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RUNGESCHER SATZ AND A CONDITION FOR STEINESS FOR THE LIMIT OF AN INCREASING SEQUENCE OF STEIN SPACES

by Alessandro SILVA

Introduction.

It has been asked sometimes whether the limit of an increasing sequence of Stein spaces is itself a Stein space. The question has been raised again recently (see [7]), and J. E. Fornaess with a striking example ([5]) based on J. Wermer's one, ([11]), very recently showed that the answer in general is no. In the positive the criterion given by Behnke-Stein (in [3], 1939) for domains of holomorphy in $\mathbf{C}^n$, and its generalization to abstract Stein spaces, (various « lemme de passage à la limite pour la cohomologie des faisceaux »; for a very general one see [6] and section (0.7)) has been, so far, the best one : (Behnke-Stein Theorem). Let X be a complex analytic space,

$$X = \bigcup_{i \in \mathbf{N}} X_i .$$

Suppose $X_i \subset\subset X_{i+1}$, X_i open and Stein for every i, $i = 0, 1, \ldots$. Then, if (X_{i+1}, X_i) is a Runge pair for every i, $i = 0, 1, \ldots$, X is Stein.

Their proof uses a method of successive approximations which goes back to Mittag-Leffler. What we wish to show here is that, although the converse of Behnke-Stein theorem does not obviously hold [1], there is nevertheless a kind of « weak » converse of it, peculiar to the countable sequence situation.

This result, proved in general for cohomology classes, in the Stein case will sound as follows: X is Stein if and only if for every i there exists j, $j > i$, such that for every k, $k > j$ every holomorphic function on X_j restricted to X_i can be approximated on the compact subset of X_i by restrictions of holomorphic functions on X_k. From that we will derive a condition for the Steiness of X: if $H^1(X, \mathcal{O}) = 0$ or (more generally), $H^1(X, \mathcal{O})$ is an Hausdorff topological vector space, then X is Stein. As a consequence of this, a further property of the Fornaess' manifold will be also shown. A. Markoe has independentely proved (see [12]) the main result of this work (th. 1.2) for $q = 1$. His technique, though, is different. While our method is devised to avoid the use of duality theorems, in order to avoid the problems of separatility arising from such use, Markoe's result relies heavily on Ramis-Ruget-Verdier duality theorems and separation criteria, see [13].

0. Preliminaries and notation.

(0.1) The complex analytic spaces considered throughout this paper will have countable topology. The property of an analytic sheaf $\mathcal{F}$ of being coherent on the complex space X will be denoted by $\mathcal{F} \in \mathrm{Coh}(X)$.

If $\mathcal{F} \in \mathrm{Coh}(X)$, the groups $H^q(X,\mathcal{F}) \overset{\text{def}}{=} \varinjlim H^q(\mathcal{U},\mathcal{F})$ are endowed with the inductive topology, (where $\mathcal{U}$ is a Cèch covering and $H^q(\mathcal{U},\mathcal{F}) \underset{\text{top}}{\cong} \ker \partial_q / \mathrm{Im}\, \partial_{q-1}$ if

$$\partial_q : \mathcal{C}^q(\mathcal{U},\mathcal{F}) \to \mathcal{C}^{q+1}(\mathcal{U},\mathcal{F})$$

is the coboundary map and the $\mathcal{C}^q(\mathcal{U},\mathcal{F})$ are endowed with their natural Frèchet-Schwartz topology). In view of [4], Prop. 4, the algebraic isomorphism $H^q(X,\mathcal{F}) \cong H^q(\mathcal{U},\mathcal{F})$, where $\mathcal{U}$ is a Leray covering is also topological. The inductive

[1] Indeed, take $X = \mathbf{C}$, $S_i = \{z \in \mathbf{C} : |z| < i + 1\}$,

$$i = 0, 1, \ldots, a_i = (2i+1)/2, \quad \text{for} \quad i = 0, 1, \ldots .$$

Let $X_i = S_i - \{a_i\}$; we have $X_i \subset\subset X_{i+1}$ and $\cup X_i = X$, but (X_{i+1}, X_i) is not a Runge pair.

topology on $H^q(X,\mathscr{F})$ is then identified with the quotient of Frèchet topology on $\ker \partial_q/\operatorname{Im} \partial_{q-1}$ and it is independent of the particular Leray covering chosen. We will denote in the following the complex of Frèchet spaces $(\mathscr{C}^\cdot(\mathscr{U},\mathscr{F});\partial.\}$ by $\{A^\cdot;\delta.\}$.

