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ON THE ČECH BICOMPLEX ASSOCIATED WITH FOLIATED STRUCTURES

by H. KITAHARA and S. YOROZU

1. We shall be in $C^\infty$-category. Let $M$ be a paracompact connected $n$-dimensional manifold with a foliation $\mathcal{F}$ of codimension $q$, and let $\mathcal{U} = \{U_\alpha\}$ be a simple covering of $M$ such that each $U_\alpha$ is a flat neighborhood with respect to $\mathcal{F}$. Then there exists a decomposable $q$-form $w = w^1 \wedge \ldots \wedge w^q$ on each $U_\alpha$ and, by Frobenius' theorem, there exists a 1-form $\eta$ on each $U_\alpha$ satisfying $dw = w \wedge \eta$, where $d$ denotes the exterior differentiation and $\wedge$ the exterior product. The 1-form $\eta$ is an interesting object; it is well known that $\eta \wedge (d\eta)^q$ defines a de Rham class in $H^{2q+1}(M, \mathbb{R})$ ([1], [2], [3]). Our aim is to show that $\eta$ itself defines a certain cohomology class, that is,

**Theorem A.** $((-1)^{q-1}/2\pi)\eta$ defines a $D$-cohomology class in $H^2(\overline{\mathcal{C}(*)}(\mathcal{U}; A^*(M)), D)$ depending only on $\mathcal{F}$.

The above theorem was announced in [4], where it contained misstatements.

**Theorem B.** Supposing $M$ admits foliations $\mathcal{F}, \mathcal{F}'$ complemen tally transversal to each other, $\eta$ defines a $D'$-cohomology class in $H^1(\overline{\mathcal{C}(*)}(\mathcal{U}; A^*(M)), D')$ (Cf. [5], [6], [7]).

2. Since $M$ has a foliation $\mathcal{F}$ of codimension $q$, the tangent bundle $TM$ of $M$ has an integrable subbundle $E$ with fibre dimension $n-q$. Let $Q = TM/E$ be a quotient bundle with fibre dimension $q$. Choosing a suitable Riemannian metric on $TM$, we obtain an isomorphism $TM \cong E \oplus Q$ (Whitney sum).
Then

\[ d\bar{w}^i = \sum_j dt_j^i \wedge w^i + \sum_j t_j^i \, dw^i \]

\[ = \sum_j \left( dt_j^i - \sum_k t_k^i \varphi_k^i \right) \wedge w^i, \]

and on the other hand

\[ d\bar{w}^i = \sum_k \bar{w}^k \wedge \bar{\varphi}_k^i \]

\[ = \sum_j \left( - \sum_k t_j^i \bar{\varphi}_k^i \right) \wedge w^i. \]

Thus

\[ - \sum_k t_j^i \bar{\varphi}_k^i = dt_j^i - \sum_k t_k^i \varphi_k^i + \sum_k f_{jk}^i \, w^k \]

where \( f_{jk}^i \) are functions on \( U_{\alpha_0} \cap U_{\alpha_1} \).

Let \( \begin{pmatrix} s_j^i & 0 \\ 0 & s_b^a \end{pmatrix} \) denote the inverse matrix of \( \begin{pmatrix} t_j^i & 0 \\ 0 & t_b^a \end{pmatrix} \).

Then

\[ \sum_{j,k} s_j^i t_j^k \bar{\varphi}_k^i = - \sum_j s_j^i \, dt_j^i + \sum_{j,k} s_j^i t_k^i \varphi_k^i - \sum_{j,k} s_j^i f_{jk}^i \, w^k \]

and we obtain

\[ \sum_i \bar{\varphi}_i^i = - \sum_{i,j} s_j^i \, dt_j^i + \sum_i \varphi_i^i - \sum_{i,j,k} s_j^i f_{jk}^i \, w^k. \]

From (4), \( dt_j^i = \sum_k t_{jk}^i \, w^k \). Thus, by (3), we obtain \( \sum_i \bar{\varphi}_i^i = \sum_i \varphi_i^i \).

Therefore we obtain \( \bar{\eta} = \eta \) on \( U_{\alpha_0} \cap U_{\alpha_1} \) and \( \eta \in \tilde{C}^{0,1} (\mathcal{U}; A^*(M)) \).

Q.E.D.

**Theorem B.** — *Supposing \( M \) admits foliations \( \mathcal{F}, \mathcal{F}' \) complementarily transversal to each other, \( \eta \) defines a \( D' \)-cohomology class in \( H^1 (\tilde{\mathcal{C}}(\mathcal{U}; A^*(M)), D') \) where \( \eta \) is defined by \( \mathcal{F} \).*
We may suppose that $U = \{U_\alpha\}$ is a simple covering such that each $U_\alpha$ is a locally trivial neighborhood of the bundle $Q \rightarrow M$.

