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On the Green type kernels on the half space in $\mathbb{R}^n$


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ON THE GREEN TYPE KERNELS
ON THE HALF SPACE IN $\mathbb{R}^n$

by Masayuki ITÔ

1. Let $\mathbb{R}^n$ be the $n(\geq 2)$-dimensional Euclidian space and $D$ be the half space $\{x=(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n; x_1 > 0\}$. For a point $x=(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$, we write

$$\bar{x} = (-x_1,x_2,\ldots,x_n) \quad \text{and} \quad |x| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}.$$ 

When $n \geq 3$, we put $G_2(x,y) = |x-y|^{2-n} - |x-\bar{y}|^{2-n}$ in $D \times D$. Then $G_2$ is the Green kernel on $D$. Analogously we set, for a number $\alpha$ with $0 < \alpha < n$,

$$G_\alpha(x,y) = |x-y|^{2-n} - |x-\bar{y}|^{\alpha-n}$$

in $D \times D$, and we call it the Green type kernel of order $\alpha$ on $D$. The following question was proposed to me in a letter by H. L. Jackson: Does $G_\alpha$ also satisfy the domination principle provided that $0 < \alpha < 2$?

This paper is inspired by this question. Let $C_\varepsilon(D)$ and $C(D)$ be the usual topological vector space of real-valued continuous functions in $D$ with compact support and the usual topological vector space of real-valued continuous functions in $D$, respectively. We set

$$C_\varepsilon^+(D) = \{f \in C_\varepsilon(D); f \geq 0\}$$

and $C^+(D) = \{f \in C(D); f \geq 0\}$. For a given Hunt convol-
olution kernel $\times$ on $\mathbb{R}^n$, we define the linear operator

$$V_{\times} : C_c(D) \ni f \mapsto (\times \ast f - \times \ast \overline{f})_D \in C(D)$$

where $\overline{f}$ is the reflection of $f$ about the boundary $\partial D$ of $D$ and where $(\times \ast f - \times \ast \overline{f})_D$ is the restriction of $\times \ast f - \times \ast \overline{f}$ to $D$. If $V_{\times}$ is positive (that is, $f \geq 0 \implies V_{\times}f \geq 0$), we say that $V_{\times}$ is the Green type kernel associated with $\times$.

The purpose of this paper is to show the following two theorems.

**Theorem 1.** Let $\times$ be a Hunt convolution kernel on $\mathbb{R}^n$ and $(\times_p)_{p \geq 0}$ be the resolvent associated with $\times$. Suppose that $\times$ is symmetric with respect to $\partial D$. Then the following two conditions are equivalent:

1. $V_{\times}$ is a Hunt kernel on $D$.
2. For each $p > 0$, $\frac{\partial}{\partial x_1} \times_p \leq 0$ in the sense of distributions in $D$.

**Theorem 2.** Let $\times$ be a Dirichlet convolution kernel on $\mathbb{R}^n$ and $\alpha$ be the singular measure (the Lévy measure) associated with $\times$. Suppose that $\times$ is also symmetric with respect to $\partial D$. Then the following two conditions are equivalent:

1. $V_{\times}$ is a Dirichlet kernel on $D$.
2. $\frac{\partial}{\partial x_1} \alpha \leq 0$ in the sense of distributions in $D$.

This theorem gives immediately that the question raised by H. L. Jackson is affirmatively solved.

2. Let $\times$ be a convolution kernel on $\mathbb{R}^n$ (2). Similarly we define $V_{\times}$. When $V_{\times}$ is positive, we set

$$\mathcal{D}^+(V_{\times}) = \{ f \in C^+(D) : V_{\times}f \in C^+(D) \},$$

where

$$V_{\times}f(x) = \sup \{ V_{\times}g(x) : g \in C_c^+(D), g \leq f \}$$

(1) An $f \in C_c(D)$ may be considered as a finite continuous function in $\mathbb{R}^n$ with compact support $\subset D$.

(2) In potential theory, a convolution kernel means a positive measure.
in $D$. Put $\mathcal{D}(V_x) = \{f \in C(D); f^+, f^- \in \mathcal{D}^+(V_x)\}$ and, for an $f \in \mathcal{D}(V_x)$, $V_xf = V_xf^+ - V_xf^-$. Then $V_x$ is a linear operator from $\mathcal{D}(V_x)$ into $C(D)$.

**Lemma 3.** Let $x$ and $x'$ be two convolution kernels on $\mathbb{R}^n$. Suppose that $x$ and $x'$ are symmetric with respect to $\partial D$ and that the convolution $x \ast x'$ is defined. If $V_x$ is positive, then, for any $f \in C_c(D)$, $V_xf \in \mathcal{D}(V_x)$ and

$$V_x(V_xf) = (x \ast x' \ast f - x \ast x' \ast \bar{f})_D.$$ 

**Proof.** We may assume that $f \geq 0$. Since $x \ast x'$ is defined and $|V_xf| \leq x' \ast f + x' \ast \bar{f}$, we have $V_xf \in \mathcal{D}(V_x)$. Our convolution kernels $x$ and $x'$ being symmetric with respect to $\partial D$, $x \ast \bar{f}(x) = x \ast f(x)$ and $x' \ast \bar{f}(x) = x' \ast f(x)$.

For the sake of simplicity, we write $h(x) = V_xf(x)$ in $D$ and $h(x) = 0$ on $\mathbb{R}^n - D$. Then, for a $g \in C_c^*(D)$, we have

$$\int_{\mathbb{R}^n} V_x(V_xf)(x)g(x) \, dx = \int_{\mathbb{R}^n} (x \ast h(x) - x \ast \bar{h}(x))g(x) \, dx$$

$$= \int_{\mathbb{R}^n} h(x)x \ast g(x) \, dx - \int_{\mathbb{R}^n} \bar{h}(x)x \ast g(x) \, dx$$

$$= \int_{\mathbb{R}^n} \left( x \ast f(x) - x \ast \bar{f}(x) \right) \bar{x} \ast g(x) \, dx$$

$$- \int_{\mathbb{R}^n} (x', x' \ast \bar{f}(x)) \bar{x} \ast g(x) \, dx$$

$$= \int x' \ast f(x)\bar{x} \ast g(x) \, dx - \int x' \ast \bar{f}(x)\bar{x} \ast g(x) \, dx$$

$$= \int x \ast x' \ast (f - \bar{f})(x)g(x) \, dx,$$

where $\bar{x}$ is the adjoint convolution kernel of $x$; that is, $\bar{x}(E) = x\{\{x; x \in E\}$ for any Borel set $E$. Since $g$ is arbitrary, we obtain the required equality.

**Remark 4.** In the above lemma, we have $V_xf \in \mathcal{D}(V_x)$ and $V_x(V_xf) = V_x(V_xf)$ provided that $V_x$ is also positive.

**Lemma 5.** Let $x$ be a convolution kernel on $\mathbb{R}^n$. Suppose that $x$ is symmetric with respect to $\partial D$. Then $V_x$ is positive if and only if $\frac{\partial}{\partial x_1}x \leq 0$ in the sense of distributions in $D$. 

