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# EXPOSÉ ON A CONJECTURE OF TOUGERON

**by Joseph BECKER**

In [8, Theoreme A] Tougeron has proven that an algebraic map of algebraic varieties induces an open map (in Krull topology) of the local analytic rings and that the image of the induced map of the completions is a closed subspace. Also [7, Theoreme B′] he asks if the image of the local analytic ring is closed and points out that this would follow from conjecture 1.9. In this note I prove this conjecture and give a different proof of Theoreme A, showing that it's really a result of algebraic geometry. Unfortunately I cannot see how to prove theorem B in a more general algebraic setting.

All the results here will be stated and proven for analytic varieties over the complex numbers; however, we could just as well work over a nondiscrete valued field of characteristic zero.

## 0. Restatement of theorem and comparison to other known results.

In [3] Artin has proven that if $f_i(x, y) \in \mathbf{C}\langle\langle x, y\rangle\rangle$ is a finite system of convergent power series equations, $c$ a positive integer, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_N)$, with a formal solution $\bar{y}(x) \in \mathbf{C}[[x]]$, then there is a convergent solution $y(x) \in \mathbf{C}\langle\langle x\rangle\rangle$ with $y \equiv \bar{y} \bmod m^c$. Also he posed the following question: if some of the $\bar{y}$'s are independent of some of the $x's$, can the new solution $y(x)$ be picked to be independent of the same $x's$. In [5] Gabrielov has given a counterexample to this conjecture. I now give a slightly different version of Gabrielov's example. Recall Gabrielov gave convergent power series $\varphi_1(x_1, x_2)$, $\varphi_2(x_1, x_2)$, $\varphi_3(x_1, x_2)$, $h(x_1, x_2)$ without constant term and formal power series $f(y_1, y_2, y_3)$ with $f(\varphi_1(x), \varphi_2(x), \varphi_3(x)) = h(x)$ and such that there is no convergent power series $g(y_1, y_2, y_3)$ with $g(\varphi_1(x), \varphi_2(x), \varphi_3(x)) = h(x)$. By Taylor expansion (i.e.

$$f(y) - f(\varphi(x)) = \Sigma(y - \varphi(x))^\alpha \frac{1}{\alpha\,!} f^{(\alpha)}(\varphi(x))\,.),$$

we have that following holds both formally and convergently:

$$\exists f \in (\mathbf{C}[[y]],\ \mathbf{C}\langle\langle y\rangle\rangle) \qquad \text{so} \qquad f(\varphi(x)) = h(x)$$

$$\longleftrightarrow \exists f \in (\mathbf{C}[[y]],\ \mathbf{C}\langle\langle y\rangle\rangle),\quad \mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{A}_3 \in (\mathbf{C}[[x, y]],\ \mathbf{C}\langle\langle x, y\rangle\rangle)$$

so

$$h(x) = f(y) + \sum_{i=1}^{3} \mathrm{A}_i(x, y)(y_i - \varphi_i(x)).$$

This gives a polynomial system of equations with indeterminants $x$ and $y$ and unknowns $f$, $\mathrm{A}_1$, $\mathrm{A}_2$, $\mathrm{A}_3$ which has a formal solution but no convergent solution. Note that one of the unknowns, $f(y)$, is required to be independent of two of the indeterminants, $x_1$ and $x_2$.

Theorem B of this paper says that this situation can not occur if $\varphi(x)$ is a polynomial. Hence if $h(x)$ is a convergent power series, $p_1(x)$, $p_2(x)$, $p_3(x)$ are polynomials then the system of equations

$$\exists f \in (\mathbf{C}[[y]],\ \mathbf{C}\langle\langle y\rangle\rangle),\ \mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{A}_3 \in (\mathbf{C}[[x, y]],\ \mathbf{C}\langle\langle x, y\rangle\rangle)$$

so

$$h(x) = f(y) + \sum_{i=1}^{3} \mathrm{A}_i(x, y)(y_i - p_i(x))$$

has a convergent solution if and only if it has a formal solution.