Let $\nabla^a$ denote a local connection on $Q | U_\alpha$, and let $\omega^a$ (resp. $\Omega^a$) denote a connection form (resp. a curvature form) of $\nabla^a$ on $U_\alpha$. Let $\Delta^p$ be a canonical $p$-simplex in $R^{p+1}$ (with coordinates $(t_0, t_1, \ldots, t_p)$). We define a connection form $\omega_{a_0 \ldots a_p}$ on $U_{a_0 \ldots a_p} \times \Delta^p$ ($U_{a_0 \ldots a_p} = U_{a_0} \cap U_{a_1} \cap \ldots \cap U_{a_p}$) by

$$\omega_{a_0 \ldots a_p} = t_0 \omega_{a_0} + \ldots + t_p \omega_{a_p}$$

and let $\Omega_{a_0 \ldots a_p}$ denote a corresponding curvature form on $U_{a_0 \ldots a_p} \times \Delta^p$.

For the set $A^k(U_{a_0 \ldots a_p} \times \Delta^p)$ of all $k$-forms on $U_{a_0 \ldots a_p} \times \Delta^p$, $\Delta^p \int : A^k(U_{a_0 \ldots a_p} \times \Delta^p) \rightarrow A^{k-p}(U_{a_0 \ldots a_p})$ denotes the integration along the fibre. Then we obtain Stokes' theorem

$$\Delta^p \int \circ d = (-1)^p \circ \partial \int + \partial \Delta^p \int \circ j^*$$

where $j : U_{a_0 \ldots a_p} \times \partial \Delta^p \rightarrow U_{a_0 \ldots a_p} \times \Delta^p$ denotes the inclusion.

We consider Čech bicomplex $\check{C}^(*) (\mathcal{U} ; A^*(M)) :$ Let $\check{C}^{p,q} = \prod_{a_0 \ldots a_p} A^q(U_{a_0 \ldots a_p})$, and let $D' : \check{C}^{p,q} \rightarrow \check{C}^{p+1,q}$ denote the ordinary simplicial differential and $D'' = (-1)^p d : \check{C}^{p,q} \rightarrow \check{C}^{p,q+1}$ the de Rham differential. A multiplication $\check{C}^{p,q} \otimes \check{C}^{p',q'} \rightarrow \check{C}^{p+p',q+q'}$ is defined by

$$(\Phi \cdot \Phi')_{a_0 \ldots a_p} = (-1)^{q'q} \Phi_{a_0 \ldots a_p} | U_{a_0 \ldots a_p} \wedge \Phi'_{a_0 \ldots a_p} | U_{a_0 \ldots a_p} \wedge U_{a_0 \ldots a_{p+q}}.$$

For $\check{C}^k(\mathcal{U} ; A^*(M)) = \sum_{p+q=k} \check{C}^{p,q}$ and $D = D' + D''$, $\check{C}^k(\mathcal{U} ; A^*(M))$, we obtain a graded algebra $(\check{C}^(*) (\mathcal{U} ; A^*(M)), D, \cdot)$.

Let $I^*(gl_q)$ denote a graded algebra of invariant polynomials on a Lie algebra $gl_q$. A characteristic homomorphism

$$\gamma : I^*(gl_q) \rightarrow C^(*) (\mathcal{U} ; A^*(M))$$

is defined by

$$\gamma \varphi = \sum_{p} (\gamma \varphi)^{p,2k-p} \varphi \in I^k(gl_q)$$
where
\[(\gamma \varphi)^{p,2k-p}_{a_0\ldots a_p} = \Delta_p \int \varphi(\Omega_{a_0\ldots a_p}, \ldots, \Omega_{a_0\ldots a_p}).\]

Then we obtain,

**Lemma 1** (Cf. [8]). - For \( \varphi \in I^k(\mathfrak{gl}_q) \),

\[
(-1)^{p(p-1)/2} \frac{(k-p)!}{k!} (\gamma \varphi)^{p,2k-p}_{a_0\ldots a_p}
\]

\[
= \begin{cases}
\Delta_p \int dt_1 \wedge \ldots \wedge dt_p \wedge \varphi(\omega_1 - \omega_a, \omega_2 - \omega_a, \ldots, \omega_p - \omega_a, \Gamma_{a_0\ldots a_p}) & p \leq k \\
0 & p > k
\end{cases}
\]

where \( \Gamma_{a_0\ldots a_p} = \Omega_{a_0\ldots a_p} - \sum_i dt_i \wedge (\omega_a - \omega_a). \)

**Remark.** - \( \gamma \) induces the Chern-Weil homomorphism

\[\gamma^*: I^*(\mathfrak{gl}_q) \longrightarrow H^*(\mathcal{C}^*(\mathfrak{g}), D) \leftarrow H^*(M).\]

The following lemma is easily proved.

