Annales de l'institut Fourier

Thomas-William Korner

On the theorem of Ivasev-Musatov. II

Annales de l'institut Fourier, tome 28, nº 3 (1978), p. 123-142

<http://www.numdam.org/item?id=AIF_1978__28_3_123_0>

Ann. Inst. Fourier, Grenoble
28, 3 (1978), 123-142

ON THE THEOREM OF IVAŠEV-MUSATOV. II

by T.W. KORNER

1. Introduction.

I shall assume an acquaintance with the notation and general spirit of [1]. Our object is to prove the following two complementary results.

THEOREM 1.1. – *Suppose* $\varphi(n)$ *is a decreasing positive sequence such that*

(A) $\sum_{n=1}^{\infty} (\varphi(n))^2$ *diverges.*

(B) *There exists a* $K \geqslant 1$ *such that* $\varphi(n) \leqslant K\varphi(r)$ *for all* $n \leqslant r \leqslant 2n$ *and all* $n \geqslant 1$.

Then we can find a positive measure $\mu \neq 0$ *with support* E *of Lebesgue measure zero yet with*

$$|\hat{\mu}(n)| = 0(\varphi(|n|)) \quad as \quad |n| \longrightarrow \infty .$$

THEOREM 1.2. – *We can find a decreasing positive convex sequence* $\varphi(n)$ *such that*

(A) $\sum_{n=1}^{\infty} (\varphi(n))^2$ *diverges*

but if μ *is a non zero measure with* $|\hat{\mu}(n)| = 0(\varphi(|n|))$ *as* $|n| \longrightarrow \infty$ *then the support of* μ *is the whole circle* $\mathbf{T}$.

The proof of Theorem 1.1 uses the same ideas as the proof in [1] and the extra complications are of a technical nature. On the other hand the proof of Theorem 1.2 is easy and quite instructive. We shall therefore begin with it.

2. The Counter Example.

There is no extra difficulty in proving the following slightly stronger result.

THEOREM 1.2$'$. – *Let* $q : (0,\infty) \longrightarrow \mathbf{R}$ *be an increasing strictly positive function. Then we can find a decreasing positive convex sequence* $\varphi(n)$ *such that*

(A) $\sum_{n=1}^{N} q(\varphi(n))$ *diverges*

but if S *is a non zero distribution with* $|\hat{S}(n)| = 0(\varphi(|n|))$ *as* $|n| \longrightarrow \infty$ *then the support of* S *is the whole circle* $\mathbf{T}$.

This follows by simple manipulations ; once we have the following lemma :

LEMMA 2.1. – *Given* $N \geqslant 1$ *and* δ, $\eta > 0$ *we can find an* $\epsilon(N,\delta,\eta) > 0$ *with the following property :*

If S *is a distribution on* $\mathbf{T}$ *and*

(i) $\sum_{n=-N}^{N} |\hat{S}(n)| \geqslant \eta$

(ii) $|\hat{S}(n)| \leqslant \epsilon$ *for* $|n| > N$

then $\sup_{x \in \mathbf{T}} \inf \{|x - y| : y \in \operatorname{supp} S\} \leqslant \delta$.

Proof. – (I should like to thank Yves Meyer for turning a long, old fashioned proof into a modern short one.) Suppose the result is false for some particular $N \geqslant 1$, $\delta, \eta > 0$. Choose closed intervals $J(1), J(2), \ldots, J(w)$ each of length less than $\delta/4$ such that $\bigcup_{k=1}^{w} J(k) = \mathbf{T}$. We know that for each $m \geqslant 1$ we can find a distribution S_m such that

$$(\mathrm{i})_m \quad \sum_{n=-\mathrm{N}}^{\mathrm{N}} |\hat{\mathrm{S}}_m(n)| \geqslant \eta$$

$$(\mathrm{ii})_m \quad |\hat{\mathrm{S}}_m(n)| \leqslant 2^{-m} \quad \text{for } |n| > \mathrm{N}$$

and $\sup\limits_{x \in \mathbf{T}} \inf \{|x - y| : y \in \operatorname{supp} \mathrm{S}\} > \delta$ so that for some $1 \leqslant k(m) \leqslant w$ we have

$$(\mathrm{iii})_m \quad \operatorname{supp} \mathrm{S}_m \cap \mathrm{J}(k(m)) = \phi \,.$$

Now there must exist a $1 \leqslant k_0 \leqslant w$ such that $k(m)$ takes the value k_0 infinitely often. We choose $m(1) \leqslant m(2) \leqslant \ldots$ such that $k(m(j)) = k_0$ and so

$$(\mathrm{iii})_{m(j)} \quad \operatorname{supp} \mathrm{S}_{m(j)} \cap \mathrm{J}(k_0) = \phi \,.$$

Set $\mathrm{T}_j = \left(\sum\limits_{n=-\mathrm{N}}^{\mathrm{N}} |\hat{\mathrm{S}}_{m(j)}(n)| \right)^{-1} \mathrm{S}_{m(j)}$ so that T_j is a distribution with

$$(\mathrm{i})'_j \quad \sum_{n=-\mathrm{N}}^{\mathrm{N}} |\hat{\mathrm{T}}_j(n)| = 1$$

$$(\mathrm{ii})'_j \quad |\hat{\mathrm{T}}_j(n)| \leqslant 2^{-m(j)} \eta^{-1} \quad \text{for } |n| > \mathrm{N}$$

$$(\mathrm{iii})'_j \quad \operatorname{supp} \mathrm{T}_j \cap \mathrm{J}(k_0) = \phi \,.$$

By inspection there must be a weakly convergent subsequence $\mathrm{T}_{j(r)} \longrightarrow \mathrm{T}$ say, and we have then

$$(\mathrm{i}) \quad \sum_{n=-\mathrm{N}}^{\mathrm{N}} |\hat{\mathrm{T}}(n)| = 1$$

$$(\mathrm{ii}) \quad \hat{\mathrm{T}}(n) = 0 \quad \text{for } |n| > \mathrm{N}$$

$$(\mathrm{iii}) \quad \operatorname{supp} \mathrm{T} \cap \mathrm{J}(k_0) = \phi \,.$$

Conditions (i) and (ii) means that T is a non zero trigonometric polynomial and can thus have only a finite number of zeros. This contradicts condition (iii) and so the result is proved by reductio ad absurdum. □

Proof of Theorem 1.2′. – We first construct integers $\mathrm{N}(r)$ and real numbers $\epsilon(r) > 0$ by the following inductive procedure.