(0.2) The projective (inverse) limit of a projective system $\mathbf{F}=\{F_\alpha;\ i_\alpha^\beta\}$ of topological vector spaces, (TVS), will be denoted by $\varprojlim$. If

$$(0.3)\qquad 0 \to \mathbf{F} \to \mathbf{E} \to \mathbf{G} \to 0$$

is an exact sequence of projective systems of TVS, by taking the projective limit one obtains a left exact sequence of TVS:

$$0 \to \varprojlim F_\alpha \to \varprojlim E_\alpha \to \varprojlim G_\alpha .$$

Following Roos, A. Ogus, in [14], has constructed the derived functors $\varprojlim^i$ (or $R^i \varprojlim$) and has shown that they are zero for $i \geqslant 2$, if α runs in a countable sequence of indices. Since the TVS category has not enough injectives, this explicit construction is needed if one wants to consider the exact sequence:

$$0 \to \varprojlim F_\alpha \to \varprojlim E_\alpha \to \varprojlim G_\alpha \\ \to \varprojlim{}^1 F_\alpha \to \varprojlim{}^1 E_\alpha \to \varprojlim{}^1 G_\alpha \to 0 .$$

In our situation, it is easy to show that, by functoriality, the derived functors of $\varprojlim$ naturally inherit a TVS structure.

(0.4) Let $\{X_i\}_{i\in\mathbf{N}}$ be a sequence of complex analytic spaces, with $X_i \subset\subset X_{i+1}$, $\mathscr{F}$, $\mathscr{F}\in \operatorname{Coh} X$ and q, $q \geqslant 1$, be a fixed integer. We will say that $(\{X_i\},\mathscr{F})$ satisfies a *weak q-Runge property* if for every i, $i=0,1,\ldots,$ there is j, $j>i$, such that for every k, $k>j$, the restriction maps $H^{q-1}(X_k,\mathscr{F}) \to H^{q-1}(X_j,\mathscr{F})$ have dense images in the topology on $H^{q-1}(X_j,\mathscr{F})$ induced from the one on $H^{q-1}(X_i,\mathscr{F})$ (2).

(2) A formal adaptation of the proof of Lemma p. 246 of [1] shows that this property is equivalent to the following, stronger in appearance: let us denote by X the union $\cup_i X_i$. Then for every i, there exists j, $j>i$, such that the restriction map $H^{q-1}(X,\mathscr{F}) \to H^{q-1}(X_j,\mathscr{F})$ has dense image in the topology on $H^{q-1}(X_j,\mathscr{F})$ induced from the one on $H^{q-1}(X_i,\mathscr{F})$. This will be understood from now on.

(0.5) This is equivalent to say that for every i , $i = 0, 1, \ldots$, there exists j , $j > i$, such that, for every k , $k > j$, every element in $\Gamma(X_j, \ker \delta_{q-1})$ restricted to $\Gamma(X_i, \ker \delta_{q-1})$ can be approximated on the compact subsets of X_i by elements in $\Gamma(X_k, \ker \delta_{q-1})$, (restricted to $\Gamma(X_i, \ker \delta_{q-1})$).

Let us show it. For $q = 1$ there is nothing to prove. Suppose then $q > 1$. The first implication is a consequence of the following simple fact:

(0.6) If in a commutative diagram of linear maps of TVS

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\
& E \arrow[rr] & & F \\
C \arrow[ur] \arrow[rr] & & D \arrow[ur] &
\end{tikzcd}$$

the horizontal maps are surjective and the image of A (in E) is dense in the image of C in E , then the image of B (in F) is dense in the image of D in F .

