3.6) \quad f(z_\nu) = 0, \quad f(z_\mu) > 1, \quad \mu \neq \nu.$$

If $f(z)$ were bounded, say $\|f\|_\infty \leq M$, (3.6) and (3.4) would be in contradiction as soon as $M\varepsilon > 1$. As we only have $\|f\|_{\text{BMO}} \leq M$, more properties of the weights λ_μ must be used. From Section 4 of [4] it also follows that $\lambda_\mu = 0$ except when $y_\mu < y_\nu$ and $|x_\mu - x_\nu| < cy_\nu/\varepsilon^2$. Let

$$J = \{t : |t - x_\nu| < 3cy_\nu/\varepsilon^2\},$$

an interval containing all x_μ with $\lambda_\mu > 0$ in its middle third. For $t \notin J$, we then have

$$(3.7) \quad |G(t)| = |P_\nu(t) - \sum \lambda_\mu P_\mu(t)| \leq CP_\nu(t).$$

By (3.6) and the John-Nirenberg Theorem,

$$\int |f(t)|^4 P_\nu(t) dt \leq CM^4.$$

Hence by Hölder's inequality

$$(3.8) \quad \int_{\mathbb{R} \setminus J} |f(t)| P_\nu(t) dt \leq CM\varepsilon^{3/4},$$

while trivially

$$(3.9) \quad \int_J |f(t)|^4 dt \leq CM^4/y_\nu \varepsilon^2.$$

By (3.7) and (3.8), $\int_{\mathbb{R} \setminus J} |fG| dt \leq CM\varepsilon^{3/4}$. By (3.9), Hölder's inequality, and our estimate on $\|G\|_{4/3}$, we also have $\int_J |fG| dt \leq CM\varepsilon^{3/4-1/2}$. Since $\left| \int fG dt \right| \geq 1$ by (3.6), there is a contradiction if $CM\varepsilon^{1/4} < 1$.

This proof for $p = \infty$, due to Peter Jones, is much simpler than my original proof.

4. We give an example of a sequence $\{z_\nu\}$ for which (C) fails but for which $T_1(L^1) \supset l_1$. At the same time we show that (C) can fail for a sequence for which

$$T_p f(\nu) = y_\nu^{1/p} f(z_\nu) = a_\nu, \quad \nu = 1, 2, \dots$$

has solution $f \in \text{Re } H^p$ whenever $\sum |a_\nu|^p < \infty$, provided $1/2 < p < 1$. Here $\text{Re } H^p$ is the space of real parts of H^p functions with the quasinorm $\|\text{Re } F\|_{H^p} = \|F\|_{H^p}$, $F \in H^p$.

LEMMA 4.1. — Let $0 < p \leq 1$, and let $\eta^p < 1/2$. Suppose there are $f_\nu \in H^p$ ($f_\nu \in L^1$ when $p = 1$) such that

$$(4.1) \quad \|f_\nu\|_{H^p}^p \leq M \quad \text{if} \quad p < 1$$

or

$$(4.2) \quad \|f_\nu\|_1 \leq M \quad \text{if} \quad p = 1, \\ |T_p f_\nu(\nu) - 1| < \eta,$$

$$(4.3) \quad \sum_{\mu, \mu \neq \nu} |T_p f_\nu(\mu)|^p < \eta^p$$

for $\nu = 1, 2, \dots$. Then $T_p(\text{Re } H^p) \supset l^p$ if $p < 1$ and $T_1(L^1) \supset l_1$ if $p = 1$.

Proof. — If $\sum |a_\nu|^p < \infty$, let $F = \sum a_\nu f_\nu$. Then by (4.1) $\|F\|_{H^p}^p \leq M \sum |a_\nu|^p$ if $p < 1$, and $\|F\|_1 \leq M \sum |a_\nu|$ if $p = 1$. And by (4.2) and (4.3),

$$\sum_{\nu=1}^{\infty} |T^p F(\nu) - a_\nu|^p \leq 2\eta^p \sum |a_\nu|^p.$$

The lemma now follows by iteration.