Set $N(0) = 0$ and $\epsilon(0) = 1$. When $N(s)$ and $\epsilon(s)$ have been constructed for all $r \geqslant s \geqslant 0$ with $\epsilon(r) < \epsilon(r-1) < \ldots < \epsilon(0)$, proceed as follows. Choose $N(r+1)$ so that

$$(1)_r \quad (N(r+1) - N(r) - 1)\, q(2^{-r-1}\epsilon(r)) \geqslant 2$$

and, if $r \geqslant 1$,

$$(2)_r \quad \frac{\epsilon(r-1) - \epsilon(r)}{N(r) - N(r-1)} \geqslant \frac{\epsilon(r)}{N(r+1) - N(r)}.$$

By Lemma 2.1 we can now find an $\epsilon(r) > \epsilon(r+1) > 0$ such that, if S is a distribution with

$$(3)_r \quad \sum_{n=-N(r+1)}^{N(r+1)} |\hat{S}(n)| \geqslant 2^{-r}$$

$$(4)_r \quad |\hat{S}(n)| \leqslant \epsilon(r+1) \quad \text{for } |n| > N(r+1), \text{ then}$$

$$(5)_r \quad \sup_{x \in \mathbf{T}} \inf \{|x - y| : y \in \operatorname{supp} S\} \leqslant 2^{-r}.$$

Set

$$\varphi(n) = \frac{N(r+1) - n}{N(r+1) - N(r)} \cdot 2^{-r}\epsilon(r) + \frac{n - N(r)}{N(r+1) - N(r)} \cdot 2^{-r-1}\epsilon(r+1)$$

for $N(r) \leqslant n < N(r+1)$ and $r \geqslant 0$. Using $(2)_r$ and the fact that $\epsilon(r) < \epsilon(r-1)$ we see that $\varphi(n)$ is a convex decreasing positive sequence. Moreover $\varphi(n) \geqslant 2^{-r-1}\epsilon(r)$ for

$$N(r) \leqslant n \leqslant (N(r+1) + N(r))/2,$$

so by $(1)_r$ and the fact that q is increasing, we have $\sum_{n=N(r)}^{N(r+1)-1} q(\varphi(n)) \geqslant 1$ and thus

$$(A) \quad \sum_{n=1}^{\infty} q(\varphi(n)) \text{ diverges.}$$

Now suppose S is a non zero distribution with $|S(n)| = 0(\varphi(|n|))$. Then conditions $(3)_r$ and $(4)_r$ will be automatically satisfied for large r. Thus $(5)_r$ will be true for all large r and so $\operatorname{supp} S = \mathbf{T}$ (since supp S is closed).

Remark. – We have seen in Theorem 1.2′ that condition (B) of Theorem 1.1 is not redundant. Could it be modified to provide a necessary and sufficient condition ? It is possible, but I cannot imagine what form such a condition would take. Note, for example, that our method of proof can be modified to give the following refinement of Theorem 1.1. (We shall not bother to prove it explicitly, but the reader should find no difficulty in convincing himself of its truth.)

THEOREM 1.1′. – *There exists a* $K(n) \longrightarrow \infty$ *such that, if condition* (B) *of Theorem 1.1 is replaced by*

(B)′ $\varphi(n) \leqslant K(n)\,\varphi(r)$ *for all* $1 \leqslant n \leqslant r \leqslant 2n$,

then the conclusion of that theorem still holds.

3. Preliminary Remarks.

There is no loss of generality in taking φ to be a decreasing continuous positive function $[1, \infty) \longrightarrow \mathbf{R}$ such that

(1) $\Sigma\, \varphi(n)^2$ is divergent

(2) $\varphi(x) \leqslant K\varphi(y)$ for all $1 \leqslant x \leqslant y \leqslant 2x$

(3) $x^M \varphi(x) \geqslant K\, y^M \varphi(y)$ for all $x \geqslant y \geqslant 1$

(4) $1 \geqslant \varphi(x)$ for all $x \geqslant 1$.

(The argument is not difficult, but details can be found in [1].)

All we need to obtain Theorem 1.1 is the following lemma.

LEMMA 3.1. – *Given* $\epsilon, \eta > 0$ *we can find an* $f \in C(\mathbf{T})$ *such that*

(i) $f(t) \geqslant 0$ *for all* $t \in \mathbf{T}$

(ii) $|\operatorname{supp} f| \leqslant \eta$

(iii) $\dfrac{1}{2\pi} \displaystyle\int_{\mathbf{T}} f(t)\, dt = 1$

(iv) $|\hat{f}(r)| \leqslant \epsilon\, \varphi(|r|)$ *for all* $r \neq 0$

(v) f *is infinitely differentiable.*

Proof of Theorem 1.1 from Lemma 3.1. – This follows Section 2 of [1]. The only difference is that in [1] we knew that

$$(\mathrm{C})_1 \quad n\varphi(n) \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty,$$

and this may no longer be true. However, condition (B) automatically implies

$$(\mathrm{C})_2 \quad n^q \varphi(n) \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty$$

for some integer $q \geqslant 1$ and this is sufficient to carry the argument through.

Specifically, we replace the statement "$|g_n(r)| \leqslant \mathrm{A}r^{-2}$" in the second paragraph of the proof of Lemma 2.2 [1] by "$|\hat{g}_n(r)| \leqslant \mathrm{A}\,|r|^{q-1}$" and this enables us to replace "$\sup\limits_{k\geqslant 1} (k\varphi(k))^{-1}$" by "$\sup (k^q \varphi(k))^{-1}$" in the formulae which conclude that proof. The easy details are left to the reader. □

In order to construct f we need, just as in [1] a function h whose derivative is never close to any particular value for very long. To simplify our later computations, we make h piecewise linear.