**Lemma 2.** - Let \( w^1, \ldots, w^q \) be 1-forms on \( U_\alpha \) such that \( w^1 \wedge \ldots \wedge w^q \neq 0 \) on \( U_\alpha \). Put \( w = w^1 \wedge \ldots \wedge w^q \). Then (i) and (ii) are equivalent:

(i) There exists a \((q,q)\)-matrix \((\varphi^j_i)\) of 1-forms on \( U_\alpha \) such that \( dw^i = \sum_j w^j \wedge \varphi^j_i. \)

(ii) There exists a 1-form \( \eta \) on \( U_\alpha \) such that \( dw = w \wedge \eta. \)

**Remark.** - The existence of the matrix \((\varphi^j_i)\) doesn't depend on choice of \( q \) 1-forms \( w^1, \ldots, w^q \) on \( U_\alpha \).

**Remark.** - In the proof of this lemma, we obtain

\[\eta = (-1)^{q-1} \sum_i \varphi^j_i. \]  \[\text{(1)}\]

Let \( \Gamma(\cdot) \) denote the space of all sections of bundle. The Bott connection \( \tilde{\nabla}: \Gamma(E) \times \Gamma(Q) \longrightarrow \Gamma(Q) \) is defined by

\[\tilde{\nabla}_X Z = \pi_*([X, \tilde{Z}]) \quad X \in \Gamma(E), Z \in \Gamma(Q),\]

where \( \tilde{Z} \in \Gamma(TM) \) such that \( \pi_* (\tilde{Z}) = Z \) and \( \pi: TM \longrightarrow Q \). Let
\{e_i, e_a\} (1 \leq i \leq q, q + 1 \leq a \leq n) be a local basis dual to \{w^i, w^a\} on \text{U}_\alpha satisfying \text{e}_i \in \Gamma(Q|_{\text{U}_\alpha}) and \text{e}_a \in \Gamma(E|_{\text{U}_\alpha}) with respect to the isomorphism \text{TM} \cong E \oplus Q. Hereafter, we suppose that the indices run the following ranges: 1 \leq i, j, k, \ldots \leq q, q + 1 \leq a, b, \ldots \leq n.

We define a connection \nabla^\alpha on \text{U}_\alpha by

\[ \nabla^\alpha X = \tilde{\nabla}_{X_E} Z + \sum_i X_Q (Z^i) e_i + \sum_{i,k} Z^i \varphi^k_i (X_Q) e_k \]  

where \(X = X_E + X_Q \in \Gamma(E|_{\text{U}_\alpha}) \oplus \Gamma(Q|_{\text{U}_\alpha})\) and \(Z = \sum_i Z^i e_i \in \Gamma(Q|_{\text{U}_\alpha})\).

We put \(\nabla^\alpha X e_j = \sum_i \omega^i_{\alpha j} (X) e_i\), that is, \(\omega^i_{\alpha j}\) denotes the connection form of \(\nabla^\alpha\) on \text{U}_\alpha.

**Lemma 3.** \(\omega^i_{\alpha j} = \varphi^i_j\) on \text{U}_\alpha.

**Proof.** We put \(\tilde{\nabla}_{X_E} e_j = \sum_i \tilde{\omega}^i_j (X_E) e_i\), then

\[ \omega^i_{\alpha j} (X) = \tilde{\omega}^i_j (X_E) + \varphi^i_j (X_Q). \]  

Now we obtain

\[ dw^i (e_a, e_j) = \frac{1}{2} \{ e_a \left( w^i (e_j) \right) - e_j \left( w^i (e_a) \right) - w^i ([e_a, e_j]) \} \]

\[ = -\frac{1}{2} \tilde{\omega}^i_j (e_a). \]

On the other hand,

\[ dw^i (e_a, e_j) = \left( \sum_k w^k \wedge \varphi^i_k \right) (e_a, e_j) \]

\[ = -\frac{1}{2} \varphi^i_j (e_a). \]

Thus we obtain \(\tilde{\omega}^i_j (X_E) = \varphi^i_j (X_E)\). Therefore, for any \(X \in \Gamma(TM|_{\text{U}_\alpha})\), \(\omega^i_{\alpha j} (X) = \varphi^i_j (X_E) + \varphi^i_j (X_Q) = \varphi^i_j (X)\). Q.E.D.