For $z_0 = x_0 + iy_0$, and for $0 < \varepsilon < y_0$, let

$$f_{z_0, \varepsilon}(t) = \frac{\varepsilon}{\pi} y_0^{1/p} (\chi_{|t-x_0| < \varepsilon} - \chi_{|t-(x_0+y_0)| < \varepsilon}),$$

where χ_s is the characteristic function of S . Then

$$|y^{1/p} f_{z_0, \varepsilon}(z_0)| < 1 \quad \text{and} \quad y^{1/p} f_{z_0, \varepsilon}(z_0) \rightarrow 1 \quad (\varepsilon \rightarrow 0).$$

Also $\|f_{z_0, \varepsilon}\|_1 \leq 4\pi$ when $p = 1$.

LEMMA 4.2. — For $1/2 < p < 1$, $\|f_{z_0, \epsilon}\|_{\mathbb{H}^p}^p \leq M_p$.

Proof. — We have

$$(4.4) \quad \|f_{z_0, \epsilon}\|_1 \leq Cy_0^{1-1/p}$$

and

$$(4.5) \quad \int f_{z_0, \epsilon}(t) dt = 0.$$

Also $f_{z_0, \epsilon}$ has support in $\{|t - x_0| < 2y_0\}$. This means that $f_{z_0, \epsilon}$ is a $(p, 1)$ atom in the sense of [6], and the lemma follows from Theorem A of that paper. A well-known elementary argument can also be given for special case at hand. Recall the non-tangential maximal function f^* from § 3. We use the theorem that $f(z) \in \text{Re } \mathbb{H}^p$ if and only if $f^* \in L^p$, and that $\|f\|_{\mathbb{H}^p} \sim \|f^*\|_p$. See [2] or [8].

When $|t - x_0| < 4y_0$, (4.4) and the Hardy-Littlewood theorem give us, for $f = f_{z_0, \epsilon}$,

$$|\{t : |t - x_0| < 4y_0, |f^*(t)|^p > \lambda\}| \leq \text{Min} \left(8y_0, \frac{Cy_0^{1-1/p}}{\lambda^{1/p}} \right).$$

Hence

$$\int_{|t-x_0| < 4y_0} |f^*(t)|^p dt \leq 8y_0 \int_0^{c/y_0} d\lambda + \int_{c/y_0}^\infty \frac{Cy_0^{1-1/p}}{\lambda^{1/p}} d\lambda = M.$$

If $|t - x_0| > 4y_0$ and if $z \in \Gamma(t)$, then

$$\left| \frac{\partial}{\partial s} P_z(s) \right| \leq \frac{c}{|z - x_0|^2} \leq \frac{c}{|t - x_0|^2}$$

on the support of $f_{z_0, \epsilon}$. Then (4.5) gives

$$|f(z)| \leq cy_0^{1-1/p} \frac{y_0}{|t - x_0|^2},$$

and so

$$\int_{|t-x_0| > 4y_0} |f^*(t)|^p dt \leq cy_0^{2p-1} \int_{4y_0}^\infty u^{-2p} du \leq C_p$$

when $p > 1/2$.

Fix η with $\eta^p < 1/2$. Let $z_1 = \frac{1}{2} + i\delta$ where $\delta = \delta(\eta)$

is to be determined, and let ε_1 be so small that

$$|f_{z_1, \varepsilon_1}(z_1) - 1| < \eta.$$

Write $f_1 = f_{z_1, \varepsilon_1}$. From $I_1 = [0, 1]$ delete the two intervals $|t - 1/2| < 2\varepsilon_1, |t - \delta - 1/2| < 2\varepsilon_1$ containing the support of f_1 , and partition the remainder of I_1 into dyadic intervals I_2, I_3, \dots, I_{m_2} of length 2^{-n_2} . (We suppose ε_1 is a negative power of 2). Let x_ν be the center of $I_\nu, 2 \leq \nu \leq m_2$ and let $y_\nu = \delta 2^{-n_2}$. The points $z_\nu = x_\nu + iy_\nu, 2 \leq \nu \leq m_2$, join z_1 in our sequence. Choose ε_2 and put $f_\nu = f_{z_\nu, \varepsilon_2}, 2 \leq \nu \leq m_2$. When n_2 is fixed, ε_2 can be chosen so that (4.2) holds for $2 \leq \nu \leq m_2$. We claim that n_2 and ε_2 can be chosen so that (4.3) holds for the finite sequence z_1, \dots, z_{m_2} . When $\nu = 1$ the left side of (4.3) is