LEMMA 3.2. – *Given* $\alpha > 0$, $10^{-1} > \beta > 0$ *and* $v \geqslant 1$, *we can find a piecewise linear function* $h : [0,2\pi] \longrightarrow \mathbf{R}$ *differentiable except on a finite set* B *with* $0, 2\pi \in \mathrm{B}$ *such that*

(5) $h(b) \equiv 0 \mod 2\pi$ *for all* $b \in \mathrm{B}$

(6) $h'(x) > h'(y) \geqslant v$ *for all* $2\pi > x > y > 0$, $x, y \notin \mathrm{B}$

(7) *Writing* $\mathrm{C} = 10^4 (\mathrm{K} + 1)^4$ *we have*

$$|\{y \in [0,2\pi]\setminus \mathrm{B} : |u - h'(y)| \leqslant (\mathrm{C}\beta\varphi(u))^{-1}\}| \leqslant \mathrm{C}\beta\varphi(u)$$

for all $u \geqslant v$ *with* $\varphi(u) \geqslant u^{-1}$.

(8) $(\beta\varphi(h'(x)))^2 \geqslant (h'(x))^{-\alpha-1}$ *for all* $x \in [0,2\pi]\setminus \mathrm{B}$.

Proof. – We construct a function h_0 (which will turn out to be, essentially, h) in a series of inductive steps. To be more precise, once we have defined $a_m \geqslant 0$ and h on the interval $[0, a_m]$ we then define an $a_{m+1} \geqslant a_m$ and construct h on the interval $[a_m, a_{m+1}]$. The details follow.

Observe first that condition (2) implies the existence of an $N(0)$ such that, if $n \geqslant N(0)$, then

(I) $(\beta\varphi(2^n))^2 \geqslant 2^{-n(1+\alpha/2)}$

implies that $(\beta\varphi(u))^2 \geqslant u^{-(1+\alpha)}$ for all $2^{n+1} \geqslant u \geqslant 2^n$ and that $2^n(\beta\varphi(2^n)) \geqslant 10^{20}$, and also

(I′) $(\beta\varphi(2^n))^2 < 2^{-n(1+\alpha/2)}$

implies that $(\beta\varphi(u))^2 < 2^{-n(1+\alpha/4)}$ for all $2^{n+1} \geqslant u \geqslant 2^n$.

Choose $N(1)$ such that

(II) $2^{N(1)} \geqslant v$, $N(1) \geqslant N(0)$ and $N(1) \geqslant 100$.

Set $a_0 = a_1 = \cdots = a_{N(1)} = 0$ and $h_0(0) = 0$.

Now suppose $a_m \geqslant 0$ and $h_0 : [0,a_m] \longrightarrow \mathbf{R}$ have been constructed (so that, in particular, $h(a_m)$ is defined) for some $m \geqslant N(1)$. If $(\beta\varphi(2^m))^2 < 2^{-m(1+\alpha/2)}$ we set $a_{m+1} = a_m$ and complete the inductive step at once.

If $(\beta\varphi(2^m))^2 \geqslant 2^{-m(1+\alpha/2)}$, we choose some integer N_m with

(III) $1\,000\, N_m \geqslant 2^m\, \beta\varphi(2^m) \geqslant 100\, N_m$

and choose l_{mr} with

(IV) $10^{-1}\,\beta\varphi(2^m) \leqslant l_{mr} \leqslant \beta\varphi(2^m)$

in such a way that, writing

$$\text{(V)}\quad h_0\left(a_m + \sum_{t=1}^{r-1} l_{mt} + x\right) = h_0\left(a_m + \sum_{t=1}^{r-1} l_{mt}\right) + (2^m + r/(\beta\varphi(2^m)))\,x$$

for $0 \leqslant x \leqslant l_{mv}$,

we have

$$\text{(VI)}\quad h_0\left(a_m + \sum_{t=1}^{r} l_{mt}\right) \equiv 0 \bmod 2\pi \text{ for all } 1 \leqslant r \leqslant N_m .$$

(Choices giving (III) and (VI) are possible since, by (I) and (II), $2^m\,\beta\varphi(2^m) \geqslant 10^{20}$.) We now set $a_{m+1} = a_m + \sum_{t=1}^{N_m} l_{mt}$ and begin again.

Next we wish to show that $a_m \longrightarrow \infty$ as $m \longrightarrow \infty$. To this end define a function φ_0 by the conditions $\varphi_0(r) = 0$ if $(\beta\varphi(r))^2 < r^{-(1+\alpha/4)}$ and $\varphi_0(r) = \varphi(r)$ otherwise. Then

$$0 \leqslant (\varphi(r))^2 - (\varphi_0(r))^2 \leqslant \beta^{-2} r^{-(1+\alpha/4)}$$

so that $\sum_{r=1}^{\infty} ((\varphi(r))^2 - (\varphi_0(r))^2)$ converges and so, using (1), $\sum_{r=1}^{\infty} (\varphi_0(r))^2$ diverges. But, if $m \geqslant \mathrm{N}(1)$ and $(\beta\varphi(2^m))^2 \geqslant 2^{-m(1+\alpha/2)}$, then

$$\begin{aligned} a_{m+1} - a_m &= \sum_{t=1}^{\mathrm{N}_m} l_{mt} \\ &\geqslant (1\,000^{-1}\, 2^m \beta\varphi(2^m))\,(10^{-1}\beta\varphi(2^m)) \\ &= 10^{-4}\beta^2\, 2^m (\varphi(2^m))^2 \\ &\geqslant \mathrm{K}^{-2}\, 10^{-4}\beta^2 \sum_{r=2^m+1}^{2^{m+1}} (\varphi(r))^2 \\ &\geqslant \mathrm{K}^{-2}\, 10^{-4}\beta^2 \sum_{r=2^m+1}^{2^{m+1}} (\varphi_0(r))^2 . \end{aligned}$$

On the other hand, if $m \geqslant \mathrm{N}(1)$ and $(\beta\varphi(2^m))^2 < 2^{-m(1+\alpha/2)}$, then, by (I′) and (II), $\varphi_0(r) = 0$ for $2^m + 1 \leqslant r \leqslant 2^{m+1}$ and so $a_{m+1} - a_m = 0 \geqslant 0 = \mathrm{K}^{-2}\, 10^{-4}\beta^2 \sum_{r=2^m+1}^{2^{m+1}} (\varphi_0(r))^2$. It follows that $a_m = \sum_{t=\mathrm{N}(1)+1}^{m} (a_t - a_{t-1}) \geqslant \mathrm{K}^{-2}\, 10^{-4}\beta^2 \sum_{r=2^{\mathrm{N}(1)}+1}^{2^m} (\varphi_0(r))^2 \longrightarrow \infty$ as required.