**Lemma 4.** If we consider the connection \(\nabla^{\alpha_0}\) defined by (2) on \text{U}_{\alpha_0} and a Riemannian connection \(\nabla^{\alpha_1}\) on \text{U}_{\alpha_1}, then, for \(\varphi_1 \in \Gamma^1 (g|_1), (\gamma \varphi_1)^{1,1}_{\alpha_0 \alpha_1} = ((-)^{q-1/2} \pi) \eta\).
Proof. — By Lemma 1,

\[(\gamma \varphi_1)_{\alpha_0 \alpha_1}^{1,1} = \Delta_1 \int dt_1 \wedge \varphi_1 (\omega_{\alpha_1} - \omega_{\alpha_0})\]

\[= \varphi_1 (\omega_{\alpha_1} - \omega_{\alpha_0}).\]

Since \(\varphi_k \in I^k (g I_q)\) is defined by

\[
det(\lambda I_q - (1/2 \pi i) X) = \sum_k \varphi_k (X) \lambda^{q-k}, \ X \in g I_q \text{ and } \text{trace} (\omega_{\alpha_1}^{\alpha_1}) = 0,
\]

we obtain \((\gamma \varphi_1)_{\alpha_0 \alpha_1}^{1,1} = (1/2 \pi i) \text{trace} (\omega_{\alpha_0}^{\alpha_0}).\) By (1) and Lemma 3,

\[\nonumber (\gamma \varphi_1)_{\alpha_0 \alpha_1}^{1,1} = ((-1)^{q-1}/2 \pi) \eta. \quad \text{Q.E.D.}\]

From this lemma, we obtain

**Theorem A.** \(((-1)^{q-1}/2 \pi) \eta\) defines a D-cohomology class in \(H^2 (\tilde{\mathcal{C}}(\eta; \mathcal{A}^*(M)), D)\) depending only on \(\mathcal{F}.\)

Proof. — If \(\gamma \varphi_1 \in \tilde{\mathcal{C}}^{(2)} (\eta; \mathcal{A}^*(M))\) is D-closed, then particular object \((\gamma \varphi_1)_{\alpha_0 \alpha_1}^{1,1}\) is D-closed and, by Lemma 4,

\[((-1)^{q-1}/2 \pi) \eta \in \tilde{\mathcal{C}}^{1,1} (\eta; \mathcal{A}^*(M))\) define a D-cohomology class in \(H^2 (\tilde{\mathcal{C}}(\eta; \mathcal{A}^*(M)), D)\). Now we prove that \(\gamma \varphi_1\) is D-closed.

\[
(D(\gamma \varphi_1))_{\alpha_0 \alpha_1}^{1,2} = (D'(\gamma \varphi_1) + D''(\gamma \varphi_1))_{\alpha_0 \alpha_1}^{1,2} = (D'(\gamma \varphi_1))_{\alpha_0 \alpha_1}^{1,2} + (D''(\gamma \varphi_1))_{\alpha_0 \alpha_1}^{1,2} = \{\gamma \varphi_1 \}_{\alpha_0 \alpha_1}^{1,2} - (\gamma \varphi_1)_{\alpha_0 \alpha_1}^{0,2} + (-1) (d(\gamma \varphi_1))_{\alpha_0 \alpha_1}^{1,2}.
\]

From Stokes' theorem,

\[
(-1) d \cdot \int_{\Delta_1} \varphi_1 (\Omega_{\alpha_0 \alpha_1}) = \Delta_1 \int \varphi_1 (\Omega_{\alpha_0 \alpha_1}) - \sum \varphi_1 (\Omega_{\alpha_0 \alpha_1}) - \int_{\partial \Delta_1} j^* \varphi_1 (\Omega_{\alpha_0 \alpha_1})
\]

and the left side of this is equal to \((-1) d \cdot (\gamma \varphi_1)_{\alpha_0 \alpha_1}^{1,1}\), the first term of the right side vanishes and the second term of the right side is equal to \(\varphi_1 (\Omega_{\alpha_1}) - \varphi_1 (\Omega_{\alpha_0})\). Thus \(d \cdot (\gamma \varphi_1)_{\alpha_0 \alpha_1}^{1,1} = \varphi_1 (\Omega_{\alpha_1}) - \varphi_1 (\Omega_{\alpha_0}).\)