Conversely, for every seminorm π on $H^{q-1}(X_i, \mathscr{F})$ and for every $\varphi \in \Gamma(X_j, \ker \delta_{q-1})$ there is a sequence $\{\varphi_n\}$ in $\Gamma(X_k, \ker \delta_{q-1})$ such that $\pi(r_i^j[\varphi], r_i^k[\varphi_n]) \to 0$. Since σ_j is a topological homomorphism (being surjective), there is a sequence $\{\psi_n\}$ in $\Gamma(X_j, \ker \delta_{q-1})$ such that

$$[\varphi] - r_j^k[\varphi_n] = [\psi_n]$$

and such that $\pi(r_i^j[\psi_n]) \to 0$. $\varphi - \rho_j^k(\varphi_n) - \psi_n$ then represents an element in $\Gamma(X_j, \mathrm{Im}\, \delta_{q-2})$, so that there is, for every n , $\chi_n \in \Gamma(X_j, A^{q-2})$ such that $\rho_j^k(\varphi_n) = \varphi + \psi_n + \delta_{q-2}(\chi_n)$. Take χ_n' in $\Gamma(X_k, A^{q-2})$ such that: $\tilde{\pi}(\rho_i^j(\chi_n), \rho_i^k(\chi_n')) \to 0$; ($\chi_n'$ exists since $A^{\cdot}$ is flabby). We have

$$\tilde{\pi}(\rho_i^j(\varphi), \quad \rho_i^k(\varphi_n - \delta_{q-2}(\chi_n'))) \to 0 ,$$

and the proof of the equivalence is complete.

If $q = 1$, we will say that $(\{X_i\}, \mathscr{F})$ satisfies a weak Runge property [3].

[3] The example in footnote (1) gives also an example of a pair $(\{X_i\}, \mathcal{O})$ with (X_{i+1}, X_i) not a Runge pair in the usual sense, but satisfying a weak Runge condition for $j = i + 1$. Indeed every holomorphic function on X_{i+1} restricted to X_i can be holomorphically extended to S_i .

(0.7) A link between projective limits of cohomology groups and weak Runge property is given by the following general Mittag-Leffler principle (see [6]):

THEOREM. — *Let* $\{X_i\}_{i\in \mathbf{N}}$ *be a sequence of complex analytic spaces with* $X_i \subset\subset X_{i+1}$ *for every* i, $i = 0, 1, \ldots$; *let* q, $q \geqslant 1$ *be a fixed integer and* $\mathcal{F}$, $\mathcal{F} \in \mathrm{Coh}(X)$. *Suppose that for every* r, $r \geqslant q$, *and for every* i, $i = 0, 1, \ldots$ *we have an exact sequence of morphisms of TVS:*

$$0 \to \Gamma(X_i, \mathrm{Ker}\ \delta_{r-1}) \to \Gamma(X_i, A^{r-1}) \xrightarrow{\delta_{r-1}} \Gamma(X_i, \ker \delta_r) \to 0$$

(*i.e.* $H^r(X_i, \mathcal{F}) = 0$). *Then, if* $X = \bigcup_{i\in \mathbf{N}} X_i$,

$$H^r(X, \mathcal{F}) \cong \varprojlim H^r(X_i, \mathcal{F}) = 0 \quad \text{for} \quad r \geqslant q+1\,.$$

If, moreover, the pair $(\{X_i\}, \mathcal{F})$ *satisfies a weak q-Runge property, one has also:*

$$H^q(X, \mathcal{F}) \cong \varprojlim H^q(X_i, \mathcal{F}) = 0\,.$$

This principle holds in more general categories (see [6]) (*).

In the analytic category the use of more or less stronger version of it has been fundamental in proving the basic results in the theory of Stein spaces (Cartan's theorems A and B) and its generalizations (Andreotti-Grauert finiteness theorems for convex-concave spaces [1]).

1. A weak converse of Behnke-Stein theorem.

(1.1) We wish to prove here the weak converse of Behnke-Stein theorem announced in the introduction and to show that it holds also for cohomology classes.

(1.2) THEOREM. — *Let* X *be a complex analytic space,* $X = \bigcup_{i\in \mathbf{N}} X_i$, X_i *open in* X *for every* i, $i = 0, 1, \ldots$; q, $q \geqslant 1$,

(*) See also: V.P. Palamodov, *Mat. Sb.*, 75 (1968), 567-603.

and $\mathscr{F}$, $\mathscr{F} \in \mathrm{Coh}(X)$ *be fixed, such that:*

$$\text{(i)} \qquad X_i \subset\subset X_{i+1}, \quad i = 0, 1, \ldots,$$
$$\text{(ii)} \qquad H^q(X_i,\mathscr{F}) = H^q(X,\mathscr{F}) = 0$$

for $i = 0, 1, \ldots$ *. The pair* $(\{X_i\},\mathscr{F})$ *then satisfies a weak q-Runge property.*