As we said above, h_0 will be essentially our h but there is one trivial technical modification that we make, simply to ensure that $h(2\pi) \equiv 0 \bmod 2\pi$. It is not central to the proof and the reader may ignore it. Observe first that by suitable choices of the N_m (subject to condition (III)) we can ensure that for some $m(0) \geqslant \mathrm{N}(1)$ and some $r(0)$ we have $2 \leqslant r(0) < \mathrm{N}_{m(0)} - 2$,

$$a_{m(0)} + \sum_{t=1}^{r(0)-1} l_{m(0)t} \leqslant 2\pi < a_m + \sum_{t=1}^{r(0)} l_{m(0)t} .$$

We write $h(y) = h_0(y)$ for $0 \leqslant y \leqslant a_{m(0)} + \sum_{t=1}^{r(0)-2} l_{m(0)t}$ and

$$h\left(a_{m(0)} + \sum_{t=1}^{r(0)-2} l_{m(0)r} + x\right) = h\left(a_{m(0)} + \sum_{t=1}^{r(0)-2} l_{m(0)t} + x\right) + v(0)x$$

for $0 \leqslant x \leqslant 2\pi - \left(a_{m(0)} + \sum_{t=1}^{r(0)-2} l_{m(0)t}\right)$ where $v(0)$ is chosen so that

(VII) $2^{m(0)+1/4} \leqslant v(0) \leqslant 2^{m(0)+1/2}$, $h(2\pi) \equiv 0 \bmod 2\pi$. (This can be done because of (I) and (II)).

The proof of the lemma will be complete if we can show that h has the properties claimed in the statement of the lemma. Clearly h is piecewise linear differentiable except on a set B with $0, 2\pi \in \mathrm{B}$ such that $h(b) \equiv 0$ for all $b \in \mathrm{B}$. (The points $b \in \mathrm{B}$, apart from 2π , are precisely points of the form $a_m + \sum_{t=1}^{r} l_{mt}$, so (5) follows from (VI) and (VII).)

If $y \in [0,2\pi]\backslash \mathrm{B}$, then $a_{m+1} > y > a_m$ for some $m \geqslant \mathrm{N}(1)$, so that $2^{m+1} \geqslant h'(y) \geqslant 2^m \geqslant v$ by (V) and (VII). Again, if $2\pi > x > y > 0$ and $x , y \notin \mathrm{B}$, then either $a_{m+1} > x > y > a_m$ for some m and (V) shows that $h'(x) \geqslant h'(y)$ or

$$a_{n+1} > x > a_n \geqslant a_{m+1} > y > a_m \text{ for some } n > m$$

and $h'(x) > 2^n \geqslant 2^{m+1} > h'(y)$. Property (6) is thus proved.

We observe also that if $2\pi > x > 0$, $x \notin \mathrm{B}$, then $a_{m+1} > x > a_m$ where $(\beta\varphi(2^m))^2 \geqslant 2^{-m(1+\alpha/2)}$ and so by (I) and the fact that $2^{m+1} \geqslant h'(x) \geqslant 2^m$ we have $(\beta\varphi(h'(x)))^2 \geqslant h(x)^{-(1+\alpha)}$ and property (8) follows.

Finally we check (7). If $2^{m-1/8} \leqslant u < 2^{m+7/8}$ we observe that by (2) $\varphi(u) \geqslant \mathrm{K}\varphi(2^{m-1})$ and $\varphi(u) \geqslant \mathrm{K}\varphi(2^m)$. Thus looking at (V) and (III), and at (VII), we see that $y \in \mathrm{B}$, $|h'(y) - u| \leqslant (\mathrm{C}\beta\varphi(u))^{-1}$ implies $a_m \neq a_{m+1}$ and $a_m < y < a_{m+1}$, so that using (V), (III) and (VII) again we have

$$\begin{aligned}
&|\{y \in [0,2\pi]\backslash \mathrm{B} : |u - h'(y)| \leqslant (\mathrm{C}\beta\varphi(u))^{-1}\}| \\
&\leqslant |\{y \in [a_m, a_{m+1}]\backslash \mathrm{B} : |u - h'(y)| \leqslant (\mathrm{C}\beta\gamma(u))^{-1}\}| \\
&\leqslant 3 \max \{l_{mr} : 1 \leqslant r \leqslant \mathrm{N}(m)\} \\
&= \mathrm{C}\beta\varphi(u) .
\end{aligned}$$

(Note that to simplify the computations we have taken C much larger than it need be.) Property (7) follows and the proof is complete. □

Although the proof above is quite long, the reader should see that the underlying idea is very simple and the details not very important.

4. The Construction.

Let h be as in Lemma 3.2. Choose $g \in C(\mathbf{T})$ such that

(9) g is infinitely differentiable

(10) $g(t) \geqslant 0$ for all $t \in \mathbf{T}$

(11) $\operatorname{supp} g \subseteq [\pi - \eta/2 , \pi + \eta/2]$

(12) $\frac{1}{2\pi} \int_{\mathbf{T}} g(t)\, dt = 1$.

We set $f = g \circ h$ and make the following claim.

LEMMA 3.1′. – *For a suitable choice of* $\alpha > 0$, $10^{-1} > \beta > 0$ *and* $v \geqslant 1$, f *satisfies the conclusions of Lemma 3.1.*

We note at once that, writing $B = \{b_0, b_1, \ldots, b_u\}$ with $0 = b_0 < b_1 < \ldots < b_u = 2\pi$, we have h linear on $[b_{t-1}, b_t]$ with $h(b_{t-1}) \equiv h(b_t) \equiv 0 \bmod 2\pi\ [1 \leqslant t \leqslant u]$ so that conditions (9), (10), (11) and (12) on g give immediately conditions (i), (ii), (iii) and (v) on f.

