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<http://www.numdam.org/item?id=AIF_1978__28_3_203_0>
RADON-NIKODYM PROPERTY
FOR VECTOR-VALUED INTEGRABLE FUNCTIONS

by Surjit Singh KHURANA

It is proved in ([6], Theorem 1) that if a Banach space E possesses Radon-Nikodym (R – N) property, then the Banach space $L_p(E, \lambda)$, $1 < p < \infty$, of Bochner $p$-integrable functions also possesses this property. In this paper we give a new proof of the corresponding result when $E$ is a Frechet space (i.e. a complete metrizable locally convex space [5]).

Let $(Y, \mathcal{B}, \mu)$ be a positive measure which is non-trivial (i.e., there exists a $B \in \mathcal{B}$ such that $0 < \mu(B) < \infty$) with $\mathcal{R} = \{A \in \mathcal{B}: (2(A)) < \infty\}$, $E$ a Frechet space with $\{P_i\}$ an increasing sequence of semi-norms on $E$ generating the topology of $E$, and $L_p(E, \lambda)$ the equivalence classes of strongly $p$-power integrable functions $Y \rightarrow E$, $1 < p < \infty$.

A function $f: X \rightarrow E$ is called strongly $p$-power integrable if there exists a sequence $\{f_n\}$ of $\mathcal{R}$-simple $E$-valued functions of $Y$ such that (i) $f_n \rightarrow f$ a.e. $[\nu]$, and (ii) $\int [P_i(f_n - f)]^p d\nu \rightarrow 0$, $\forall i$. The increasing sequence of semi-norms

$$N_{i,p}, N_{i,p}(f) = \left[ \int [P_i(f)]^p d\nu \right]^{1/p}$$

makes $L_p(E, \lambda)$ a Frechet space. We use the definition of [4] for a Frechet space to have $R – N$ property.

$E = K$, we denote $L_p(E, \nu)$ by $L_p(\nu)$ and the corresponding norm by $\| \cdot \|_p$.

**Theorem.** Suppose $E$ is a Frechet space with $R – N$ property and $(Y, \mathcal{B}, \nu)$ a non-trivial positive measure space. Then $L_p(E, \nu)$ has $R – N$ property for $1 < p < \infty$. 
Proof. — Using ([3], Theorem 5 (iv)) it is sufficient to prove the R – N property for every separable closed subspace; this means we can assume that E is separable ([3], Theorem 5). Let (X, \mathcal{U}, \lambda) be a finite measure space, \mu : \mathcal{U} \rightarrow L_p(E, \nu) a measure of finite variation (i.e., \forall i, the variation of \mu relative to N_{p,i} is finite, [4]), absolutely continuous with respect to \lambda. Assume first that \nu(Y) < \infty and let \lambda \times \nu be the product of \lambda and \nu on the \sigma-algebra \mathcal{U} \times \mathcal{B}.

For an A \in \mathcal{U}, B \in \mathcal{B}, define \omega(A \times B) = \int_B \mu(A) d\nu \in E (since \nu(Y) < \infty, P_i(\mu(A)) \in L_p(\nu), \forall i, implies P_i(\mu(A)) \in L_i(\nu)). Take \{A_i \times B_i\} a disjoint sequence in X \times Y (A_i \in \mathcal{U}, B_i \in \mathcal{B}) and let \bigcup A_i \times B_i = A \times B (A \in \mathcal{U}, B \in \mathcal{B}). Fix an f \in E'. f \circ \mu : \mathcal{U} \rightarrow L_p(\nu) is of bounded variation and absolutely continuous relative to \lambda. Since L_p(\nu) has R – N property, there exists a function \phi : X \times Y \rightarrow K such that

\[
\int_A \phi(x, y) d\lambda(x), \forall A \in \mathcal{U};
\]
it is routine verification that \phi(x, y) \in L_1(\lambda \times \nu). Thus