Proof. — The proof being rather long let us describe here its main steps. In step (α) we will put on $K_i^r = \Gamma(X_i, \ker \delta_r)$ a topology τ_i, besides the usual one induced by $A_i^r = \Gamma(X_i, A^r)$ that will be denoted by σ_i (4). In step (β) we will prove that the topologies τ_i are topologies of complete TVS, under the assumptions of the theorem. In the final step (γ), a contradiction will be found, if we negate the Runge property, based on the fact that every sequence $\{\omega_j\}$ in K_j, with ω_j the restriction to K_i of an element in K_j, $j > i$, is a Cauchy sequence in the topology τ_i.

(α) τ_i is described as follows: consider the filtration

$$K_i \supset \rho_i^{i+1}(K_{i+1}) \supset \ldots \supset \rho_i^{i+k}(K_{i+k}) \supset \ldots$$

of K_i, where the ρ_i^l are the restriction maps. The $\rho_i^l(K_l)$ being subspaces, one can consider them as a basis of 0-neighborhoods of a vector space topology on K_i. We will denote it by τ_i.

(β) Let us show that, under the assumptions (i) and (ii) of the theorem, τ_i are topologies of complete TVS. For every i, $i = 0, 1, \ldots$, we have the exact sequences of morphisms of Frèchet spaces:

$$0 \to K_i \to A_i \xrightarrow{\delta_{q-1}} K_i^q \to 0 .$$

The restrictions being continuous one has then the exact sequence of inverse systems:

$$0 \to \{K_i, \rho_i^j\} \to \{A_i, \rho_i^j\} \to \{K_i^q, \rho_i^j\} \to 0 .$$

Hence one can apply the functor $\varprojlim$ to obtain the long exact sequence:

$$(1.3) \qquad 0 \to \varprojlim K_i \to \varprojlim A_i \to \varprojlim K_i^q \to \varprojlim{}^1 K_i \to \ldots .$$

(4) We will write in the sequel K_i or A_i for K_i^{q-1} or A_i^{q-1}.

Consider now K_i with the topology τ_i. For every i, $i = 0, 1, \ldots,$ we have an exact sequence of TVS:

$$0 \to K_i \xrightarrow{\varepsilon_i^i} \prod_{0}^{i}{}_j K_j \xrightarrow{\pi} \operatorname{coker} \varepsilon_i^i \to 0,$$

where $\varepsilon_i^i(\omega) = (\rho_0^i(\omega), \rho_1^i(\omega), \ldots, \rho_j^i(\omega), \ldots, \omega)$ and π is the canonical surjection. Let $B_i = \prod_0^i{}_j K_j$, and define ε_i^l: $K_l \to B_l$ by $\varepsilon_i^l(\omega) = (\rho_0^l(\omega), \ldots, \rho_i^l(\omega), 0, \ldots, 0)\, l > i$. Then the diagram

$$\begin{array}{ccc} K_l & \xrightarrow{\varepsilon_i^l} & B_l \\ {\scriptstyle \rho_i^l}\downarrow & & \downarrow{\scriptstyle p_i^l} \\ K_i & \xrightarrow{\varepsilon_i^i} & B_i \end{array}$$

where p_i^l is the projection on the product of the first i factors, commutes and all the morphisms involved are homomorphisms of TVS. We have so the inverse systems of TVS: $\{K_i; \rho_i^j\}$, $\{B_i; p_i^j\}$, $\{\operatorname{coker} \varepsilon_i^i; \tilde{p}_i^j\}$, where the $\tilde{p}_i^j$'s are the maps induced by taking the quotients and we can consider also an exact sequence of inverse systems linking them. Applying $\varprojlim$ we obtain the sequence:

$$(1.4) \quad 0 \to \varprojlim K_i \to \varprojlim B_i \xleftarrow{\varprojlim \pi} \varprojlim \operatorname{coker} \varepsilon_i^i \to \varprojlim{}^1 (K_i) \to \varprojlim{}^1 (B_i) \to \ldots,$$

and let us remark that $\varprojlim B_i \cong \prod_0^{+\infty}{}_i K_i$, so that, if we start with the topologies τ_i on K_i, we can endow $\varprojlim (B_i)$ with the product topology, that we will denote by τ.