Thus Lemma 3.1′ reduces to the following claim :

LEMMA 3.1″. – *For a suitable choice of* $\alpha > 0$, $10^{-1} > \beta > 0$ *and* $v \geqslant 1$, f *satisfies condition (iv) of Lemma 3.1,* i.e.

(iv) $|\hat{f}(r)| \leqslant \epsilon\varphi(|r|)$ for all $r \neq 0$.

If we can prove this, then Theorem 1.1 will follow. The rest of the paper will thus be devoted to estimating $|\hat{f}(r)|$.

We note first that, since f is real values, we need only consider the case $r \geqslant 1$. We choose some $1/10 > \delta > 0$ and set $X = [0, b_k]$, $Y = [b_k, 2\pi]$ where $0 \leqslant k \leqslant u$ is chosen so that $h'(x) \leqslant r^{1-\delta}$ for $x \in X \backslash B$ and $h'(x) > r^{1-\delta}$ for $x \in Y$.

We write

$$f_1(x) = f(x)\,,\ f_2(x) = 0 \qquad \text{for } x \in X$$

$$f_1(x) = 0 \quad ,\ f_2(x) = f(x) \quad \text{for } x \in \mathrm{Y}$$

so that $f = f_1 + f_2$ and $\hat{f}(r) = \hat{f}_1(r) + \hat{f}_2(r)$.

Because h is linear on $[b_{t-1}, b_t]$ and $h(b_{t-1}) \equiv h(b_t) \equiv 0 \bmod 2\pi$, it is easy to see that

$$(13) \quad \sup_{x \in \mathbf{T}} |f_1^{(w)}(x)| \leqslant \Big(\sup_{x \in \mathrm{X}} |h'(x)|\Big)^w \sup_{x \in \mathbf{T}} |g^{(w)}(x)|\,.$$

Thus

$$\begin{aligned} |\hat{f}_1(r)| &\leqslant r^{-w} \sup_{x \in \mathbf{T}} |f_1^{(w)}(x)| \\ &\leqslant r^{-w}(r^{1-\delta})^w \sup_{x \in \mathbf{T}} |g^{(w)}(x)| \\ &= r^{-\delta w} \mathrm{A}(g, w) \end{aligned}$$

where $\mathrm{A}(g, w)$ depends on g and w only. In particular, taking w large enough (depending on δ and M) we have $|f_1(r)| \leqslant \dfrac{\mathrm{A}_1}{r^{\mathrm{M}+1}}$ where A_1 is some constant depending on M, δ and ϵ. Thus (by (3)) we have

$$(14) \quad |\hat{f}_1(r)| \leqslant \frac{\mathrm{A}_2\, \varphi(r)}{r}$$

where A_2 depends on K, M, δ and g. But $f_1 = 0$ if $r^{1-\delta} < v$, so, provided only that v is large enough (depending on K, M, δ and g), we have

$$(15) \quad |\hat{f}_1(r)| \leqslant \frac{\epsilon}{2}\, \varphi(r) \quad \text{for all } r \geqslant 1\,.$$

The problem of estimating $|\hat{f}(r)|$ thus reduces to that of estimating $|\hat{f}_2(r)|$.

Before starting this, we remind ourselves of which quantities are fixed and which are free.

(16) The function g and the number M, K (and consequently C) will be taken as fixed once and for all.

(17) The quantities β , $\delta > 0$ may be chosen arbitrarily subject to being sufficiently small.

The number $\alpha > 0$ was taken arbitrarily, but we now make the restriction

(18) $1 > \alpha > 0$.

For the purposes of the main discussion the reader may take $\alpha = 1/2$. Finally we recall that

(19) The quantity v may be taken arbitrarily large, subject to the condition $v \geqslant v_0(\delta)$, where $v_0(\delta)$ depends on δ .

How are we going to estimate $|\hat{f}_2(r)|$? Observe that, since g is infinitely differentiable, we have $g(x) = \sum_{q=-\infty}^{\infty} \hat{g}(q) \exp iqx$ with $\sum_{q=-\infty}^{\infty} |\hat{g}(q)| < \infty$. Thus by elementary theorems of analysis, we have $f(x) = \sum_{q=-\infty}^{\infty} \hat{g}(q) \exp iqh(x)$, and (if $r \neq 0$), $\hat{f}_2(r) = \sum_{q \neq 0} \hat{g}(q) \frac{1}{2\pi} \int_y \exp(i(qu(x) - rx))\, dx$ and so

(20) $$|\hat{f}_2(r)| \leqslant \sum_{q \neq 0} |\hat{g}(q)|\, \Psi(q, r)$$

where

(21) $$\Psi(q,r) = \left| \frac{1}{2\pi} \int_y \exp(i(qh(x) - rx))\, dx \right| .$$

To estimate $\Psi(q,r)$ we shall use the following version of Van der Corput's Lemma :

LEMMA 4.1. – *Let* $\lambda > 0$. *If* $\mathcal{K} : (a,b) \longrightarrow \mathbf{R}$ *is continuous and piecewise differentiable with*

$$\mathcal{K}'(u) \leqslant \mathcal{K}'(w) \quad \textit{for all} \quad a < u \leqslant w < b$$

or

$$\mathcal{K}'(u) \geqslant \mathcal{K}'(w) \quad \textit{for all} \quad a < u \leqslant w < b$$

with $\mathcal{K}'(u)$, $\mathcal{K}'(w)$ *defined, further* $|\mathcal{K}'(u)| \geqslant \chi$ *for all* $a < u < b$ *with* $\mathcal{K}'(u)$ *defined, then*

$$\left| \int_a^b \exp(i\mathcal{K}t))\, dt \right| \leqslant \frac{\pi}{\chi} \cdot$$

Proof. – As for Lemma 3.2(i) in [1]. □

The second form of Van der Corput's Lemma (Lemma 3.2(ii) [1]) has been absorbed into the estimates that follow.