$$\begin{aligned} \delta 2^{-n_2} \sum_{\mu=2}^{m_2} |f_1(z_\mu)|^p &\leq C 2^{-n_2} \sum_{k=\varepsilon_1 2^{n_2}}^{\infty} \left(\frac{\delta 2^{-n_2}}{k^2 2^{-2n_2} + \delta^2 2^{-2n_2}} \right)^p \\ &\leq C \delta^{1+p} 2^{-n_2(1-p)} (\varepsilon_1 2^{n_2})^{1-2p} \\ &= C \delta^{1+p} \frac{2^{-n_2 p}}{\varepsilon_1^{2p-1}}, \end{aligned}$$

which is small if $n_2 > n_2(\varepsilon_1)$. For $\nu > 1$, one term in the left side of (4.3) is

$$\begin{aligned} |T_\nu f_\nu(1)|^p &\leq C y_\nu^{p-1} |P_1(x_\nu) - P_1(x_\nu + y_\nu)|^p \\ &\leq C \delta^{p-1} 2^{-n_2(2p-1)} \sup_{I_\nu} \left| \frac{\partial P_1}{\partial s} \right|^p, \end{aligned}$$

and since $p > 1/2$ this is small if 2^{-n_2} is small. For $\nu > 1$ we also have the sum

$$\begin{aligned} \sum_{\substack{\mu=2 \\ \mu \neq \nu}}^{m_2} |T_\nu f_\nu(\mu)|^p &\leq C (\delta 2^{-n_2})^p \sum_{\substack{\mu=2 \\ \mu \neq \nu}}^{m_2} |P_\mu(x_\nu) - P_\mu(x_\nu + y_\nu)|^p \\ &\leq C (\delta 2^{-n_2})^{2p} \sum_{k=1}^{\infty} \frac{1}{(k^2 2^{-2n_2} + \delta^2 2^{-2n_2})^p} \\ &\leq C \delta^{2p} < \eta/2 \end{aligned}$$

if δ is chosen correctly.

From each $I_\nu, 2 \leq \nu \leq m_2$, delete the two intervals of length $4\varepsilon_2$ whose middle halves support f_ν . The remaining

parts of $\bigcup_2^{m_2} I_\nu$ partition into dyadic intervals I_μ ,

$$m_2 + 1 \leq \mu \leq m_3,$$

of length 2^{-n_3} and with centers x_μ . Let $z_\mu = x_\mu + i\delta 2^{-n_3}$, and let $f_\mu = f_{z_\mu, \varepsilon_3}$, $m_2 + 1 \leq \mu \leq m_3$. Taking n_3 large and ε_3 small, we can use the above reasoning to obtain (4.2) and (4.3) for $1 \leq \nu \leq m_3$. This process can be continued to get an infinite sequence of points for which by Lemma 4.1, $T_p(\text{Re } H^p) \supset l^p$ if $p < 1$ and $T_1(L^1) \supset l^1$ if $p = 1$.

The sequence lies in the unit square so that (C) will fail if $\Sigma y_\nu = \infty$. However

$$\frac{1}{8} \Sigma y_\nu = \Sigma |I_\nu| = 1 + (1 - 8\varepsilon_1) + (1 - 8\varepsilon_1)(1 - 8\varepsilon_2) + \dots$$

and this sum diverges if $\Sigma \varepsilon_j < \infty$.

By using functions f_ν with several vanishing moments, one can obtain similar examples for $0 < p \leq 1/2$.

Added in Proof. — Peter Jones has proved $T_1(\text{Re } H^1) = \text{Re } l^1$ implies (C) by refining the proof of Lemma 3.1.

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