HARUO KITAHARA
SHINSUKE YOROZU

On the Čech bicomplex associated with foliated structures


<http://www.numdam.org/item?id=AIF_1978__28_3_217_0>
ON THE ČECH BICOMPLEX ASSOCIATED WITH FOLIATED STRUCTURES

by H. KITAHARA and S. YOROZU

1. We shall be in $C^\infty$-category. Let $M$ be a paracompact connected $n$-dimensional manifold with a foliation $\mathcal{F}$ of codimension $q$, and let $\mathcal{U} = \{U_\alpha\}$ be a simple covering of $M$ such that each $U_\alpha$ is a flat neighborhood with respect to $\mathcal{F}$. Then there exists a decomposable $q$-form $w = w_1 \wedge \ldots \wedge w_q$ on each $U_\alpha$ and, by Frobenius' theorem, there exists a 1-form $\eta$ on each $U_\alpha$ satisfying $dw = w \wedge \eta$, where $d$ denotes the exterior differentiation and $\wedge$ the exterior product. The 1-form $\eta$ is an interesting object; it is well known that $\eta \wedge (d\eta)^q$ defines a de Rham class in $H^{2q+1}(M, \mathbb{R})$ ([1], [2], [3]). Our aim is to show that $\eta$ itself defines a certain cohomology class, that is,

**Theorem A.** $(-1)^{q+1/2}\pi \eta$ defines a $D$-cohomology class in $H^2(\tilde{C}(\mathcal{U}; A^\ast(M)), D)$ depending only on $\mathcal{F}$.

The above theorem was announced in [4], where it contained misstatements.

**Theorem B.** Supposing $M$ admits foliations $\mathcal{F}, \mathcal{F}'$ complementally transversal to each other, $\eta$ defines a $D'$-cohomology class in $H^1(\tilde{C}(\mathcal{U}; A^\ast(M)), D')$ (Cf. [5], [6], [7]).

2. Since $M$ has a foliation $\mathcal{F}$ of codimension $q$, the tangent bundle $TM$ of $M$ has an integrable subbundle $E$ with fibre dimension $n-q$. Let $Q = TM/E$ be a quotient bundle with fibre dimension $q$. Choosing a suitable Riemannian metric on $TM$, we obtain an isomorphism $TM \cong E \oplus Q$ (Whitney sum).
Then
\[ d\bar{w}^j = \sum_i dt^i_j \wedge w^j + \sum_i t^i_j dw^j \]
\[ = \sum_i \left( dt^i_j - \sum_k t^i_k \varphi^i_k \right) \wedge w^j, \]
and on the other hand
\[ d\bar{w}^j = \sum_k \bar{w}^k \wedge \bar{\varphi}^i_k \]
\[ = \sum_i \left( - \sum_k t^i_k \bar{\varphi}^i_k \right) \wedge w^j. \]

Thus
\[ - \sum_k t^i_k \bar{\varphi}^i_k = dt^i_j - \sum_k t^i_k \varphi^i_k + \sum_k f^i_{jk} w^k \]
where \( f^i_{jk} \) are functions on \( U_{\alpha_0} \cap U_{\alpha_1} \).

Let \( \begin{pmatrix} s^i_j & 0 \\ 0 & s^a_b \end{pmatrix} \) denote the inverse matrix of \( \begin{pmatrix} t^i_j & 0 \\ 0 & t^a_b \end{pmatrix} \).

Then
\[ \sum_{i,k} s^i_j t^k_{ij} \bar{\varphi}^i_k = - \sum_j s^i_j dt^i_j + \sum_{j,k} s^i_j t^k_{ij} \varphi^i_k - \sum_{i,k} s^i_j f^i_{jk} w^k \]
and we obtain
\[ \sum_{i} \bar{\varphi}^i_i = - \sum_{i,j} s^i_j dt^i_j + \sum_i \varphi^i_i - \sum_{i,j,k} s^i_j f^i_{jk} w^k. \]

From (4), \( dt^i_j = \sum_k t^i_{jk} w^k \). Thus, by (3), we obtain \( \sum_{i} \bar{\varphi}^i_i = \sum_i \varphi^i_i \).

Therefore we obtain \( \bar{\eta} = \eta \) on \( U_{\alpha_0} \cap U_{\alpha_1} \) and \( \eta \in \tilde{C}^{0,1} (U ; A^*(M)). \)

Q.E.D.