From this and \((\gamma \varphi_1)_{\alpha_0}^{0,2} = \varphi_1 (\Omega_{\alpha_0})\), we obtain

\[
(D(\gamma \varphi_1))_{\alpha_0 \alpha_1}^{1,2} = \{\varphi_1 (\Omega_{\alpha_1}) - \varphi_1 (\Omega_{\alpha_0})\} + (-1) \{\varphi_1 (\Omega_{\alpha_1}) - \varphi_1 (\Omega_{\alpha_0})\}
\]

\[= 0.\]

From \((\gamma \varphi_1)_{\alpha_0}^{0,2} = \varphi_1 (\Omega_{\alpha_0})\) and \(d \circ \varphi_1 (\Omega_{\alpha_0}) = 0\), we obtain

\[
(D(\gamma \varphi_1))_{\alpha_0}^{0,3} = (D''(\gamma \varphi_1))_{\alpha_0}^{0,3} = (-1)^0 (d(\gamma \varphi_1))_{\alpha_0}^{0,3} = 0.
\]
Now, from lemma 1, \((\gamma \varphi_1) \in \mathfrak{C}^{0,2} + \mathfrak{C}^{1,1}\). Thus we obtain
\[
(D(\gamma \varphi_1))_{a_0 a_1 a_2 a_3}^{3,0} = 0
\]
and
\[
(D(\gamma \varphi_1))_{a_0 a_1 a_2}^{2,1} = (D'(\gamma \varphi_1))_{a_0 a_1 a_2}^{2,1}
= \varphi_1(\omega_{a_2}) - \varphi_1(\omega_{a_1}) - \varphi_1(\omega_{a_2}) + \varphi_1(\omega_{a_0})
+ \varphi_1(\omega_{a_1}) - \varphi_1(\omega_{a_0})
= 0.
\]
Therefore we obtain \(D(\gamma \varphi_1) = 0\). Q.E.D.

3. For \(q\) 1-forms \(w^1, \ldots, w^q\) on \(U_\alpha\) \((w^1 \wedge \ldots \wedge w^q \neq 0)\) we may choose \(n-q\) 1-forms \(w^{q+1}, \ldots, w^n\) on \(U_\alpha\) such that
\[
w^1 \wedge \ldots \wedge w^q \wedge w^{q+1} \wedge \ldots \wedge w^n \neq 0 \quad \text{on} \quad U_\alpha.
\]
Thus we obtain expressions
\[
\varphi^i_j = \sum_k \varphi^i_{jk} w^k + \sum_a \varphi^i_a w^a,
\]
\[
\eta = \sum_k \eta_k w^k + \sum_a \eta_a w^a.
\]
Using same letters \(\varphi^i_j, \eta\) to simplify, we put
\[
\varphi^i_j = \sum a \varphi^i_{ja} w^a \quad \text{and} \quad \eta = \sum a \eta_a w^a \quad (3)
\]
on \(U_\alpha\).

Hereafter, we suppose that the manifold \(M\) admits foliations \(\mathcal{F}, \mathcal{F}'\) complementally transversal to each other and that \(\mathcal{F}\) (resp. \(\mathcal{F}'\)) is of codimension \(q\) (resp. \(n-q\)). Then we may consider that \(w^1, \ldots, w^q\) are defined by \(\mathcal{F}\) and that \(w^{q+1}, \ldots, w^n\) are defined by \(\mathcal{F}'\).

Let 1-forms \(\bar{w}^i, \bar{w}^a, \bar{\varphi}^i_j, \bar{\eta}\) on \(U_{\alpha_1}\) correspond to 1-forms \(w^i, w^a, \varphi^i_j, \eta\) on \(U_{\alpha_0}\) respectively. Then we obtain

**Lemma 5.** - On \(U_{\alpha_0} \cap U_{\alpha_1} (\neq \phi)\), \(\bar{\eta} = \eta\).

**Proof.** - On \(U_{\alpha_0} \cap U_{\alpha_1}\), we may put
\[
\bar{w}^i = \sum_j t^i_j w^j, \quad \bar{w}^a = \sum_b t^a_b w^b. \quad (4)
\]
Proof. - We take $\eta_{a_0}$ (resp. $\eta_{a_1}$) for $\eta$ on $U_{a_0}$ (resp. $\eta$ on $U_{a_1}$). Then we obtain
$$
\eta_{a_1} - \eta_{a_0} = 0 \quad \text{on} \quad U_{a_0} \cap U_{a_1},
$$
and
$$
(D' \eta)_{a_0 a_1}^{1,1} = \eta_{a_1} - \eta_{a_0} = 0.
$$
Thus $\eta$ is $D'$-closed. Q.E.D.

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