5. The Estimates.

From condition (3) near the beginning of § 3 it is clear that if δ is small enough depending on α or, more formally,

(22) $0 < \delta < \delta_0(\alpha)$ for some $\delta_0(\alpha)$ depending on α then

(23) For any $A \geqslant 1$ the condition $(\varphi(t))^2 \geqslant A\, t^{-(\alpha+1)}$ for some $r^{1-\delta} \leqslant t \leqslant 4r$ implies that $(\varphi(s))^2 \geqslant A\, s^{-(\alpha+3)/2}$ for all $r^{1-\delta} \leqslant s \leqslant 4r$.

Specialising, we draw two conclusions. Firstly

(24) $(\beta\varphi(t))^2 \geqslant t^{-(\alpha+1)}$ for some $r^{1-\delta} \leqslant t \leqslant 4r$ implies that $\varphi(s) \geqslant s^{-1}$ for all $r^{1-\delta} \leqslant s \leqslant 4r$.

(25) $(\beta\varphi(t))^2 \geqslant t^{-(\alpha+1)}$ for some $r^{1-\delta} \leqslant t \leqslant 4r$ implies that $(\beta\varphi(r))^2 \geqslant r^{-(\alpha+3)/2}$.

Let us make the further condition

(22) $0 < \delta < (1-\alpha)/4$

so that $\dfrac{\alpha+3}{4(1-\delta)} < 1$.

We are now in a position to prove the following useful estimate.

LEMMA 5.1. – *Writing* $\lambda_0 = \inf\limits_{x \in Y \setminus B} h'(x)$ *we have*

(26) $\lambda_0^{-1} \leqslant \beta\varphi(r)$.

Proof. – By the definition of Y we have, using condition (8) of Lemma 3.2

$$\begin{aligned}\lambda_0 &= \inf\{h'(x) : h'(x) > r^{1-\delta},\ x \in [0,2]\setminus B\}\\ &= \inf\{h'(x) : h'(x) > r^{1-\delta},\ (\beta\varphi(h'(x))^2 \geqslant (h'(x))^{-\alpha-1},\\ &\qquad\qquad x \in [0,2\pi]\setminus B\}\\ &\geqslant \inf\{u : u > r^{1-\delta},\ (\beta\varphi(u))^2 \geqslant u^{-\alpha-1}\}\end{aligned}$$

$$\lambda_0 = \inf \{u : u \geqslant r^{1-} \ , \ (\beta\varphi(u))^2 = u^{-\alpha-1}\}$$
$$= u_0 \ , \ \text{say} \ .$$

There are 2 cases to consider according as $u_0 \leqslant r$ or $u_0 > r$.

If $u_0 \leqslant r$, then (25) shows us that $(\beta\varphi(r))^2 \geqslant r^{-(\alpha+3)/2}$ and so, using (2), we have

$$\lambda_0 \geqslant u_0 \geqslant r^{1-\delta} \geqslant (r^{(\alpha+3)4})^{4(1-\delta)/(\alpha+3)}$$
$$\geqslant r^{(\alpha+3)/4} \geqslant (\beta\varphi(r))^{-1}$$

which is the desired result.

On the other hand, if $u_0 \geqslant r$, then by continuity $(\beta\varphi(u_0))^2 = u_0^{-\alpha-1}$ and so $u_0 = (\beta\varphi(u_0))^{-2/(\alpha+1)} \geqslant (\beta\varphi(u_0))^{-1}$ (since $1 > \beta\varphi(u_0) > 0$ and $2 > \alpha + 1 > 1$). But φ is decreasing, so $\lambda_0 \geqslant u_0 \geqslant (\beta\varphi(r))^{-1}$ in this case also. □

Our estimation of the terms $g(q) \mid \Psi(q,r)$ will split into 4 cases. (Recall that we are considering the case $r \geqslant 1$).

Case (i) $\quad q < 0$

Case (ii) $\quad q \geqslant r^\delta \ , \ r^{1-\delta} \geqslant v$

Case (iii) $\quad r^\delta > q \geqslant 1 \ , \ r^{1-\delta} \geqslant v$

Case (iv) $\quad 1 \leqslant r^{1-\delta} \leqslant v$.

Let us start with the first case.

LEMMA 5.2. – *There exists a* $\beta_1(\epsilon)$ *such that if*

(27) $\quad 0 < \beta < \beta_1(\epsilon)$

we have

$$\sum_{q<0} \mid \hat{g}(q) \mid \Psi(q,r) \leqslant \epsilon\varphi(r)/4 \ . \tag{28}$$

Proof. – Set $Y = [a,b]$, $\mathscr{K}(x) = qh(x) - rx$. Then $\mathscr{K}$ satisfies the conditions of Lemma 4.1 with $\lambda = -q\lambda_0 + r$. Thus

$$\Psi(q,r) = \left| \frac{1}{2\pi} \int_a^b exp(i\mathscr{K}(x))dx \right| \leqslant \frac{1}{-q\lambda_0 + r} \leqslant \frac{1}{\lambda_0}$$

and, by Lemma 5.1, $\Psi(q,r) \leqslant \beta\varphi(r)$. It follows that

$$\sum_{q<0} |\hat{g}(q)|\, \Psi(q,r) \leqslant \sum_{q<0} |\hat{g}(q)|\, \beta\varphi(r)$$

$$\leqslant \beta\Big(\sum_{q=-\infty}^{\infty} |\hat{g}(q)|\Big)\, \varphi(r)$$

and the result follows on taking $\beta = \epsilon/4 \sum_{q=-\infty}^{\infty} |\hat{g}(q)|$. □

We move on to the second case.

LEMMA 5.3. – *There exists a* $v_1(\epsilon,\delta)$ *such that if*

(29) $v \leqslant v_1(\epsilon,\delta)$

we have

(30) $\sum_{q \geqslant r^\delta} |\hat{g}(q)|\, \Psi(q.r) \leqslant \epsilon\varphi(r)/8$ *for all* $r^{1-\delta} \geqslant v$.