**Theorem B.** — Supposing \( M \) admits foliations \( \mathcal{F}, \mathcal{F}' \) complementarily transversal to each other, \( \eta \) defines a \( D' \)-cohomology class in \( H^1 (\tilde{C}^{\infty} (U ; A^*(M)), D') \) where \( \eta \) is defined by \( \mathcal{F} \).
We may suppose that $\mathcal{U} = \{U_\alpha\}$ is a simple covering such that each $U_\alpha$ is a locally trivial neighborhood of the bundle $Q \to M$.

Let $\nabla^\alpha$ denote a local connection on $Q|_{U_\alpha}$, and let $\omega^\alpha$ (resp. $\Omega^\alpha$) denote a connection form (resp. a curvature form) of $\nabla^\alpha$ on $U_\alpha$. Let $\Delta^p$ be a canonical $p$-simplex in $\mathbb{R}^{p+1}$ (with coordinates $(t_0, t_1, \ldots, t_p)$). We define a connection form $\omega_{\alpha_0 \cdots \alpha_p}$ on $U_{\alpha_0 \cdots \alpha_p} \times \Delta^p$ ($U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_p}$) by

$$\omega_{\alpha_0 \cdots \alpha_p} = t_0 \omega_{\alpha_0} + \ldots + t_p \omega_{\alpha_p}$$

and let $\Omega_{\alpha_0 \cdots \alpha_p}$ denote a corresponding curvature form on $U_{\alpha_0 \cdots \alpha_p} \times \Delta^p$.

For the set $A^k(U_{\alpha_0 \cdots \alpha_p} \times \Delta^p)$ of all $k$-forms on $U_{\alpha_0 \cdots \alpha_p} \times \Delta^p$, $\Delta^p \int : A^k(U_{\alpha_0 \cdots \alpha_p} \times \Delta^p) \to A^{k-p}(U_{\alpha_0 \cdots \alpha_p})$ denotes the integration along the fibre. Then we obtain Stokes' theorem

$$\Delta^p \int \circ d = (-1)^p d \circ \Delta^p \int + \partial \Delta^p \int \circ j^*$$

where $j : U_{\alpha_0 \cdots \alpha_p} \times \partial \Delta^p \to U_{\alpha_0 \cdots \alpha_p} \times \Delta^p$ denotes the inclusion.

We consider Čech bicomplex $\check{C}^{(*)}(\mathcal{U} ; A^*(M))$ : Let $\check{C}^{p,q} = \prod_{\alpha_0 \cdots \alpha_p} A^q(U_{\alpha_0 \cdots \alpha_p})$, and let $D' : \check{C}^{p,q} \to \check{C}^{p+1,q}$ denote the ordinary simplicial differential and $D'' = (-1)^p d : \check{C}^{p,q} \to \check{C}^{p,q+1}$ the de Rham differential. A multiplication $\cdot \check{C}^{p,q} \otimes \check{C}^{p',q'} \to \check{C}^{p+p',q+q'}$ is defined by

$$(\Phi \cdot \Phi')_{\alpha_0 \cdots \alpha_{p+q}} = (-1)^{p'q'} \Phi_{\alpha_0 \cdots \alpha_p}|_{U_{\alpha_0 \cdots \alpha_{p+q}}} \wedge \Phi'_{\alpha_{p+q} \cdots \alpha_{p+q}}|_{U_{\alpha_0 \cdots \alpha_{p+q}}}.$$ 

For $\check{C}^{(k)}(\mathcal{U} ; A^*(M)) = \sum_{p+q=k} \check{C}^{p,q}$ and $D = D' + D'' : \check{C}^{(k)} \to \check{C}^{(k+1)}$, we obtain a graded algebra $(\check{C}^{(*)}(\mathcal{U} ; A^*(M)), D, \cdot)$.

Let $I^*_{\mathfrak{gl}_q}$ denote a graded algebra of invariant polynomials on a Lie algebra $\mathfrak{gl}_q$. A characteristic homomorphism

$$\gamma : I^*_{\mathfrak{gl}_q} \to C^{(*)}(\mathcal{U} ; A^*(M))$$

is defined by

$$\gamma \varphi = \sum_{p} (\gamma \varphi)^{p,k-p} \varphi \in I^k_{\mathfrak{gl}_q}$$
where
\[(\gamma \varphi)_{\alpha_0 \ldots \alpha_p}^{p,2k-p} = \Delta_p \int \varphi(\Omega_{\alpha_0 \ldots \alpha_p}, \ldots, \Omega_{\alpha_0 \ldots \alpha_p}).\]