By construction, we have:

$$(1.5) \qquad \varprojlim{}^1 (B_i) = 0.$$

If we put (1.5) into (1.4) we obtain:

$$(1.6) \qquad \varprojlim{}^1 (K_i) \cong \operatorname{coker} \varprojlim \pi,$$

and from (1.3) and assumption (ii) in (1.2), we get

$$\text{coker} \varprojlim \pi = 0 . \tag{1.7}$$

To conclude the proof of (β) we will show that (1.7) implies that K_i^{q-1} is complete in the topology τ_i. To see that consider the isomorphisms :

$$\varprojlim B_i \cong \prod_0^{+\infty}{}_i K_i ,$$

and

$$\varprojlim \text{coker } \varepsilon_i^i \cong \varprojlim \prod_0^{+\infty}{}_i K_i/\text{Im}\left(\varepsilon_i : \prod_i^{+\infty}{}_j K_j \to \prod_0^{+\infty}{}_j K_j\right),$$

ε_i being given by ε_i^i on the factor K_i and the identity on the other factors. Then, if $\varprojlim B_i$ is complete when it is endowed with the topology τ, the topology τ_i which is the image topology under the projection on the i^{th} factor, K_i, will be itself complete (Bourbaki, Top. Gen. IX, § 2, 3rd ed.), and (β) will be proved. But (1.7) implies that $\varprojlim \pi$ is an epimorphism of TVS, then the completeness of $\varprojlim B_i$ in the topology τ will follow if we prove that $\varprojlim$ coker ε_i^i is isomorphic to the completion (in the sense of Bourbaki, Top. Gen. II, § 3, 3rd ed.) $[\varprojlim B_i]$ of $\varprojlim B_i$.

By definition of τ, any Cauchy sequence in $\varprojlim B_i$ gives a Cauchy sequence in coker ε_i^i for every i, $i = 0, 1, \ldots,$ so that a map $\varphi : [\varprojlim B_i] \to \varprojlim$ coker ε_i^i is defined. We want to show that φ is an isomorphism of TVS; let us determine φ^{-1}. Let $\tilde{t} \in \varinjlim$ coker ε_i^i, $\tilde{t} = (\tilde{t}_i)_{i \in \mathbf{N}}$, $\tilde{t}_i \in$ coker ε_i^i, and let $t_i \in B_i$ such that $\pi(t_i) = \tilde{t}_i$, and $(t_i)_{i\in\mathbf{N}} \in \varprojlim B_i$. Since $\tilde{p}_i^j(\tilde{t}_j) = \tilde{t}_i$, $t_i - t_j$ belongs to B_i for $j > i$: the sequence $\{t_i\}$ is then a Cauchy sequence in $\varprojlim B_i$; an element $[t]$ on $[\varprojlim B_i]$ corresponds to it and it is easy to check that the map $\varphi^{-1} : \tilde{t} \to [t]$ is independent of the choice of the $t_i' s$: φ^{-1} is then well defined and continuous and φ is an isomorphism of vector spaces. The topology $[\tau]$ on $[\varprojlim B_i]$ is moreover the weakest of all topologies that make the maps $\varphi_i : [\varprojlim B_i] \to$ coker ε_i^i induced by φ continuous. Indeed, let τ' be a weaker topology with the same property and U a 0-neighborhood in $[\tau]$. Then $\varphi_i(U)$ is a 0-neighborhood in coker ε_i^i, and the set $\varphi_i^{-1}\varphi_i(U) = U + B_i$ is a 0-neighborhood in $[\varinjlim B_i]$ in the topology τ'. Letting U and i vary the set $\{U + B_i\}$ is a fundamental 0-neighborhood system for the topology $[\tau]$. Hence, τ' is finer than $[\tau]$ so that $[\tau] = \tau'$. In view of Bourbaki, Top. Gen., I, 4, 4, 3rd, this is enough to conclude that φ is an isomorphism of TVS and (β) is then proved.