\[
\int_{B_i} f \circ \mu(A_i) d\nu = \int_{A_i \times B_i} \phi(x, y) d(\lambda \times \nu)
\]
(Fubini's theorem) and so

\[
\sum \int_{B_i} f \circ \mu(A_i) d\nu = \int_B f \circ \mu(A) d\nu
\]
(unconditional convergence). Since (f, \int_{B_i} \mu(A_i) d\nu) = \int_{B_i} f \circ \mu(A_i) d\nu,

\[
\sum \int_{B_i} \mu(A_i) d\nu = \int_B \mu(A) d\nu.
\]
Also for a finite disjoint collection \{C_i \times D_i\} in X \times Y (C_i \in \mathcal{U}, D_i \in \mathcal{B}),

\[
\sum \int_{D_i} f \circ \mu(C_i) d\nu = \int_{\bigcup C_i \times D_i} \phi(x, y) d(\lambda \times \nu) \quad \text{(previous notation)}
\]
and

\[
|f \circ \sum \int_{D_i} \mu(C_i) d\nu| \leq \int |\phi(x, y)| d(\lambda \times \nu).
\]
Combining these results we see that \omega can be uniquely extended to a finitely additive set function \omega : \theta \rightarrow E, \theta being the algebra generated by \{A \times B : A \in \mathcal{U}, B \in \mathcal{B}\}. \omega is countably additive, and \omega(\theta) is bounded in E. Since E has R – N property it cannot contain a subspace isomorphic to c_0 ([11]; [3], Theorem 5). From this it easily follows that \omega is exhaustive ([2], II; [7], Theorem 4). Thus \omega can be uniquely extended to a countably additive measure on the \sigma-algebra \mathcal{U} \times \mathcal{B} ([2], III). We claim that \omega \ll \lambda \times \nu. For an f \in E', A \in \mathcal{U}, B \in \mathcal{B}, f \circ \omega(A \times B) = \int_{A \times B} \phi(x, y) d(\lambda \times \nu)
rious notations) and so \( f \circ \omega(H) = \int_H \phi d(\lambda \times \nu), \quad \forall H \in \mathcal{U} \times \mathcal{B}. \)

If \((\lambda \times \nu)(H) = 0\) we get \( f \circ \omega(H) = 0 \) and so \( \omega(H) = 0 \).

We now prove that \( \omega \) is of finite variation. Fix \( i \in \mathbb{N} \) and let \( \lambda_0 = \text{the finite variation of } \mu \text{ relative to the semi-norm } N_{i,p}. \)

\( H = \{ f \in \mathcal{E}', |f(x)| \leqslant P_i(x), \forall x \in E \} \)
is a metrizable compact subset of \((\mathcal{E}', \sigma(\mathcal{E}', \mathcal{E}))\) and so have a countable dense subset \( \{ f_j \} \). Let \( \phi_j \in L_1(\lambda_0 \times \nu) \) such that
\[
f_j \circ \mu(A) = \int_A \phi_j(x, y) d\lambda_0(x)
\]
(same reasoning as before).

Let \( \varphi_0 = \sup (|\varphi_1|, |\varphi_2|) \) and fix \( x \in X \). Take
\( B_1 = \{ y \in Y : |\varphi_1(x, y)| = \varphi_0(x, y) \} \) and \( B_2 = Y \setminus B_1 \).

We claim the variation of \( \xi = \chi_{B_1} f_1 \circ \mu + \chi_{B_2} f_2 \circ \mu \), in \( L_p(\nu) \), does not exceed \( \lambda_0 \).

From
\[
|\xi(A)|^p = |(\chi_{B_1} f_1 \circ \mu + \chi_{B_2} f_2 \circ \mu)(A)|^p \\
\leqslant \chi_{B_1} (P_i(\mu(1)))^p + \chi_{B_2} (P_i(\mu(A)))^p = (P_i(\mu(A)))^p,
\]
we get \( \| \xi(A) \| _p \leqslant N_{i,p}(\mu(A)) \leqslant \lambda_0(A) \) and so the claim is established.