Then we obtain,

**Lemma 1** (Cf. [8]). For \(\varphi \in \mathbb{R}(g_{l_q})\),

\[(-1)^{\lfloor p(p-1)/2 \rfloor} \frac{(k-p)!}{k!} (\gamma \varphi)_{\alpha_0 \ldots \alpha_p}^{p,2k-p}.\]

\[
\begin{cases}
\Delta_p \int dt_1 \wedge \ldots \wedge dt_p \wedge \varphi(\omega_1 - \omega_0, \omega_2 - \omega_0, \ldots, \omega_p - \omega_0, \Gamma_{\alpha_0 \ldots \alpha_p}) \\
0
\end{cases}
\]

where \(\Gamma_{\alpha_0 \ldots \alpha_p} = \Omega_{\alpha_0 \ldots \alpha_p} - \sum_i dt_i \wedge (\omega_i - \omega_0).\)

**Remark.** \(\gamma \) induces the Chern-Weil homomorphism

\[\gamma^*: \mathbb{R}^*(g_{l_q}) \rightarrow \mathbb{H}^*(\mathbb{C}^*, D) \cong \mathbb{H}^*(M).\]

The following lemma is easily proved.

**Lemma 2.** Let \(w^1, \ldots, w^q\) be 1-forms on \(U_\alpha\) such that \(w^1 \wedge \ldots \wedge w^q \neq 0\) on \(U_\alpha\). Put \(w = w^1 \wedge \ldots \wedge w^q\). Then (i) and (ii) are equivalent:

(i) There exists a \((q, q)\)-matrix \((\varphi_i^j)\) of 1-forms on \(U_\alpha\) such that \(dw^i = \sum_j w^j \wedge \varphi_i^j\).

(ii) There exists a 1-form \(\eta\) on \(U_\alpha\) such that \(dw = w \wedge \eta\).

**Remark.** The existence of the matrix \((\varphi_i^j)\) doesn’t depend on choice of \(q\) 1-forms \(w^1, \ldots, w^q\) on \(U_\alpha\).

**Remark.** In the proof of this lemma, we obtain

\[\eta = (-1)^q \sum_i \varphi_i^i.\]  

(1)

Let \(\Gamma(\cdot)\) denote the space of all sections of bundle. The Bott connection \(\tilde{\nabla}: \Gamma(E) \times \Gamma(Q) \rightarrow \Gamma(Q)\) is defined by

\[\tilde{\nabla}_X Z = \pi_*([X, \tilde{Z}]) \quad X \in \Gamma(E), Z \in \Gamma(Q)\]

where \(\tilde{Z} \in \Gamma(TM)\) such that \(\pi_* (\tilde{Z}) = Z\) and \(\pi: TM \rightarrow Q\). Let
\{e_i, e_a\} (1 \leq i \leq q, q + 1 \leq a \leq n) be a local basis dual to \{w^i, w^a\} on U_\alpha satisfying \( e_i \in \Gamma(\mathcal{E}|U_\alpha) \) and \( e_a \in \Gamma(\mathcal{E}|U_\alpha) \) with respect to the isomorphism \( TM \cong 
abla_\mathcal{E} \oplus \mathcal{Q} \). Hereafter, we suppose that the indices run the following ranges: \( 1 \leq i, j, k, \ldots \leq q, q + 1 \leq a, b, \ldots \leq n \).

We define a connection \( \nabla^\alpha \) on \( U_\alpha \) by

\[
\nabla^\alpha_X Z = \tilde{\nabla}_{X_E} Z + \sum_i X_Q(Z^i)e_i + \sum_{i,k} Z^i \varphi_i^k(X_Q)e_k
\]

(2)

where \( X = X_E + X_Q \in \Gamma(\mathcal{E}|U_\alpha) \oplus \Gamma(\mathcal{Q}|U_\alpha) \) and \( Z = \sum_i Z^i e_i \in \Gamma(\mathcal{Q}|U_\alpha) \).

We put \( \nabla^\alpha_X e_i = \sum_i \omega^i_{aj}(X)e_i \), that is, \( \omega^i_{aj} \) denotes the connection form of \( \nabla^\alpha \) on \( U_\alpha \).

**Lemma 3.** \( \omega^i_{aj} = \varphi_i \) on \( U_\alpha \).

**Proof.** We put \( \tilde{\nabla}_{X_E} e_j = \sum_i \tilde{\omega}^i_{aj}(X_E)e_i \), then

\[
\omega^i_{aj}(X) = \tilde{\omega}^i_{aj}(X_E) + \varphi_j(X_Q).
\]

Now we obtain

\[
d\omega^i (e_a, e_j) = \frac{1}{2} \{ e_a(w^i(e_j)) - e_j(w^i(e_a)) - w^i([e_a, e_j]) \}
\]

\[
= -\frac{1}{2} \tilde{\omega}^i_{aj}(e_a).
\]

On the other hand,

\[
d\omega^i (e_a, e_j) = \left( \sum_k w^k \wedge \varphi^i_k \right) (e_a, e_j)
\]

\[
= -\frac{1}{2} \varphi_i(e_a).
\]

Thus we obtain \( \tilde{\omega}^i_{aj}(X_E) = \varphi_i(X_E) \). Therefore, for any \( X \in \Gamma(TM|U_\alpha) \), \( \omega^i_{aj}(X) = \varphi_i(X_E) + \varphi_j(X_Q) = \varphi_j(X) \). Q.E.D.