Proof. — First we shall show the « if » part. For a \( t \in (0, \infty) \), put \( H_t = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n; x_1 = t \} \) and
\[
D' = \{ x = (x_1, x_2, \ldots, x_n) \in D; \int_{H_t} dx = 0 \}.
\]
It suffices to prove that, for any \( f \in C^\infty_c(D) \) and any \( x \in D' \),
\[
x \ast f(x) \geq x \ast f(\overline{x}),
\]
because \( \int_{D-D'} dx = 0 \) and
\[
x \ast f(\overline{x}) = x \ast f(x).
\]
We choose a sequence \( (\varphi_k)_{k=1}^\infty \) of non-negative, spherically symmetric and infinitely differentiable functions such that \( \int \varphi_k dx = 1 \) and that the support of \( \varphi_k \), \( \text{supp}(\varphi_k) \), is contained in \( \{ x \in \mathbb{R}^n; |x| < 1/k \} \). Then \( x \ast \varphi_k \) is symmetric with respect to \( \partial D \) and
\[
\frac{\partial}{\partial x_1} x \ast \varphi_k(x) \leq 0 \quad \text{in}
\]
\[
\{ x \in \mathbb{R}^n; x_1 \geq 1/k \}.
\]
Let \( f \in C^\infty_c(D) \) and \( x = (x_1, x_2, \ldots, x_n) \in D' \). Then
\[
\int_{|y-x| \geq 1/m} f(y) x \ast \varphi_k(x - y) dy \geq \int_{|y-x| \geq 1/m} f(y) x \ast \varphi_k(\overline{x} - y) dy
\]
provided with \( 0 < m \leq k \). By letting \( k \to \infty \) and \( m \to \infty \), we obtain that
\[
x \ast f(x) = \int f(y) d\lambda \ast \varepsilon_x(y)
\]
\[
\geq \int_{\mathbb{R}^n-H_t} f(y) d\lambda \ast \varepsilon_x(y)
\]
\[
\geq \int_{\mathbb{R}^n-H_t} f(y) d\lambda \ast \varepsilon_x(y)
\]
\[
\geq x \ast f(\overline{x}) - \left( \sup_{z \in \mathbb{R}^n} |f(z)| \right) \int_{H_t} dx = x \ast f(\overline{x})
\]
where \( \varepsilon_x \) denote the unit measure at \( x \). Since \( f \) and \( x \) are arbitrary, the « if » part is true.

Next we shall show the « only if » part. Suppose that the « only if » part is false. Then there exist a number \( t > 0 \), a point \( x = (x_1, x_2, \ldots, x_n) \in D \) with \( x_1 > t \) and a non-negative, spherically symmetric and infinitely differentiable function \( \varphi \) in \( \mathbb{R}^n \) with \( \text{supp}(\varphi) \subset \{ x \in \mathbb{R}^n; |x| < t \} \) such that
\[
\frac{\partial}{\partial x_1} x \ast \varphi(x) > 0.
\]
Hence we can choose a number
$s > 0$ such that $s < x_1 - t$ and that, for every $y \in D$ with $|y| < s$, $x * \varphi(x - y) < x * \varphi(x - \bar{y})$. Since

$$x * \varphi(x - \bar{y}) = x * \varphi(x - y),$$

we have, for an $f \neq 0 \in C_c^+(D)$ satisfying

$$\text{supp}(f) \subseteq \{y \in \mathbb{R}^n; |y| < s\},$$

$$x * f * \varphi(x) < x * f * \varphi(x) = x * f * \varphi(x).$$

But this contradicts the inequality $x * f \geq x * \overline{f}$ in $D$. Thus we see that the « only if » part is true.

In the same manner as above, we obtain the following

**Lemma 6.** — Let $\alpha$ be a positive measure in $\mathbb{R}^n - \{0\}$. Suppose that $\alpha$ is symmetric with respect to $\partial D$. If $\frac{\partial}{\partial x_1} \alpha \leq 0$ in the sense of distributions in $D$, then, for any $f \in C_c^+(D)$,

$$\int f(x - y) \, d\alpha(y) \geq \int \overline{f}(x - y) \, d\alpha(y)$$

in $D \cap C \text{ supp } (f)$.

3. We say that a convolution kernel $x$ on $\mathbb{R}^n$ is a Hunt convolution kernel if $x = \int_0^\infty x_t \, dt$, where $(x_t)_{t \geq 0}$ is a vaguely continuous semi-group of positive measures in $\mathbb{R}^n$; that is, $\alpha_0 = \varepsilon$ (the Dirac measure), $\alpha_t * \alpha_s = \alpha_{t+s}$ ($\forall t \geq 0, \forall s \geq 0$) and the application $\mathbb{R}^+ = [0, \infty) \ni t \to \alpha_t$ is vaguely continuous. In this case, $(x_t)_{t \geq 0}$ is uniquely determined (see, for example, [3]) and called the vaguely continuous semi-group associated with $x$. For a $p \in \mathbb{R}^+$, put

$$x_p = \int_0^\infty \exp(-pt) x_t \, dt;$$

then $(x_p)_{p \geq 0}$ is called the resolvent associated with $x$. This is characterized by a family $(x_p)_{p \geq 0}$ of convolution kernels on $\mathbb{R}^n$ satisfying

$$x_p - x_q = (q - p)x_p * x_q (\forall p \geq 0, \forall q > 0)$$

and $\lim_{p \to 0} x_p = x_0 = x$ (vaguely).
Lemma 7 (see [3] or Theorem 5 in [6]). — Let \( x , (\alpha_i)_{t \geq 0} \) and \( (x_p)_{p \geq 0} \) be the same as above. For a \( p > 0 \) and a \( t > 0 \), put

\[
\alpha_{p,t} = \exp (-pt) \sum_{k=0}^{\infty} \frac{p^k t^k}{k!} (px_p)^k \quad \text{and} \quad \alpha_{p,0} = \varepsilon;
\]

then \( (\alpha_{p,t})_{t \geq 0} \) is a vaguely continuous semi-group of positive measures and we have

\[
x + \frac{1}{p} \varepsilon = \int_0^\infty \alpha_{p,t} \, dt \quad \text{and} \quad \lim_{p \to 0} \alpha_{p,t} = \alpha_t \quad \text{(vaguely)} \quad (t \geq 0).
\]

Lemma 8. — Let \( x = \int_0^\infty \alpha_t \, dt \) be a Hunt convolution kernel on \( \mathbb{R}^n \) and \( (x_p)_{p \geq 0} \) be the resolvent associated with \( x \). If \( x \) is symmetric with respect to \( \partial D \), then, for any \( p \) and any \( t \), \( x_p \) and \( \alpha_t \) are also symmetric with respect to \( \partial D \).