Now \( \xi(A) = \int_A (\chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2) d\lambda_0 \). If \( |\xi| \) is the variation of \( \xi \) relative to \( L_p(\nu) \), then \( |\xi| (A) = \int_A \| \chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2 \| _p d\lambda_0 \).

If \( \| \chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2 \| _p \geqslant 1 + \eta \) for some \( \eta > 0 \) on \( A \in \mathcal{U} \), then \( \lambda_0(A) \geqslant |\xi| (A) \geqslant (1 + \eta) \lambda_0(A) \) which means \( \lambda_0(A) = 0 \) and so \( \| \chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2 \| _p \leqslant 1 \) a.e. \( [\lambda_0] \).

Now
\[
\| \chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2 \| _1 \leqslant \| \chi_{B_1} \varphi_1 + \chi_{B_2} \varphi_2 \| _p (\lambda_0(X))^{1/q} \leqslant (\lambda_0(X))^{1/q}
\]
(Holder's inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \)) means
\[
\int \varphi_0(x, y) d\nu(y) \leqslant (\lambda_0(X))^{1/q}, \quad \text{a.e. } [\lambda_0].
\]
Since \( x \in X \) was arbitrary we see \( \varphi_0(x, y) \in L_1(X \times Y, \lambda_0 \times \nu) \).

If \( \varphi = \sup (|\varphi_1|, |\varphi_2|, \ldots) \), then proceeding in the same way we prove that \( \int \varphi(x, y) d\nu(y) \leqslant (\lambda_0(X))^{1/q}, \quad \text{a.e. } [\lambda_0] \) and so \( \varphi \in L_1(X \times Y, \lambda_0 \times \nu) \) (the set where \( \varphi \) takes values \( +\infty \) has measure zero; we put \( \varphi \equiv 0 \) on that set). Fix \( \epsilon > 0 \) and let \( \{ H_j \} \) be a finite disjoint collection in \( \mathcal{U} \times \mathcal{B} \).

\[
\sum_l P_f(\omega(H_j)) = \epsilon \leqslant \sum_j [f_k(j) \circ \omega(H_j)] \leqslant \sum_j \int_{H_j} |f_k(j)| d(\lambda_0 \times \nu) \\
\leqslant \sum_j \int_{H_j} \varphi d(\lambda_0 \times \nu) \leqslant \int \varphi d(\lambda_0 \times \nu)
\]
for some finite sequence \( \{k(j)\} \subseteq \mathbb{N} \). This proves \( \omega \) is of finite variation. Since \( E \) has \( R - N \) property we get a \( g \in L_1(E, \lambda \times \nu) \) such that \( \int_B \mu(A) d\nu = \int_B \int_A g(x, y) d\lambda(x) d\nu(y) \).

Put \( \psi = \mu(A) - \int_A g(x, y) d\lambda(x) \). We get \( \int_B \psi d\nu = 0 \), \( \forall B \in \mathcal{B} \). Fix \( i \in \mathbb{N} \) and let \( \{f_j\} \) be a countable dense set in the compact metric space \( H = \{f \in E' : |f(x)| \leq P_f(x), \forall x \in E \} \subseteq (E', \sigma(E', E)) \).

We get \( \int_B f_j \circ \psi = 0 \), \( \forall B \in \mathcal{B} \) and so \( P_f(\psi) = 0 \) a.e. \( [\nu] \). Thus \( \psi = 0 \) a.e. \( [\nu] \). Thus \( \mu(A) = \int_A g(x, y) d\lambda(x) \). It is easy to verify that \( g(\cdot, x) \in L_1(L_p(E, \nu), \lambda) \).