Proof. – Pick an integer $P \geqslant (\delta^{-1}+1)(M+1)+10$. Since g is infinitely differentiable, we have

$$|\hat{g}(q)| \leqslant B_0(g,P)\, q^{-P}$$

where $B_0(g,P)$ depends on g and P only. Thus, since $\Psi(q,r) \leqslant 1$ (trivially) we have

$$\sum_{q \geqslant r^\delta} |\hat{g}(q)|\, \Psi(q,r) \leqslant B_0(g,P) \sum_{q \geqslant r} q^{-P}$$

$$\leqslant B_1(g,P)\, r^{-\delta(P-1)}$$

$$\leqslant B_2(g,P)\, r^{-M+1}$$

$$\leqslant \frac{B_3(g,P)}{r}\, \varphi(r)$$

where $B_j(g,P)$ depends on g and P only $[j = 1,2,3]$. Thus setting $v_1(\epsilon,\delta) = \epsilon/(8\, B_3(g,P))$ we have the desired result. □

The third case is the central estimate.

LEMMA 5.4. – *There exists a* $\beta_2(\epsilon)$ *such that if*

(31) $0 < \beta < \beta_2(\epsilon)$

we have

$$(32)\qquad \sum_{r^\delta > q \geq 1} |\hat{g}(q)|\,\Psi(q,r) \leq \epsilon\varphi(r)/16 \text{ for all } r \geq v\,.$$

Proof. – We distinguish 2 possibilities according as

$$(\beta\varphi(t))^2 < t^{-(\alpha+1)} \text{ for all } r^{1-\delta} \leq t \leq 4r \text{ or not.}$$

Case(iii a) : Suppose $(\beta\varphi(t))^2 < t^{-(\alpha+1)}$ for all $r^{1-\delta} \leq t \leq 4r$. Then, using the notation and some of the arguments of Lemma 5.1, we have $\lambda_0 = \inf_{x\in Y\setminus B} h'(x) \geq 4r$. Thus, writing $Y = [a,b]$ and $\mathcal{K}(x) = qh(x) - rx$, we see that $\mathcal{K}$ satisfies the conditions of Lemma 4.1 with $\lambda = q\lambda_0 - r \geq \lambda_0 - r \geq \lambda_0/2$. Hence

$$\Psi(q,r) = \left|\frac{1}{2\pi}\int_a^b \exp(i\mathcal{K}(x))dx\right| \leq \frac{1}{2\pi} \leq \frac{1}{\lambda_0}$$

and, using Lemma 5.1, $\Psi(q,r) \leq \beta\varphi(r)$. It follows that

$$\sum_{r^\delta > q \geq 1} |\hat{g}(q)|\,\Psi(q,r) \leq \beta \sum_{q=-\infty}^{\infty} |\hat{g}(q)|\ \varphi(r)\,.$$

Condition (32) will thus be verified if $\beta_2(\epsilon) \leq \epsilon\Big/\Big(16\sum_{q=-\infty}^{\infty}|\hat{g}(q)|\Big)$.

Case(iii b) : Suppose $(\beta\varphi(t))^2 \geq t^{-(\alpha+1)}$ for some $r^{1-\delta} \leq t \leq 4r$. Then, by condition (24), $\varphi(s) \geq s^{-1}$ for all $r^{1-\delta} \leq s \leq 4r$. Condition (7) of Lemma 3.2 now given

$(33)'\quad |\{y \in [0,2\pi]\setminus B : |u - h'(y)| \leq (C\beta\varphi(u))^{-1}\}| \leq C\beta\varphi(u)$ for all $r^{1-\delta} \leq u \leq 4r$,

and so, in particular,

$(33)\quad |\{y \in Y\setminus B : |u - h'(y)| \leq (C\beta\varphi(u))^{-1}\}| \leq C\beta\varphi(u)$ for all $r^{1-\delta} \leq u \leq 4r$.

Since h' is increasing on $Y\setminus B$, we can find intervals $Y(1)$, $Y(2)$, $(Y(3)$ (some, possibly, empty) such that $Y(1)\cup U(2)\cup Y(3) = Y$ and

$$h'(y) - \frac{r}{q} < -(C\beta\varphi(r/q))^{-1} \qquad \text{for all} \quad y \in Y(1)\setminus B$$

$$\left|h'(y) - \frac{r}{q}\right| < (C\beta\varphi(r/q))^{-1} \qquad \text{for all} \quad y \in Y(2)\setminus B$$

$$h'(y) - \frac{r}{q} > (C\beta\varphi(r/q))^{-1} \quad \text{for all} \quad y \in Y(3)\setminus B\,.$$

Set $\mathscr{K}(x) = qh(x) - r$. Then $\mathscr{K}$ satisfies the conditions of Lemma 4.1 on $Y(1)$ and $Y(3)$

with

$$\mathscr{K}'(x) \leqslant -q(C\beta\varphi(r/q))^{-1} \quad \text{for} \quad x \in Y(1)\setminus B$$

$$\mathscr{K}'(x) \geqslant \quad q(C\beta\varphi(r/q))^{-1} \quad \text{for} \quad x \in Y(3)\setminus B$$

so that

$$\left|\frac{1}{2\pi}\int_{Y(1)} \exp(i(qh(x) - rx))\,dx\right| \leqslant \frac{C\beta\varphi(r/q)}{2q}$$

$$\left|\frac{1}{2\pi}\int_{Y(3)} \exp(i(qh(x) - rx))\,dx\right| \leqslant \frac{C\beta\varphi(r/q)}{2q}\,.$$

Further (provided that $r^{\delta} \geqslant q \geqslant 1$ and so $r \geqslant r/q \geqslant r^{1-\delta}$) we have from (33) that $|Y(2)| \leqslant C\beta\varphi(r/q)$ and so

$$\left|\frac{1}{2\pi}\int_{Y(2)} \exp(i(qh(x) - rx))\,dx\right| \leqslant C\beta\varphi(r/q)\,.$$