Proof. — For a \( p \geq 0 \), we denote by \( \tilde{x}_p \) the reflection of \( x_p \) about \( \partial D \). Evidently \( (\tilde{x}_p)_{p \geq 0} \) is the resolvent associated with \( \tilde{x} \). By using \( x = \tilde{x} \) and the unicity of the resolvent associated with \( x \), we have, for each \( p \geq 0 \), \( x_p = \tilde{x}_p \). This means that \( x_p \) is symmetric with respect to \( \partial D \). This gives also that, for any \( f \in C_c(D) \),

\[
\int_0^\infty \exp (-pt) f \, dx_t \, dt = \int_0^\infty \exp (-pt) \tilde{f} \, dx_t \, dt \quad (\forall p \geq 0).
\]

The Laplace transformation being injective, we have, for each \( t \geq 0 \), \( \int f \, dx_t = \int \tilde{f} \, dx_t \). Hence, \( f \) being arbitrary, we see that \( x_t \) is symmetric with respect to \( \partial D \).

Similarly we have the following

Remark 9. — If \( x \) is symmetric with respect to the origin 0 (resp. spherically symmetric), then \( x_p \) and \( x_t \) are also symmetric with respect to 0 (resp. spherically symmetric).

Let \( x \) be a convolution kernel on \( \mathbb{R}^n \). We say that \( x \) is a Dirichlet convolution kernel if the (generalised) Fourier transformation \( \hat{x} \) of \( x \) is defined and equal to \( \frac{1}{\psi} \), where \( \psi \) is a real-valued negative definite function in \( \mathbb{R}^n \) such that \( \frac{1}{\psi} \)

is locally summable. By virtue of the Lévy-Khinchine theorem, we have, for any \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \),

\[
\psi(x) = c + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \int (1 - \cos (2\pi x \cdot y)) \, d\alpha(y),
\]

where \( c \) is a non-negative constant, \( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \) is a positive semi-definite form, \( x \cdot y \) is the inner product in \( \mathbb{R}^n \) and where \( \alpha \) is a positive measure in \( \mathbb{R}^n - \{0\} \) symmetric with respect to 0 and satisfying \( \int |x|^2/(1 + |x|^2) \, dx(x) < \infty \).

It is well-known that the above decomposition of \( \psi \) is unique. The positive measure \( \alpha \) in \( \mathbb{R}^n - \{0\} \) is called the singular measure associated with \( x \). Since, for each \( t \geq 0 \), \( \exp(-t\psi) \) is of positive type in \( \mathbb{R}^n \), there exists a positive measure \( \alpha_t \) in \( \mathbb{R}^n \) such that \( \alpha_t = \exp(-t\psi) \). Evidently \( (\alpha_t)_{t \geq 0} \) is a vaguely continuous semi-group of positive measures and \( \alpha = \int_0^\infty \alpha_t \, dt \). Hence a Dirichlet convolution kernel is a Hunt convolution kernel and symmetric with respect to 0.

4. A positive linear operator \( V : C_c(D) \to C(D) \) is called a continuous kernel on \( D \) (Evidently \( V \) is continuous). Similarly as in the section 2, we define \( \mathcal{D}(V) \) and \( \mathcal{D}(V) \).

We say that \( V \) is a Hunt kernel on \( D \) if \( V = \int_0^\infty \tilde{V}_t \, dt \) (that is, for any \( f \in C_c(D) \), \( Vf(x) = \int_0^\infty \tilde{V}_t f(x) \, dt \) in \( D \)), where \( \tilde{(\tilde{V}_t)}_{t \geq 0} \) is a continuous semi-group of continuous kernels on \( D \); that is, \( \tilde{V}_0 = I \) (the identity), for any \( t \geq 0 \), \( s \geq 0 \) and any \( f \in C_c(D) \), \( \tilde{V}_t f \in \mathcal{D}(\tilde{V}_s) \), \( \tilde{V}_s (\tilde{V}_t f) = \tilde{V}_t (\tilde{V}_s f) = \tilde{V}_{t+s} f \) and the application \( \mathbb{R}^+ \ni t \to \tilde{V}_t f \) is continuous in \( C(D) \). Similarly as in [3], we see that \( \tilde{(\tilde{V}_t)}_{t \geq 0} \) is uniquely determined, and we call it the continuous semi-group associated with \( V \).

For a \( p > 0 \), put \( V_p = \int_0^\infty \exp(-pt) \tilde{V}_t \, dt \); then we call \( (V_p)_{p \geq 0} \) the resolvent associated with \( V \). It is known that, for any \( p > 0 \), \( q > 0 \) and any \( f \in C_c(D) \), \( V_p f \in \mathcal{D}(V_q) \), \( V_q f \in \mathcal{D}(V_p) \),

\[
V_p f - V_q f = (q - p) V_q (V_p f) = (q - p) V_p (V_q f)
\]

(the resolvent equation) and \( \lim_{p \to 0} V_p f = V_0 f = Vf \) in \( C(D) \).
Let $V_1$ and $V_2$ two continuous kernels on $D$. If, for any $f \in C_c(D)$, $V_2f \in \mathcal{D}(V_1)$, the application $C_c(D) \ni f \to V_1(V_2f)$ is positive linear, we denote it by $V_1 \cdot V_2$.

**Remark 10** (see [2]). — A Hunt kernel $V$ on $D$ satisfies the domination principle; that is, for two $f, g \in C_c^+(D)$, $Vf \leq Vg$ on $\text{supp}(f)$ implies the same inequality on $D$.

5. We shall show Theorem 1 mentioned in the section 1. 

(1) $\implies$ (2). By Lemmas 5 and 8, it suffices to prove that, for each $p > 0$, $V_{zp}$ is positive. Let $(V_p)_{p \geq 0}$ be the resolvent associated with $V_x$. Then, for an $f \in C_c^+(D)$ and a $p > 0$, $V_pf = (pV_x + I)(V_pf)$. On the other hand, Lemmas 3 and 8 give the $V_{zp}f \in \mathcal{D}(V_x)$ and

$$V_xf = (x \ast (f - f))(D) = ((px + \varepsilon) \ast x_p \ast (f - f))_D = (pV_x + I)(V_{zp}f).$$

By using the resolvent equation, we have

$$V_pf - V_{zp}f = (I - pV_p)(pV_x + I)(V_pf - V_{zp}f) = 0.$$

The function $f$ being arbitrary, we have $V_p = V_{zp}$, and hence $V_{zp}$ is positive.

(2) $\implies$ (1). By Lemma 5, $V_{zp}$ is positive ($\forall p > 0$). Let $\alpha_p$, be the positive measure defined in Lemma 7 ($\forall p > 0$, $\forall t > 0$) and $(\alpha_t)_{t > 0}$ be the vaguely continuous semi-group associated with $x$. By Lemmas 3 and 7,

$$V_{zp_t} = \exp(-pt) \sum_{k=0}^{\infty} \frac{p^{sk}}{k!} (pV_x)^k,$$

where $(pV_x)^0 = I$, $(pV_x)^1 = pV_x$ and

$$(pV_{zp})^{n+1} = (pV_{zp})^n \cdot (pV_{zp}).$$

Therefore $V_{zp_t}$ is positive. From Lemma 7, it follows that, for any $f \in C_c(D)$, $\lim_{p \to \infty} V_{zp_t}f = V_xf$ in $C(D)$ ($\forall t > 0$). Hence $V_{zp_t}$ is a continuous semi-group of continuous kernels on $D$ and that

$$V_x = \int_0^\infty V_{zp_t} dt.$$ Consequently $V_x$ is a Hunt kernel on $D$. This completes the proof.
Question 11. — Let \( x \) be a Hunt convolution kernel on \( \mathbb{R}^n \) satisfying \( x = \overline{x} \). Is it true that \( V_x \) is a Hunt kernel on \( D \) provided that \( V_x \) is positive?