Now we consider the case when \( \nu(Y) = +\infty \). By ([3], Theorem 5) it is enough to prove the result for every closed separable subspace of \( L_p(E, \nu) \). Let \( F \) be a closed separable subspace of \( L_p(E, \nu) \). It is a simple verification that there exists a \( B \in \mathcal{B} \) with \( \sigma \)-finite \( \nu \)-measure such that \( f = 0 \) a.e. \( [\nu] \) outside \( B \), \( \forall f \in F \). Thus ([3], Theorem 5) we can assume that \( \nu \) is \( \sigma \)-finite. Let \( \{K_n\} \) be a \( \mathcal{B} \)-measurable partition of \( Y \), such that \( 0 < \nu(K_n) < \infty \), \( \forall n \).

Define \( \nu_n = \chi_{K_n} \nu \), \( \nu_n : \mathcal{B}_n \to [0, \infty) \), \( \mathcal{B}_n = \mathcal{B} \cap K_n \). Given \( \mu \) as before, we get \( \mu_n : \mathcal{U} \to L_p(E, \nu_n) \), \( \mu_n(A) = \chi_{K_n} \mu(A) \in L_p(E, \nu_n) \). It is easy to verify that \( \mu_n \) is of finite variation relative to \( L_p(E, \nu_n) \) and absolutely continuous relative to \( \lambda \). Proceeding as before we get \( g_n : X \times K_n \to E \) such that

\[
\mu_n(A) = \int_A g_n(x, y) d\lambda(x) = \int_A \chi_{K_n} g_n(x, y) d\lambda(x).
\]

Define \( g(x, y) = g_n(x, y) \), \( y \in K_n \), we claim

\[
\mu(A) = \int_A g(x, y) d\lambda(x).
\]

If \( \mu(A) = f \in L_p(E, \nu) \), then

\[
(P_f(\mu(A) - \sum_{j=1}^k \mu_j(A)))^p \leq (P_f(f))^p
\]

and so by dominated convergence theorem \( \sum_{j=1}^k \mu_j(A) \) converges to \( \mu(A) \) in \( L_p(E, \nu) \). Let \( |\mu| \) and \( \sum_{j=1}^k \mu_j \) be the variations of \( \mu \) and \( \sum_{j=1}^k \mu_j \) relative to \( N_{t,p} \). Then
\[ |\mu|(A) \geq \left| \sum_{j=1}^{n} \mu_j \right|(A) = \int_A \sum_{j=1}^{n} N_{i,p} \left( \sum_{K_j} g_j \right) d\lambda. \]

By monotone convergence theorem \( N_{i,p}(g) < \infty, \) a.e. \([\lambda]\) and \( \int N_{i,p}(g) d\lambda < \infty. \) On the set where \( N_{i,p}(g) = +\infty \) we change its value and the value of each of \( g_n \) to 0 and so \( g(\cdot, x) \in L_p(E, \nu), \forall x \in X. \) Now it is easy to verify that \( h_n = \sum_{j=1}^{n} \chi_{K_j} g_j \) converges to \( g \) in \( L_p(E, \nu), \) a.e. \([\lambda]\) and \( N_{i,p}(g - h_n), \) as a function of \( x, \) is decreasing as \( n \) increases. By monotone convergence theorem, \( \int N_{i,p}(g - h_n) d\lambda \longrightarrow 0. \) Thus
\[
N_{i,p} \left( \int_A g d\lambda - \int_A h_n d\lambda \right) \leq \int N_{i,p}(g - h_n) d\lambda \longrightarrow 0
\]
and so \( \int_A h_n d\lambda \longrightarrow \int_A g d\lambda \) in \( L_p(E, \nu). \) But
\[
\int_A h_n d\lambda = \sum_{j=1}^{n} \mu_j(A) \longrightarrow \mu(A)
\]
and so \( \mu(A) = \int_A g d\lambda. \) The result now follows easily.

**BIBLIOGRAPHY**


Manuscrit reçu le 21 septembre 1977
Proposé par G. Choquet.

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