(γ) We can now finish the proof of the theorem. We have to show that for every i, $i = 0, 1, \ldots$, there is j, $j > i$, such that for every convex 0-neighborhood U in K_i, $\rho_i^j(K_j) \subset U + \rho_i^k(K_k)$, for every $k > j$. Suppose the contrary; then there exists U as above, and a rearrangement $\{\nu_k\}_{k \geqslant 0}$ of the sequence $\{k\}_{k \geqslant i}$, with $\nu_0 = i$, such that $\rho_{\nu_0}^{\nu_k}(K_{\nu_k}) \not\subset U + \rho_{\nu_0}^{\nu_{k+1}}(K_{\nu_{k+1}})$, for every k. Let us define inductively a sequence $\{\omega_k\}$, $\omega_k \in \rho_{\nu_0}^{\nu_k}(K_{\nu_k})$ for every k, and a familly $\{\omega_k'\}$ of linear functionals on K_i, satisfying the following properties :

$$\omega_k' = 0 \quad \text{on} \quad K_{\nu_{k+1}}, \tag{1}$$

$$\sup_{\omega \in U} |\langle \omega_k', \omega \rangle| \leqslant 1, \tag{2}$$

$$|\langle \omega_k', \sum_{\nu_0}^{\nu_k}{}_l\, \omega_l \rangle| \geqslant k. \tag{3}$$

Suppose $\omega_0, \ldots, \omega_{k-1}$ and $\omega_0', \ldots, \omega_{k-1}'$ have been chosen accordingly. Pick up $\eta \in \rho_{\nu_0}^{\nu_k}(K_{\nu_k})$ but $\eta \notin U + \rho_{\nu_0}^{\nu_{k+1}}(K_{\nu_{k+1}})$. By the convexity of $U + \rho_{\nu_0}^{\nu_{k+1}}(K_{\nu_{k+1}})$ and by (a consequence of) the Hahn-Banach theorem (see f.i. [9], chapter 15) we can find a linear functional ω_k' on K_i such that $|\langle \omega_k', \eta \rangle| \geqslant 1$ and satisfying (1) and (2). By, if necessary, replacing η by $c\eta$ for c large enough, (3) is also verified if we set $\omega_k = c\eta$.

The series $\sum_{\nu_0}^{+\infty}{}_l\, \omega_l$ converges to ω, $\omega \in K_i$, in the topology τ_i. Indeed, $\{\rho_{\nu_k}^{\nu_{k+1}}(K_{\nu_{k+1}})\}$, is a 0-neighborhood basis for τ_i so that, for every k, $\rho_{\nu_k}^{\nu_{k+1}}(K_{\nu_{k+1}}) + \rho_{\nu_k}^{\nu_{k+1}}(K_{\nu_{k+1}}) \subset \rho_{\nu_{k-1}}^{\nu_k}(K_{\nu_k})$ (after having rearranged $\{\nu_k\}$ if necessary). Then for every k and m the sum $\sum_{\nu_{k+1}}^{\nu_{k+m}}{}_l\, \omega_l$ belongs to $\rho_{\nu_k}^{\nu_{k+1}}(K_{\nu_{k+1}})$, hence it is a Cauchy series in the topology τ_i and converges to an element $\omega \in K_i$.

Consider now $\{\omega'_k\}$. By (2) it is a bounded sequence on U, but by (1) and (3), we have:

$$|\langle\omega'_k,\omega\rangle| = \left|\left\langle\omega'_k, \sum_{\nu_0}^{\nu_k} \omega_k\right\rangle\right| \geqslant k\,, \quad \text{contradiction.}$$

The proof of theorem (1.2) is then completed.

2. A criterion for Steiness of the limit of an increasing sequence of Stein spaces.

(2.1) Let us recall that a complex space X is said to be cohomologically q-complete if $H^r(X, \mathscr{F}) = 0$ for $r \geqslant q$ and for every $\mathscr{F}$, $\mathscr{F} \in \mathrm{Coh}(X)$. X is then Stein if and only if is cohomologically 1-complete. We have:

(2.2) Proposition. — *Let $\{X_i\}$ be a sequence of cohomologically q-complete spaces with $X_i \subset\subset X_{i+1}$ for $i = 0, 1, \ldots$. $X = \bigcup_{i\in\mathbf{N}} X_i$ is cohomologically q-complete if and only if for every $\mathscr{F}$, $\mathscr{F} \in \mathrm{Coh}\,(X)$, the pair $(\{X_i\},\mathscr{F})$ satisfies a weak q-Runge property.*

Proof. — Immediate, from (0.7) and (1.2).