Remark 12. — Let \( k(x) \) be a non-negative continuous function in the wide sense in \( \mathbb{R}^n \) satisfying \( k(x) = k(\overline{x}) \).
Suppose that \( x = k(x) \, dx \) is a Hunt convolution kernel and that \( V_x \) is also a Hunt kernel on \( D \). Put

\[
G(x,y) = k(x-y) - k(x-y)
\]
in \( D \times D \).

If the function kernel \( k(x-y) \) satisfies the continuity principle (\(^3\)), then \( G \) satisfies the domination principle; that is, for two positive measures \( \mu \) and \( \nu \) in \( D \) with compact support and with \( \int G \mu \, d\mu < \infty \), then \( G\mu \leq G\nu \) on \( \text{supp}(\mu) \) implies the same inequality in \( D \), where

\[
G\mu(x) = \int G(x,y) \, d\mu(y).
\]

It is known that \( k(x-y) \) satisfies the continuity principle when \( x \) is a Dirichlet convolution kernel (see [4]).

We show this remark. We see that \( G \) also satisfies the continuity principle. Therefore it suffices to prove that, for a positive measure \( \mu \) in \( D \) with compact support and an \( x \in D \), \( G\mu \leq G\varepsilon_x \) in \( D \) provided that \( G\mu \leq G\varepsilon_x \) on \( \text{supp(\mu)} \) and that \( G\mu \) is finite continuous (see [8]). Since \( V_x \) is a Hunt kernel, there exists \( f \in C_c^+ (D) \) such that \( V_x f = Gf \geq 1 \) on \( \text{supp}(\mu) \), where \( Gf(y) = \int G(y,z) f(z) \, dz \). Here we remark that \( \mu \) is considered as a positive measure in \( \mathbb{R}^n \). For a given positive number \( \delta \), there exists a neighborhood \( U \) of 0 such that, for any finite continuous function \( \varphi \geq 0 \) in \( \mathbb{R}^n \) with \( \text{supp}(\varphi) \subset U \) with \( \int \varphi \, dx = 1 \), \( \mu * \varphi, \varepsilon_x * \varphi \in C_c^+ (D) \) and \( G(\mu * \varphi) \leq G(\varepsilon_x * \varphi) + \delta Gf \) on \( \text{supp}(\mu * \varphi) \). By letting \( \varphi \, dx \to \varepsilon \) (vaguely) and \( \delta \downarrow 0 \), we have \( G\mu \leq G\varepsilon_x \).

(\(^3\)) This means that, for a positive measure \( \mu \) in \( \mathbb{R}^n \) with compact support, the function \( \int k(x-y) \, d\mu(y) \) of \( x \) is finite continuous provided that its restriction to \( \text{supp}(\mu) \) is finite continuous.
6. Theorem 1 gives the following

**Corollary 13.** — Let \( x = \int_0^\infty \alpha_t \, dt \) be a Hunt convolution kernel on \( \mathbb{R}^n \). Then \( x \) is symmetric with respect to \( \partial D \) and \( V_x \) is a Hunt kernel on \( D \) if and only if, for each \( t \geq 0 \), \( \alpha_t \) is symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} \alpha_t \leq 0 \) in the sense of distribution in \( D \).

**Corollary 14.** — Let \( x = \int_0^\infty \alpha_t \, dt \) be a Hunt convolution kernel on \( \mathbb{R}^n \) and \( \mu \) be a Hunt convolution kernel on \( \mathbb{R}^1 \) supported by \( \mathbb{R}^+ \). Suppose that \( x_\mu = \int_0^\infty \alpha_t \, d\mu(t) \) is defined (in the sense of measures) and that \( x \) is symmetric with respect to \( \partial D \). If \( V_x \) is a Hunt kernel on \( D \), then \( V_{x_\mu} \) is also a Hunt kernel on \( D \).

**Proof.** — We denote by \( (\mu_p)_{p \geq 0} \) the resolvent associated with \( \mu \). Since \( \mu_p \leq \mu \), \( x_{\mu,p} = \int \alpha_t \, d\mu_p(t) \) is defined (\( \forall p \geq 0 \)). It is known that \( x_{\mu,p} \) is a Hunt convolution kernel on \( \mathbb{R}^n \) and that \( (x_{\mu,p})_{p \geq 0} \) is the resolvent associated with \( x_\mu \) (see Theorem 1 in [5]). By Theorem 1 and Corollary 13, \( \alpha_t \) is symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} \alpha_t \leq 0 \) in the sense of distributions in \( D \). Hence \( x_\mu \) is also symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} x_{\mu,p} \leq 0 \) in the sense of distributions in \( D \) (\( \forall p \geq 0 \)). Consequently Theorem 1 gives this corollary.

In the same manner as above, we have the following

**Corollary 15.** — Let \( (\alpha_t)_{t \geq 0} \) be a vaguely continuous semi-group of positive measures in \( \mathbb{R}^n \) and \( \mu \) be a Hunt convolution kernel on \( \mathbb{R}^1 \) supported by \( \mathbb{R}^+ \). Suppose that \( \int_0^\infty \alpha_t \, d\mu(t) \) is defined and that, for each \( t \geq 0 \), \( \alpha_t \) is symmetric with respect to \( \partial D \) and \( \frac{\partial}{\partial x_1} \alpha_t \leq 0 \) in the sense of distributions in \( D \). Then \( V_{x_\mu} \) is a Hunt kernel on \( D \), where

\[
x_\mu = \int_0^\infty \alpha_t \, d\mu(t).
\]
We shall show that the question raised by H. L. Jackson is affirmatively solved.

Remark 16. — Let \( \nu \) be a positive measure in \((0, 2)\) such that \( \int_0^2 \frac{1}{x} \, d\nu(x) < \infty \) and \( c_0, c_1 \) be non-negative constants. Put

\[
\chi = \begin{cases} 
  c_0 \varepsilon + \left( \int |x|^{s-n} \, d\nu(x) \right) \, dx & \text{if } n = 2 \\
  c_0 \varepsilon + \left( \int |x|^{s-n} \, d\nu(x) + c_1 |x|^{2-n} \right) \, dx & \text{if } n \geq 3.
\end{cases}
\]

Then \( V_\chi \) is a Hunt kernel.