**Lemma 4.** If we consider the connection \( \nabla^{\alpha_0} \) defined by (2) on \( U_{\alpha_0} \) and a Riemannian connection \( \nabla^{\alpha_1} \) on \( U_{\alpha_1} \), then, for \( \varphi_1 \in T^1(\mathfrak{g}_\mathcal{F}(q), (\gamma \varphi_1)^{1,1}_{\alpha_0 \alpha_1} = ((-)^{d-1/2}\pi) \eta. \)
Proof. — By Lemma 1,
\[(\gamma \varphi_1)^{1,1}_{a_0 a_1} = \Delta_1 \int dt_1 \wedge \varphi_1 (\omega_{a_1} - \omega_{a_0}) = \varphi_1 (\omega_{a_1} - \omega_{a_0}).\]
Since \(\varphi_k \in \Gamma^k (g I_q)\) is defined by
\[
\det (\lambda I_q - (1/2\pi) X) = \sum_k \varphi_k (X) \lambda^{q-k}, \quad X \in g I_q \quad \text{and} \quad \text{trace} (\omega_{a_1}^i) = 0,
\]
we obtain \((\gamma \varphi_1)^{1,1}_{a_0 a_1} = (1/2\pi) \text{trace} (\omega_{a_0})\). By (1) and Lemma 3,
\[(\gamma \varphi_1)^{1,1}_{a_0 a_1} = ((-1)^{q-1}/2\pi) \eta. \quad \text{Q.E.D.}\]

From this lemma, we obtain

**Theorem A.** — \(((1/2\pi) \eta\) defines a D-cohomology class in \(H^2 (\check{C}^0 (\mathfrak{U}; A^* (M)), D)\) depending only on \(\mathfrak{F}\).**

**Proof.** — If \(\gamma \varphi_1 \in \check{C}^{(2)} (\mathfrak{U}; A^* (M))\) is D-closed, then particular object \((\gamma \varphi_1)^{1,1}_{a_0 a_1}\) is D-closed and, by Lemma 4,
\[
((1/2\pi) \eta \in \check{C}^{(2)} (\mathfrak{U}; A^* (M)) \) define a D-cohomology class in \(H^2 (\check{C}^0 (\mathfrak{U}; A^* (M)), D)\). Now we prove that \(\gamma \varphi_1\) is D-closed.
\[
(D(\gamma \varphi_1))^{1,2}_{a_0 a_1} = (D'(\gamma \varphi_1) + D''(\gamma \varphi_1))^{1,2}_{a_0 a_1} = (D' (\gamma \varphi_1))^{1,2}_{a_0 a_1} + (D'' (\gamma \varphi_1))^{1,2}_{a_0 a_1} = ((\gamma \varphi_1)^{0,2}_{a_0} - (\gamma \varphi_1)^{0,2}_{a_0}) + (-1) (d(\gamma \varphi_1))^{1,2}_{a_0 a_1}.
\]
From Stokes' theorem,
\[
(-1) d \cdot \Delta_1 \int \varphi_1 (\Omega_{a_0 a_1}) = \Delta_1 \int d \varphi_1 (\Omega_{a_0 a_1}) - \int_{\partial \Delta_1} j^* \varphi_1 (\Omega_{a_0 a_1})
\]
and the left side of this is equal to \((-1) d \cdot (\gamma \varphi_1)^{1,1}_{a_0 a_1}\), the first term of the right side vanishes and the second term of the right side is equal to \(\varphi_1 (\Omega_{a_1}) - \varphi_1 (\Omega_{a_0})\). Thus \(d \cdot (\gamma \varphi_1)^{1,1}_{a_0 a_1} = \varphi_1 (\Omega_{a_1}) - \varphi_1 (\Omega_{a_0})\).
From this and \((\gamma \varphi_1)^{0,2}_{a_0} = \varphi_1 (\Omega_{a_0})\), we obtain
\[
(D(\gamma \varphi_1))^{1,2}_{a_0 a_1} = \{\varphi_1 (\Omega_{a_1}) - \varphi_1 (\Omega_{a_0})\} + (-1) \{\varphi_1 (\Omega_{a_1}) - \varphi_1 (\Omega_{a_0})\}
\]
\[= 0.\]
From \((\gamma \varphi_1)^{0,2}_{a_0} = \varphi_1 (\Omega_{a_0})\) and \(d \circ \varphi_1 (\Omega_{a_0}) = 0\), we obtain
\[
(D(\gamma \varphi_1))^{0,3}_{a_0} = (D'' (\gamma \varphi_1))^{0,3}_{a_0} = (-1)^0 (d(\gamma \varphi_1))^{0,3}_{a_0} = 0.
\]
Now, from lemma 1, \((\gamma \phi_1) \in \check{\mathcal{C}}^{0,2} + \check{\mathcal{C}}^{1,1}\). Thus we obtain
\[
(D(\gamma \phi_1))^{3,0}_{a_0 a_1 a_2 a_3} = 0
\]
and
\[
(D(\gamma \phi_1))^{2,1}_{a_0 a_1 a_2} = (D'(\gamma \phi_1))^{2,1}_{a_0 a_1 a_2}
= \varphi_1(\omega_{a_2}) - \varphi_1(\omega_{a_1}) - \varphi_1(\omega_{a_2}) + \varphi_1(\omega_{a_0})
+ \varphi_1(\omega_{a_1}) - \varphi_1(\omega_{a_0})
= 0.
\]
Therefore we obtain \(D(\gamma \phi_1) = 0\). Q.E.D.