For $q = 1$ the above necessary and sufficient condition can be relaxed:

(2.3) Proposition. — *Let $\{X_i\}$ be a sequence of Stein spaces with $X_i \subset\subset X_{i+1}$ for $i = 0, 1, \ldots$. Then $X = \bigcup_{i\in\mathbf{N}} X_i$ is a Stein space if and only if $(\{X_i\},\mathcal{O})$ satisfies a weak Runge property.*

Proof. — In view of (0.7), (1.2) and (2.2), we need only to show that if $(\{X_i\},\mathcal{O})$ satisfies a weak Runge property, then $(\{X_i\},\mathscr{F})$ does as well, for every $\mathscr{F}$, $\mathscr{F} \in \mathrm{Coh}\,(X)$. This can be seen by the following adaptation of a standard argument: for every i, $i = 0, 1, \ldots,$ let j, $j > i$, be the least integer such that the elements in $\rho_i^j(\Gamma(X_j,\mathcal{O}))$ can be approximated (on the compact subsets of X_i) by elements in $\rho_i^k(\Gamma(X_k,\mathcal{O}))$ for $k > j$, and let $\mathscr{F}$ be any coherent sheaf on X. X_k being Stein, Cartan's theorem A holds on X_k for the sheaf $\mathscr{F}|_{X_k}$, for every k. Hence we can consider the diagram of sheaf homomorphisms :

$$\begin{array}{ccc} \mathcal{O}^{p_k} & \xrightarrow{\tilde{\sigma}_k} & \mathscr{F}|_{X_k} \to 0 \\ \downarrow & & \downarrow \\ \mathcal{O}^{p_j} & \xrightarrow{\tilde{\sigma}_j} & \mathscr{F}|_{X_j} \to 0 \end{array}$$

which has exact rows in a neighborhood of $\overline{X}_k$, since X_k is relatively compact. We obtain an induced commutative diagram of continuous linear maps of Frèchet spaces :

$$(*) \qquad \begin{array}{ccc} \Gamma(X_k,\mathcal{O}^{p_k}) & \xrightarrow{\sigma_k} & \Gamma(X_k,\mathscr{F}) \to 0 \\ \rho_j^k \downarrow & & \downarrow r_j^k \\ \Gamma(X_j,\mathcal{O}^{p_j}) & \xrightarrow{\sigma_j} & \Gamma(X_j,\mathscr{F}) \to 0 \end{array}$$

which has exact rows since X_l is Stein for every l, $l = 0, 1, \ldots$. Moreover we can take $p_j = p_k = p$ (see f.i. [2], 6.3).

The conclusion then follows from (0,6).

(2.4) We can now prove the main result of this section :

THEOREM. — *Let* $\{X_i\}$ *be a sequence of Stein spaces with* $X_i \subset\subset X_{i+1}$, *for* $i=0, 1, \ldots$ *and let* $X = \bigcup_{i\in\mathbf{N}} X_i$. *Then if* $H^1(X,\mathcal{O}) = 0$, X *is Stein. More generally, if* $H^1(X,\mathcal{O})$ *has an Hausdorff topology (for instance if* $\dim_{\mathbf{C}} H^1(X,\mathcal{O}) < +\infty$), X *is Stein.*

Proof. — Since

$$H^1(X_0,\mathcal{O}) = \ldots = H^1(X_i,\mathcal{O}) = \ldots = H^1(X,\mathcal{O}) = 0,$$

the first assertion follows from (1.2) and (2.3). Let us prove the second one.

Let E a topological vector space. A Hausdorff topological vector space $\check{E}$ is, canonically, associated to it, by setting

$\check{E} = E/\overline{\{0\}}$, where $\overline{\{0\}}$ denotes the topological closure, in E, of the zero of E.

We have (cf. [4]):

(2.5) Let X be a complex analytic space, $\mathscr{F} \in \mathrm{Coh}(X)$, and q, $q \geqslant 1$, be a fixed integer. Suppose $X = \bigcup_{i \in \mathbf{N}} X_i$, $X_i \subset X_{i+1}$, X_i open for every i, $i = 0, 1, \dots$. Then if $\check{H}^q(X_i, \mathscr{F}) = 0$ for every i, $i = 0, 1, \dots$, we have: $\check{H}^q(X, \mathscr{F}) = 0$.