In fact, we have, with a positive constant \( c(\alpha) \),

\[
|x|^{s-n} = c(\alpha) \int_0^\infty \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{2t} \right) t^{\alpha-1} \, dt
\]

\((0 < \alpha < 2 \text{ if } n = 2, 0 < \alpha \leq 2 \text{ if } n \geq 3)\). Evidently the function \( c(\alpha) \) of \( \alpha \) is finite continuous. Put

\[
\mu = \begin{cases} 
  c_0 \varepsilon + \left( \int c(\alpha) t^{s/2-1} \, d\nu(x) \right) \, dt & \text{if } n = 2 \\
  c_0 \varepsilon + \left( \int c(\alpha) t^{s/2-1} \, d\nu(x) + c_1 c(2) \right) \, dt & \text{if } n \geq 3
\end{cases}
\]

in \( \mathbb{R}^1 \). Since \( \int_0^2 \frac{1}{x} \, d\nu(x) < \infty \), \( \chi_\mu \) is a convolution kernel on \( \mathbb{R}^n \) and

\[
\chi_\mu = \left( \int \frac{1}{(2\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{2t} \right) d\mu(t) \right) \, dx.
\]

Hence \( \mu \) is a convolution kernel on \( \mathbb{R}^1 \) supported by \( \mathbb{R}^+ \). Then \( \mu \) is a Hunt convolution kernel on \( \mathbb{R}^1 \) (cf. [5]), and Corollary 14 gives our remark.

Let \( G_\alpha \) be the Green type kernel of order \( \alpha \) in \( D \). Put

\[
G(x,y) = \begin{cases} 
  G_\alpha(x,y) \, d\nu(x) & \text{if } n = 2 \\
  G_\alpha(x,y) \, d\nu(x) + c_1 G_2(x,y) & \text{if } n \geq 3
\end{cases}
\]

Then Remarks 12 and 16 give that \( G \) satisfies the domination principle.

7. Let \( L_{\text{loc}}(D) \) be the usual Fréchet space of real-valued locally summable functions in \( D \). A Hilbert space \( H(D) \)
contained in $L_{\text{loc}}(D)$ is called a Dirichlet space on $D$ if the following three conditions are satisfied:

1. For each compact set $K$ in $D$, there exists a constant $A(K) > 0$ such that, for any $u \in D$, $\int_K |u| \, dx \leq A(K)\|u\|$. 
2. $C_c(D) \cap H(D)$ is dense both in $C_c(D)$ and in $H(D)$.
3. For any normalized contraction $T$ on $\mathbb{R}^1$ (4) and any $u \in H(D)$, $T \cdot u \in H(D)$ and $\|T \cdot u\| \leq \|u\|$. 

This is the definition by A. Beurling and J. Deny (see [1]). Here we denote by $\|\cdot\|$ and by $(\cdot, \cdot)$ the norm in $H(D)$ and the associated inner product, respectively. For an $f \in C_c(D)$, (1) gives that there exists uniquely $u^*_f \in H(D)$ such that, for any $u \in H(D)$, $(u^*_f, u) = \int uf \, dx$.

Let $V$ be a linear operator from $C_c(D)$ into $L_{\text{loc}}(D)$. We say that $V$ is a Dirichlet kernel on $D$ if there exists a Dirichlet space $H(D; V)$ on $D$ such that, for any $f \in C_c(D)$, $Vf = u^*$. 

Evidently $H(D; V)$ is uniquely determined. We call $H(D; V)$ the Dirichlet space associated with $V$ and $V$ the kernel of $H(D; V)$. For a Dirichlet kernel $V$ on $D$, we set

$$\mathcal{D}(V) = \left\{ f \in L_{\text{loc}}(D); \sup \left\{ \frac{\int uf \, dx}{\|u\|} ; \quad u \neq 0 \in C_c(D) \cap H(D; V) \right\} < \infty \right\}$$

and $\mathcal{D}^+(V) = \{ f \in \mathcal{D}(V); f \geq 0 \}$, where $\|\cdot\|$ denote the norm in $H(D; V)$. By virtue of (2), for an $f \in \mathcal{D}(V)$, there exists uniquely $Vf \in H(D; V)$ such that, for any

$$u \in C_c(D) \cap H(D; V), \quad (Vf, u) = \int uf \, dx,$$

where $(\cdot, \cdot)$ denote the inner product in $H(D; V)$. Thus $V$ may be considered as a linear operator from $\mathcal{D}(V)$ into $H(D; V)$. It is known that $V$ is positive (that is, $f \in \mathcal{D}^+(V) \Rightarrow Vf \geq 0$ a.e.) (see [1]).

(4) This means that $T$ is an application: $\mathbb{R}^1 \to \mathbb{R}^1$ such that $R(0) = 0$ and $|Ta - Tb| \leq |a - b|$ ($\forall a, \forall b \in \mathbb{R}^1$).
Lemma 17. — Let $x$ be a Hunt convolution kernel on $\mathbb{R}^n$ satisfying $x = \bar{x}$. If $V_x$ is a Dirichlet kernel on $D$, then $V_x$ is a Hunt kernel.

Proof. — For the sake of simplicity, we write $H = H(D; V_x)$. Denote by $\| \cdot \|$ and by $(\cdot , \cdot )$ the norm in $H$ and the inner product in $H$, respectively. Let $L^2(D)$ be the Hilbert space of real-valued square summable functions in $D$. For a $p \geq 0$, $H_p$ denotes the Hilbert space associated to the norm $\| u \| _p = \left( p \int |u|^2 \, dx + \| u \|^2 \right)^{1/2}$ on $H \cap L^2(D)$. Evidently $H_p$ is a Dirichlet space on $D$. Let $f \in C_c(D)$. For any $u \in C_c(D) \cap H$, we have

$$
\int V_p f(x) u(x) \, dx = \frac{1}{p} ((V_p f, u)_p - (V_p f, u))
$$

$$
= \frac{1}{p} ((V_x f, u) - (V_p f, u))
$$

$$
\leq \frac{1}{p} (\| V_x f \| + \| V_p f \|) \| u \|,
$$

where $V_p$ is the kernel of $H_p$ and where $(\cdot , \cdot )_p$ is the inner product in $H_p$. Hence $V_p f \in L^2(V)$. Since, for any $u \in C_c(D) \cap H$,

$$p(V_x(V_p f), u) = p \int u(x) V_p f(x) \, dx
$$

$$
= (V_p f, u)_p - (V_p f, u) = (V_x f - V_p f, u),
$$

(2) gives $V_x f - V_p f = p V_x(V_p f)$ a.e. in $D$. Let $(x_p)_{p \geq 0}$ be the resolvent associated with $x$. By Lemmas 3 and 8, we have $V_x f - V_{x_p} f = p V_x(V_{x_p} f)$. In the same manner as in the proof of Theorem 1, we have $V_p f = V_{x_p} f$ a.e. in $D$, and hence $V_{x_p}$ is positive ($\forall p > 0$). By Theorem 1 and Lemma 5, we see that $V_x$ is a Hunt kernel.

We shall prove Theorem 2 mentioned in the section 1.

(1) $\Rightarrow$ (2). Let $(x_p)_{p \geq 0}$ be the resolvent associated with $x$. Then it is known that $p^2 x_p \to x$ vaguely in $\mathbb{R}^n - \{0\}$ as $p \to \infty$ (see [1]), and hence theorem 1 and Lemma 17 give that $\frac{\partial}{\partial x_1} x \leq 0$ in the sense of distributions in $D$.