3. For \(q\) 1-forms \(w^1, \ldots, w^q\) on \(U_\alpha\) \((w^1 \wedge \ldots \wedge w^q \neq 0)\) we may choose \(n-q\) 1-forms \(w^{q+1}, \ldots, w^n\) on \(U_\alpha\) such that
\[
w^1 \wedge \ldots \wedge w^q \wedge w^{q+1} \wedge \ldots \wedge w^n \neq 0\] on \(U_\alpha\).
Thus we obtain expressions
\[
\varphi_j^a = \sum_k \varphi_{jk}^i w^k + \sum_a \varphi_{ja}^i w^a,
\]
and
\[
\eta = \sum_k \eta_k w^k + \sum_a \eta_a w^a.
\]
Using same letters \(\varphi_j^a, \eta\) to simplify, we put
\[
\varphi_j^a = \sum_a \varphi_{ja}^i w^a \quad \text{and} \quad \eta = \sum_a \eta_a w^a
\] (3)
on \(U_\alpha\).

Hereafter, we suppose that the manifold \(M\) admits foliations \(\mathcal{F}, \mathcal{F}'\) complementally transversal to each other and that \(\mathcal{F}\) (resp. \(\mathcal{F}'\)) is of codimension \(q\) (resp. \(n-q\)). Then we may consider that \(w^1, \ldots, w^q\) are defined by \(\mathcal{F}\) and that \(w^{q+1}, \ldots, w^n\) are defined by \(\mathcal{F}'\).

Let 1-forms \(\bar{w}^i, \bar{w}^a, \bar{\varphi}_j^i \), \(\bar{\eta}\) on \(U_{\alpha_1}\) correspond to 1-forms \(w^i, w^a, \varphi_j^i, \eta\) on \(U_{\alpha_0}\) respectively. Then we obtain

**Lemma 5.** - On \(U_{\alpha_0} \cap U_{\alpha_1} (\neq \emptyset)\), \(\bar{\eta} = \eta\).

**Proof.** - On \(U_{\alpha_0} \cap U_{\alpha_1}\), we may put
\[
\bar{w}^i = \sum_j t_j^i w^l, \quad \bar{w}^a = \sum_b t_b^a w^b.
\] (4)
Proof. - We take \( \eta_{\alpha_0} \) (resp. \( \eta_{\alpha_1} \)) for \( \eta \) on \( U_{\alpha_0} \) (resp. \( \eta \) on \( U_{\alpha_1} \)). Then we obtain
\[
\eta_{\alpha_1} - \eta_{\alpha_0} = 0 \quad \text{on} \quad U_{\alpha_0} \cap U_{\alpha_1},
\]
and
\[
(D' \eta)^{1,1}_{\alpha_0 \alpha_1} = \eta_{\alpha_1} - \eta_{\alpha_0} = 0.
\]
Thus \( \eta \) is \( D' \)-closed. Q.E.D.

BIBLIOGRAPHY


Manuscrit reçu le 12 avril 1977
Proposé par G. Reeb.

H. Kitahara and S. Yorozu,
Department of Mathematics
College of Liberal Arts
Kanazawa University
Kanazawa 920, Japan.