In particular, if $\{X_i\}$ is a sequence of Stein spaces, $\check{H}^1(X, \mathscr{F}) = 0$ for every $\mathscr{F}$, $\mathscr{F} \in \mathrm{Coh}(X)$. So that, since by assumption we have $\check{H}^1(X, \mathcal{O}) = H^1(X, \mathcal{O})$, we are reduced to the first case and the proof of (2.4) is completed.

(2.6) *Remark.* — Theorem (2.4) sharpens a result of Villani, [10], Cor. 2 where he proves that X is Stein if $H^1(X, \mathscr{I})$ is Hausdorff for every sheaf $\mathscr{I}$ of ideals of $\mathcal{O}_X$.

(2.7) *Remark.* — The assumption $X_i \subset\subset X_{i+1}$ in Theorem (2.4) *is not restrictive.* Indeed suppose $X = \cup Y_i$, $Y_i \subset Y_{i+1}$, Y_i open and Stein for every $i = 0, 1, \dots$.

We show that the sequence $\{Y_i\}$ can be replaced by a sequence $\{X_i\}_{i \in \mathbf{N}}$, with $X = \bigcup_{i \in \mathbf{N}} X_i$, $X_i \subset\subset X_{i+1}$, X_i open and Stein for every i, $i = 0, 1, \dots$.

Indeed, by the solution of the Levi problem, let us write $Y_i = \bigcup_{j \in \mathbf{N}} B_j^{(i)}$, where $B_j^{(i)}$ are the level sets of the strictly 1-convex function $\varphi^{(i)}$. Recursively define a sequence $j_0 < \dots < j_i < \dots$ such that $B_j^{(k)} \subset B_{j_i}^{(i)}$ for $k \leqslant i$ and $0 \leqslant j \leqslant \max(j_i, i)$. The sequence $\{X_i\} = \{B_{j_i}^{(i)}\}$ then satisfies the requirements. Moreover, if $(\{Y_i\}, \mathcal{O})$ satisfied a weak Runge property, so does $(\{X_i\}, \mathcal{O})$.

3. A property of Fornaess' manifold.

(3.1) Let us describe briefly Fornaess' manifold. A « Wermer's arrow » ψ is an holomorphic map $\psi : \mathbf{C}^3 \to \mathbf{C}^3$ such that $\psi|_\Delta$, where Δ is a bounded open subset, is one to one and such that the holomorphic convex hull of $\psi(K)$ with respect to any convex open set Ω, $\Omega \supset \psi(\Delta)$, is *not* contained in $\psi(\Delta)$ for every compact subset K of Δ. Fornaess in [5] shows the existence of such a map, following [11], and constructs a manifold F, $F = \cup F_i$, $F_i \subset\subset F_{i+1}$, F_i Stein for every i, $i = 0, 1, \ldots$ in such a way that the following diagram

$$\begin{array}{ccc} \Delta_i & \xrightarrow{\gamma_i} & F_i \\ {\scriptstyle \psi_i}\downarrow & & \downarrow \cap \\ \Delta_{i+1} & \xrightarrow{\gamma_{i+1}} & F_{i+1} \end{array}$$

where $\Delta_{i+1} \supset \psi_i(\Delta_i)$, Δ_i are bounded convex open subsets of $\mathbf{C}^3$, the γ_i are biholomorphic maps and the ψ_i are Wermer's arrows, commutes.

He shows that F *is not Stein.* From the results of the preceeding sections we obtain an extra property of F:

(3.2) PROPOSITION. — $H^1(F, \mathcal{O})$ *is non Hausdorff; it is (uncountably) infinite dimensional* [5], (*in particular the* $\bar{\partial}$-*problem is not solvable on* F).

Proof. — Since the F_i's are Stein and F is not, $H^1(F, \mathcal{O})$ cannot be Hausdorff in view of (2.4), hence it cannot be finite dimensional. The non-countability follows from [8], th. A.

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[5] Remark that one has also $H^2(F, \mathcal{F}) = H^3(F, \mathcal{F}) = 0$, for every $\mathcal{F}$, $\mathcal{F} \in \mathrm{Coh}(F)$.

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