(2) $\Rightarrow$ (1). Since $p^2 x_p \to x$ vaguely in $\mathbb{R}^n - \{0\}$ as $p \to \infty$, Lemma 8 gives that $x$ is symmetric with respect to $\partial D$. Let $A$ be the diagonal set of $D \times D$ and $\beta$ be the
positive measure in \( D \times D - A \) defined by

\[
\int \int f(x)g(y) \, d\beta(x,y) = \int \int (f(x-y) - f(x-y))g(x) \, d\alpha(y) \, dx
\]

for any couple \( f, g \in C_c(D) \) with \( \text{supp}(f) \cap \text{supp}(g) = \emptyset \) (see Lemma 6). For any \( p, x_p \) being symmetric with respect to the origin, we have \( \alpha = \widetilde{\alpha} \), and hence \( \beta \) is symmetric with respect to \( A \). Let \( C_c^\infty(D) \) be the topological vector space of real-valued and infinitely differentiable functions in \( D \) with compact support (we identify an element of \( C_c^\infty(D) \) and an infinitely differentiable function in \( \mathbb{R}^n \) with compact support in \( D \)).

Let \( f \in C_c^\infty(D) \). Consider the approximation of the function \( |f(x) - f(y)|^2 \) of \( (x,y) \) by the functions of form \( \sum_i \varphi_i(x)\psi_i(y) \) in \( D \times D \), where \( \varphi_i \in C_c^\infty(D) \) and \( \psi_i \in C_c^\infty(D) \) with \( \text{supp}(\varphi_i) \cap \text{supp}(\psi_i) = \emptyset \).

Then we see that

\[
0 \leq \int \int |f(x) - f(y)|^2 \, d\beta(x,y) + \int |f(x)|^2 a(x) \, dx
\]

\[
= \int \int |f(x-y) - f(x)|^2 \, d\alpha(y) \, dx
\]

\[
- \int \int (\widetilde{f}(x-y) - \widetilde{f}(x))(f(x-y) - f(x)) \, d\alpha(y) \, dx < \infty \tag{5}
\]

where, for \( x = (x_1, x_2, \ldots, x_n) \in D \), \( a(x) = 2 \int_{x_i \geq x_i} d\alpha(y) \).

Let \( \tilde{H} \) be the specialized Dirichlet space with the kernel \( x \) (see [1]). We denote by \( ||| \cdot ||| \) and by \( \langle \cdot, \cdot \rangle \) the norm in \( \tilde{H} \) and the associated inner product. For a couple \( f, g \in C_c^\infty(D) \), we put

\[
(f, g) = \int f g \left( \frac{a}{2} + c \right) \, dx + \frac{1}{4\pi^2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \, dx
\]

\[
+ \frac{1}{2} \int (f(x) - f(y))(g(x) - g(y)) \, d\beta(x,y)
\]

\[
= \langle (f - \widetilde{f}, g) \rangle = \langle (f, g - \widetilde{g}) \rangle = \frac{1}{2} \langle (f - \widetilde{f}, g - \widetilde{g}) \rangle,
\]

(\( \dagger \)) The author would like to express his hearty thanks to Prof. F. Hirsch for the correction of this formula.
where $\hat{x} = \left( c + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{i} x_{j} + \int (1 - \cos (2\pi x \cdot y)) \, dx(y) \right)^{-1}$.

Then $(\cdot, \cdot)$ is an inner product in $C_c^\infty(D)$. For a compact set $K$ in $D$, we have

$$\sup_{u \in C_c^\infty(D)} \frac{\int_K |u| \, dx}{\|u\|} = \sup_{u \in C_c^\infty(D) \backslash \{0\}} \frac{\sqrt{2} \int_K |u - \bar{u}| \, dx}{\|u - \bar{u}\|} < \infty,$$

where $\|u\| = (u, u)^{1/2}$. Hence the completion $H$ of $C_c^\infty(D)$ by $\|\cdot\|$ is contained in $L_{loc}(D)$. Evidently, for any $u \in C_c^\infty(D)$ and any normalized contraction $T$ on $R^1$, $T \cdot u \in H$ and $\|T \cdot u\| \leq \|u\|$. For a $u \in H$, we choose a sequence $(u_k)_{k=1}^\infty \subset C_c^\infty(D)$ such that

$$\lim_{k \to \infty} \|u_k - u\| = 0.$$

Since $(T \cdot u_k)_{k=1}^\infty$ converges weakly to $T \cdot u$ in $H$ as $k \to \infty$ (see [1]), we have $T \cdot u \in H$ and $\|T \cdot u\| \leq \|u\|$. Hence $H$ is a Dirichlet space on $D$. We shall show that $V_x$ is the kernel of $H$. For an integer $m \geq 1$, let $T_m$ denote the projection from $R^1$ into $\left[ -\frac{1}{m}, \frac{1}{m} \right]$. Let $f \in C_c(D)$; then $x \ast (f - \bar{f}) - T_m \cdot x \ast (f - \bar{f}) \in \hat{H}$ and

$$V_x f - T_m \cdot V_x f \in C_c(D),$$

because $x \ast (f - \bar{f}) = 0$ on $\partial D$ and $\lim_{|x| \to \infty} x \ast (f - \bar{f})(x) = 0$.

Therefore there exists a neighborhood $V_m$ of the origin such that, for any non-negative, spherically symmetric and infinitely differentiable function $\varphi$ in $R^1$ with supp $(\varphi) \subset V_m$ and $\int \varphi \, dx = 1$, $f \ast \varphi \in C_c^\infty(D)$ and

$$(V_x f - T_m \cdot V_x f) \ast \varphi \in C_c^\infty(D).$$

Since

$$(x \ast (f - \bar{f}) - T_m \cdot x \ast (f - \bar{f})) \ast \varphi = (V_x f - T_m \cdot V_x f) \ast \varphi - (V_x f - T_m \cdot V_x f) \ast \varphi$$

and, for a $u \in \hat{H}$,

$$\|u \ast \varphi\|^2 = \int \int ((u \ast \varepsilon_x, u \ast \varepsilon_y)) \varphi(x) \varphi(y) \, dx \, dy \leq \|u\|^2,$$
we have

\[ \| (V_x f - T_m \cdot V_x f) * \varphi \|^2 \]

\[ \leq \frac{1}{2} \| \| \chi * (f - \bar{f}) - T_m \cdot \chi * (f - \bar{f}) \| \|^2 \leq 2 \| \chi * (f - \bar{f}) \| \|^2. \]

By letting \( \varphi d\mu \rightarrow \varepsilon \) (vaguely) and \( m \rightarrow \infty \), we see that \( V_x f \in H \) and, for any \( u \in C_c^\infty(D) \),

\[ (V_x f, u) = ((\chi * (f - \bar{f}), u)) = \int u(f - \bar{f}) \, dx = \int uf \, dx. \]

This implies immediately that, for any \( u \in H \),

\[ (V_x f, u) = \int uf \, dx. \]

Consequently \( V_x \) is the kernel of the Dirichlet space \( H \). This completes the proof.