Summing we obtain

$$\begin{aligned}\Psi(r,q) &= \left|\frac{1}{2\pi}\int_{Y} \exp(i(qh(x) - rx))\,dx\right| \\ &\leqslant \frac{C\beta\varphi(r/q)}{2q} + \frac{C\beta\varphi(r/q)}{2q} + C\beta\varphi(r/q) \\ &\leqslant 2\,C\beta\varphi(r/q)\,,\end{aligned}$$

and so, using condition (3) stated near the beginning of section § 3, we have

$$\Psi(r,q) \leqslant 2\,CK\beta q^{M}\varphi(r)\,.$$

But, as we have noted before, g is infinitely differentiable and so $|\hat{g}(q)| \leqslant B_0(g,M+2)\,q^{-(M+2)}$, where $B_0(g,M+2)$ depends on g and M only. Thus

$$\begin{aligned}\sum_{r^{\delta} > q \geqslant 1} |\hat{g}(q)|\,\Psi(q,r) &\leqslant 2\,CK\beta\varphi(r) \sum_{r^{\delta} > q \geqslant 1} q^{-2} \\ &\leqslant 16\,CK\beta\varphi(r)\end{aligned}$$

and condition (32) will be verified if $\beta_2(\epsilon) \leqslant \epsilon/(16^2\,\mathrm{CK})$. The lemma is thus proved. □

The fourth and final case is dealt with by a lemma whose proof is even more trivial than it looks.

LEMMA 5.5. – *There exists a* $\beta_3(\epsilon,v)$ *depending on* ϵ *and* v *such that if*

$$(34) \quad 0 < \beta < \beta_3(\epsilon,v)$$

we have

$$(35) \quad \sum_{q=1}^{\infty} |\hat{g}(q)|\, \Psi(q,r) \leqslant \epsilon\varphi(r)/16 \textit{ for all } 1 \leqslant r^{1-\delta} \leqslant v.$$

Proof. – Using the notation and the simplest ideas of Lemma 5.1, we see that $\lambda_0 \longrightarrow \infty$ as $\beta \longrightarrow 0$. Thus, writing $\mathcal{H}(x) = ph(x) - rx$, we have, for any fixed p, that $\inf_{x\in Y\setminus B} \mathcal{H}'(x) = p\lambda_0 - r \longrightarrow \infty$ as $\beta \longrightarrow 0$ and so, applying Lemma 5.1 we have

$$\Psi(p,r) \longrightarrow 0 \quad \text{as} \quad \beta \longrightarrow 0 \quad \text{for each } p\,.$$

But $0 \leqslant \Psi(q,r) \leqslant 1$ for all q, and $\sum_{q=1}^{\infty} |\hat{g}(q)| < \infty$, so $\sum_{q=1}^{\infty} |\hat{g}(q)|\, \Psi(q,r) \longrightarrow 0$ as $\beta \longrightarrow 0$ for each fixed r. In particular, since there are only a finite number of integers r with $1 \leqslant r^{1-\delta} \leqslant v$, we have

$$\sup_{1\leqslant r^{1-\delta}\leqslant v} \sum_{q\leqslant 1}^{\infty} |\hat{g}(q)|\, \Psi(q,r) \longrightarrow 0 \text{ as } \beta \longrightarrow 0$$ which is the desired result. □

Combining Lemmas 5.2, 5.3, 5.4 and 5.5 we see that for v large enough (depending on ϵ) and β small enough (depending on v and ϵ), we have (by (20))

$$|\hat{f}_2(r)| \leqslant \sum_{q\neq 0} |\hat{g}(q)|\, \Psi(q,r) \leqslant \epsilon\varphi(r)/2\,.$$

Thus using (15) and (19) we see that for suitable choice of α (in fact any $1 > \alpha > 0$), v and β we have

$$|\hat{f}(r)| \leqslant |\hat{f}_1(r)| + |\hat{f}_2(r)| \leqslant \epsilon\varphi(r) \quad \text{for all} \quad r \geqslant 1$$

and so (since f is real)

$$|\hat{f}(r)| \leqslant \epsilon\varphi(|r|) \quad \text{for all} \quad r \neq 1 .$$

Thus Lemma 3.1″ is proved and Theorem 1.1 follows. We have achieved the object of this paper.

6. A remark.

The only place where we needed the fact that φ is decreasing is in the last paragraph of the proof of Lemma 5.1. However, Lemma 5.1 is essential for our estimates (for example in Lemma 5.2).

Thus, unless we change the proof quite a lot, we can only replace the hypothesis φ decreasing by a hypothesis which implies, for some suitable $1 > \alpha > 0$,

(36) If $r \geqslant v$ and $(\beta\varphi(r))^2 < r^{-\alpha-1}$ then, if $u > r$ and $(\beta\varphi(u))^2 \geqslant u^{-\alpha-1}$, it follows that $u \geqslant (\beta\varphi(r))^{-1}$.

If we take v large enough and α small enough, (36) is implied by

(B)′ There exists a $K \geqslant 1$ such that

$$2^{-1/2}\,\varphi(y) \leqslant \varphi(x) \leqslant K\varphi(y) \quad \text{for all} \quad x \leqslant y \leqslant 2x \quad \text{and all} \quad x \geqslant 1 .$$

We have thus the following strengthening of Theorem 1.1.

THEOREM 1.1″. – *Suppose* $\varphi(n)$ *is a positive sequence such that*

(A) $\sum_{n=1}^{\infty} (\varphi(n))^2$ *diverges*

(B)′ *There exists a* $K \geqslant 1$ *such that* $K^{-1}\varphi(n) \leqslant \varphi(r) \leqslant \sqrt{2\varphi}(n)$ *whenever* $n \leqslant r \leqslant 2n$.

Then we can find a positive measure $\mu \neq 0$ *with support* E *of Lebesgue measure zero yet with*

$$|\hat{\mu}(n)| = 0(\varphi(|n|)) \quad \text{as} \quad |n| \longrightarrow \infty .$$

I do not know whether condition (B)$'$ can be replaced by

(B)$''$ There exists a $K \geqslant 1$ such that

$$K^{-1}\varphi(n) \leqslant \varphi(r) \leqslant K\varphi(n) \quad \text{whenever} \quad n \leqslant r \leqslant 2n .$$

BIBLIOGRAPHY

[1] T.W. KORNER, On the Theorem of Ivašev-Musatov I, *Annales de l'Institut Fourier*, 27, 3 (1977), 97-115.

Manuscrit reçu le 22 juillet 1977
Proposé par J.P. Kahane.

T.W. KORNER,
Department of Mathematics
Trinity Hall
Cambridge (Gde Bretagne).