Theorem 2 gives also that the question raised by H. L. Jackson is affirmatively solved. In fact, the singular measure associated with the convolution kernel \( r^\alpha \) is equal to \( c_\alpha |x|^{2-n} \, dx \) provided that \( 0 < \alpha < 2 \), where \( c_\alpha \) is a positive constant, where \( |x|^{2-n} \, dx \) is symbolically denoted by \( r^\alpha \) \((0 < \alpha < n)\).

We denote now by \( \Delta \) the laplacian on \( \mathbb{R}^n \). We say that a convolution kernel \( \chi \) on \( \mathbb{R}^n \) is a Frostman-Kunugui kernel if \( \chi \) is spherically symmetric, vanishes at infinity \(^6\), and if \( \Delta \chi \geq 0 \) in the sense of distributions outside the origin 0. Theorem 2 and Theorem 1 in [7] give the following

Corollary 18. — Suppose \( n \geq 3 \). Then the following two statements hold.

1. For a Frostman-Kunugui kernel \( \chi \neq 0 \) on \( \mathbb{R}^n \) satisfying

\[ \frac{\partial}{\partial x_1} \Delta \chi \leq 0 \] in the sense of distributions in \( D \), there exists uniquely a spherically symmetric Dirichlet convolution kernel \( \chi' \) on \( \mathbb{R}^n \) such that \( V_{\chi'} \) is a Dirichlet kernel on \( D \) and that, for any \( f \in C_c(D) \), \( V_x(V_{\chi'} f)(x) = V_x(V_{\chi'} f)(x) = G_{\chi'}(x) \) in \( D \).

2. For a spherically symmetric Dirichlet kernel \( \chi \) on \( \mathbb{R}^n \) such that \( V_{\chi} \) is a Dirichlet kernel on \( D \), there exists uniquely

\(^6\) This means that, for any finite continuous function \( f \) in \( \mathbb{R}^n \) with compact support, \( \chi * f(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \).
a Frostman-Kunugui kernel \( x' \) on \( \mathbb{R}^n \) such that \( \frac{\partial}{\partial x_1} \Delta x \leq 0 \) in the sense of distributions in \( D \) and that, for any \( f \in C_c(D) \), \( V_x(V_x f)(x) = V_x(V_x f)(x) = G_x f(x) \) in \( D \).

**Proof.** — First we shall show (1). By Theorem 1 in [7], there exists uniquely a spherically symmetric Dirichlet kernel \( x' \) on \( \mathbb{R}^n \) such that \( x \ast x' = r^{2-n} \). We have, with a positive constant \( c \), \( (\Delta x) \ast x' = -c \varepsilon \) in the sense of distributions in \( \mathbb{R}^n \). This implies that the singular measure associated with \( x' \) is equal to \( \frac{1}{c} \Delta x \) outside 0. Theorem 2 and our assumption give that \( V_x \) is a Dirichlet kernel on \( D \). Since \( \Delta x \geq 0 \) in the sense of distributions in \( \mathbb{R}^n - \{0\} \) and \( x \) vanishes at infinity, \( \frac{\partial}{\partial x_1} x \leq 0 \) in the sense of distributions in \( D \). By Lemma 5, \( V_x \) is positive, and by Lemma 3 and Remark 4, we obtain the required equality. Let’s show the uniqueness of \( x' \). Let \( x'' \) be a Dirichlet convolution kernel on \( \mathbb{R}^n \) which is possessed of the same properties as of \( x' \). Since \( x \) is injective (see Theorem 1 in [7]) \((7)\) and
\[
x \ast (V_{x'} f - V_{x''} f) = x \ast (V_{x'} f - V_{x''} f)
\]
in \( \mathbb{R}^n \) \((8)\), we have \( V_{x'} f = V_{x''} f \) \( \forall f \in C_c(D) \). This implies that, for any \( f \in C_c(D) \), \( (x' - x'') f = (x' - x'') \ast \tilde{f} \). In the same manner as in Lemma 5, we have \( \frac{\partial}{\partial x_1} (x' - x'') = 0 \) in the sense of distributions in \( D \). Since \( x' - x'' \) is spherically symmetric and vanishes at the infinity, we have \( x' = x'' \). Thus we see that (1) holds.

Next we shall show (2). By Theorem 1 in [7], there exists uniquely a Frostman-Kunugui kernel \( x' \) on \( \mathbb{R}^n \) such that \( x \ast x' = r^{2-n} \). Since the singular measure associated with \( x \) is equal to \( \frac{1}{c} \Delta x' \) outside 0, Theorem 2 gives that \( \frac{\partial}{\partial x_1} \Delta x' \leq 0 \) in the sense of distributions in \( D \). Similarly as

\((7)\) This means that, for an \( f \in C(D) \), \( f = 0 \) provided that \( x \ast |f| \) is defined and that \( x \ast f = 0 \).

\((8)\) We may assume that \( V_{x'} f \) is a continuous function in \( \mathbb{R}^n \) with support \( \subset D \).
above, we see that $V_{\kappa'}$ is positive and the required equality holds. Since $\kappa$ is also injective (see, for example, [1]), we can similarly show the uniqueness of $\kappa'$.

Remember the Riesz decomposition formula

$$r^{a-n} \ast r^{(2-a)-n} = a_{2}r^{2-n} \quad (0 < a < 2),$$

where $a_{2}$ is a positive constant (see [9]). Then, by this corollary, we see that $G_{\kappa}$ satisfies the domination principle provided with $n \geq 3$ and $0 < a < 2$.

**Remark 19.** — For a spherically symmetric convolution kernel $\kappa$ on $\mathbb{R}^{n}$, $\frac{\partial}{\partial x_{1}} \kappa \leq 0$ in the sense of distributions in $D$ if and only if $\frac{\partial}{\partial r} \kappa \leq 0$ in the sense of distributions in $\mathbb{R}^{n} - \{0\}$, where $r = |x|$. In this case, $\kappa$ is absolutely continuous outside $0$.

By using Theorem 1, Corollary 13 and this remark 19, we have the following

**Remark 20.** — Let $\kappa = \int_{0}^{\infty} \kappa_{t} dt$ be a spherically symmetric Dirichlet kernel on $\mathbb{R}^{n}$. Then $V_{\kappa}$ is a Dirichlet kernel on $D$ if and only if, for any $t > 0$, $\kappa_{t}$ is of form

$$\kappa_{t} = c_{t}e + k_{t}(|x|) dx,$$

where $c_{t}$ is a non-negative constant and $k_{t}$ is a non-negative decreasing (in the wide sense) function on $\mathbb{R}^{+}$.

8. First we shall show that the inverse of the question raised by H. L. Jackson is also affirmative.

**Proposition 21.** — If the Green type kernel $G_{\kappa}$ $(0 < \kappa < n)$ on $D$ satisfies the domination principle, then $0 < \kappa \leq 2$.

**Proof.** — Since $G_{\kappa}$ satisfies the domination principle, $G_{\kappa}$ also satisfies the balayage principle (see, for example, [8]); that is, for a positive measure $\mu$ in $D$ with compact support and a compact set $F$ in $D$, there exists a positive measure $\mu'$ supported by $F$ such that $G_{\kappa} \mu \geq G_{\kappa} \mu'$ in $D$ and
ON THE GREEN TYPES KERNELS ON THE HALF SPACE

$G_\alpha \mu = G_\alpha \mu_F$ $G_\alpha$-n.e. on $F$ (*) Let $\mu \not= 0$ and $F$ be a closed ball contained in $D$ such that $\text{supp} (\mu) \cap F = \emptyset$. Suppose that $\alpha > 2$. Let $t$ be positive integer satisfying $0 < \alpha - 2t \leq 2$ and $\beta = \alpha - 2t$. Then

$$G_\alpha(x,y) = \int G_{2t}(x,z)G_{2t}(z,y) \, dz$$

(see Lemma 3). Since $G_{2t}(G_{\beta} \mu) = G_{2t}(G_{\beta} \mu_F)$ a.e. on $F$, we have $G_{\beta} \mu = G_{\beta} \mu_F$ a.e. on $F$, because

$$\Delta^t(G_{2t}(G_{\beta} \mu) - G_{2t}(G_{\beta} \mu_F)) = (-c)^t(G_{\beta} \mu - G_{\beta} \mu_F)$$

in the sense of distributions in $D$, where $c$ is the positive constant satisfying $\Delta r^{a-n} = -cr$. Since $G_{\beta} \mu$ is continuous on $F$ and $G_{\beta} \mu_F$ is lower semi-continuous, we have $G_{\beta} \mu \geq G_{\beta} \mu_F$ on $F$, and so $\int G_{\beta} \mu_F \, d\mu_F < \infty$. The function kernel $G_\beta$ satisfying the domination principle, we have $G_{\beta} \mu \geq G_{\beta} \mu_F$ in $D$. By virtue of the injectivity of $G_\beta$, we have $G_{\beta} \mu \not= G_{\beta} \mu_F$. But this contradicts the equality $G_{2t}(G_{\beta} \mu) = G_{2t}(G_{\beta} \mu_F)$ $G_\alpha$-n.e. on $F$. Thus we achieve the proof.

We raise a question.

**Question 22.** — Let $x$ be a convolution kernel on $\mathbb{R}^n$ satisfying $x = x$. Suppose that $V_x$ is a Hunt kernel on $D$. Then is it true that $x$ is the sum of a Hunt convolution kernel and of a non-negative constant?

The following proposition shows that the answer is "yes" in a special case.

**Proposition 23.** — Let $x$ be a convolution kernel on $\mathbb{R}^n$ satisfying $x = \bar{x}$. Suppose that $V_x$ is a Hunt kernel on $D$. If $\int d\alpha < \infty$ and $x$ is absolutely continuous outside $0$, then $x$ is a Hunt convolution kernel.

**Proof.** — We may assume that $\int d\alpha < 1$. For a $p \in (0,1]$, we put

$$x_p = \sum_{k=0}^{\infty} (-p)^k(x)^{k+1};$$

(*) We write $G_\alpha \mu = G_\alpha \mu_F$ $G_\alpha$-n.e. on $F$ if, for any positive measure $\nu$ in $D$ with $\text{supp} (\nu) \subset F$ and $\int G_\alpha \nu \, d\nu < \infty$, $\int G_\alpha \mu \, d\nu = \int G_\alpha \mu_F \, d\nu$. 


then $\mu_p$ is a real measure in $\mathbb{R}^n$, absolutely continuous outside 0, $\mu_p = \overline{\mu}_p$ and $\int d|\mu_p| < \infty$, where $|\mu_p|$ denote the total variation of $\mu_p$. Since $(p\chi + \varepsilon) * \mu_p = \mu$, Lemma 3 gives that, for any $f \in C_c(D)$, $(pV_\mu + 1)(V_{\mu}*f) = V_xf$. Let $(V_\mu)_{\mu \geq 0}$ the resolvent associated with $V_x$. In the same manner as in Theorem 1, we have, for any $f \in C_c(D)$, $V_\mu f = V_{\mu}*f$ in $D$. Hence $V_{\mu}$ is positive. In the same manner as in Lemma 5, we have $\frac{d}{dx_1}\mu_p \leq 0$ in the sense of distributions in $D$. We show that $\mu_p$ is a convolution kernel. It suffices to prove that, for any $f \in C_c(D)$, $\int_D f d\mu_p > 0$, because

$$\mu_p(\{0\}) = \frac{\chi(\{0\})}{1 + p\chi(\{0\})} > 0,$$

and $\mu_p$ is absolutely continuous outside 0. For each integer $k \geq 1$, we choose a non-negative, spherically symmetric and infinitely differentiable function $\varphi_k$ in $\mathbb{R}^n$ such that $\int \varphi_k dx = 1$ and $\text{supp} (\varphi_k) \subset \left\{ x \in \mathbb{R}^n ; |x| < \frac{1}{k} \right\}$. Since $\frac{d}{dx_1} \mu_p * \varphi_k(x) \leq 0$ in the set

$$\left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n ; x_1 \geq \frac{1}{k} \right\}$$

and $\lim_{|x| \to \infty} \mu_p * \varphi_k(x) = 0$, we have $\mu_p * \varphi_k(x) \geq 0$ in the above set. Hence, for any $f \in C_c(D)$,

$$\int_D f d\mu_p = \lim_{k \to \infty} \int_{x_1 \geq \frac{1}{k}} f(x) \mu_p * \varphi_k(x) dx > 0.$$

Consequently $\mu_p$ is a convolution kernel ($\forall p \in (0,1]$). Since $\mu - \mu_p = p\mu * \mu_p$, $\mu \geq \mu_p$. For a $p \in (1, 2]$, we put

$$\mu_p = \sum_{k=0}^{\infty} (1 - p)^k (\mu_1)^{k+1};$$

then $\mu_p$ is also a real measure in $\mathbb{R}^n$, absolutely continuous outside 0, $\mu_p = \overline{\mu}_p$, $\int d|\mu_p| < \infty$ and $\mu - \mu_p = p\mu * \mu_p$. In the same manner as above, $\mu_p$ is a convolution kernel. Inductively we obtain a family $(\mu_p)_{p \geq 0}$ of convolution ker-
nels satisfying $x - x_p = p_x * x_p$ and $\lim_{p \to 0} x_p = x$ (vaguely). By Lemma 3.2 in [6], we obtain that, for each $p \geq 0$ and $q > 0$, $x_p - x_q = (q - p)x_p * x_q$ and $\lim_{p \to 0} x_p = x$ (vaguely), where $x_0 = x$. Since $V_x$ is a Hunt kernel on $D$, $x \neq 0$, and hence, for any $x \neq 0 \in \mathbb{R}^n$, $x \neq x * \varepsilon_x$, because

$$\lim_{|x| \to \infty} x * f(x) = 0$$

for any finite continuous function $f$ in $\mathbb{R}^n$ with compact support. Hence, by Corollary 1 of Theorem 5 in [6], $x$ is a Hunt convolution kernel. This completes the proof.

**Remark 24.** — In the above proposition, if $x$ is spherically symmetric, the same conclusion holds without the assumption that $x$ is absolutely continuous outside 0. See Remark 19.

**BIBLIOGRAPHY**


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