## 1. Open mappings of Krull topologies.

Let R, S and T be commutative local rings with unit element and with maximal ideals $m_{\mathrm{R}}$ and $m_{\mathrm{S}}$, respectively.

Let $\varphi : R \to S$ be a local ring homomorphism, i.e. $\varphi(m_R) \subset m_S$. Then $\varphi$ is continuous in the $m$-adic topologies on R and S. Such a map is open if and only if for every integer $j > 0$, there exists integer $k_j > 0$ such that $\varphi(m_R^j) \supset m_S^{k_j} \cap \varphi(R)$. We say that R is a subspace of S if for every $j > 0$, there exists $k_j > 0$ such that $m_R^j \supset \varphi^{-1}(m_S^{k_j})$. The following are obvious :

1.1. — If R is a subspace of S, then $\varphi$ is injective and open.

1.2. — If $\varphi$ is open and injective, then R is a subspace of S.

1.3. — If $\varphi$ is open and $p = \ker \varphi$, then $R/p$ is a subspace of S.

1.4. — If a local hom $\psi : R \to T$ is surjective, then $\psi$ is open.

1.5. — If $\psi : R \to T$, $\eta : T \to S$ are local, $\psi$ is open and T is a subspace of S, then $\eta \circ \psi$ is open.

1.6. — If $\psi$ is onto and $\eta$ is open, then $\eta \circ \psi$ is open.

1.7. — If R is a subspace of T and T is a subspace of S, then R is a subspace of S.

1.8. — If $\psi : R \to T$, $\eta : T \to S$ are local, and R is a subspace of S, then R is a subspace of T.

1.9. — If $\psi : R \to T$, $\eta : T \to S$ are local, $\eta$ injective, and $\eta \circ \psi$ is open, then $\psi$ is open.

1.10. — If R is Noetherian, then R is a subspace of $\hat{R}$, the completion in the $m$ adic topology.

1.11. — If $\varphi : R \to S$ is open, then $(\ker \varphi)\hat{} = \ker \hat{\varphi}$.

1.12. — If R is a subspace of S, then the induced map $\varphi : \hat{R} \to \hat{S}$ is an injection.

1.13. — $\varphi(R) \neq 0$ if and only if $\varphi^{-1}(m_S) = m_R$. We denote this condition by saying that S dominates R.

By a theorem of Chevalley [9, p. 270, Vol. II].

1.14. — If R and S are complete Noetherian local rings and S dominates R, then R is a subspace of S. More generally if R is a complete Noetherian local ring and $(a_n)$

a descending sequence of ideals of R such that $\bigcap_{n=0}^{\infty} a_n = (0)$, then there is an integer valued function $s(n)$ which tends to infinity with $n$ such that $a_n \subset m_R^{s(n)}$.

It follows immediately from 1.4, 1.14, and 1.5 that:

1.15. — If R and S are complete Noetherian local rings, then $\varphi$ is open.

By considering the commutative diagram:

$$\begin{tikzcd} R \arrow[r, two heads] & R/\ker \varphi \arrow[r, tail] \arrow[d] & S \arrow[d] \\ & \hat{R}/\ker \hat{\varphi} \arrow[r, tail] & \hat{S} \end{tikzcd}$$

and applying 1.12, 1.15, 1.10, 1.7, 1.8, 1.4 and 1.5 in that order:

1.16. — If $(\ker \varphi)^\wedge = \ker \hat{\varphi}$, then $\varphi$ is open.

If R is a ring, we say that S is a spot over R if S is the localization at a prime ideal of a finitely generated algebra over R. By the Zariski subspace theorem [1, 10.3 + 10.13]:

1.17. — If R is a spot over a field $k$, S is a spot over R, S dominates R and $\hat{R}$ is a domain, then R is a subspace of S.

We complete our list with a few observations, in which all the rings are Noetherian.

1.18. — If T is a finite R module, I is a nonzero ideal in T, and T is a domain, then $\psi^{-1}(I) = I \cap R$ is a nonzero ideal in R.

*Proof.* — T is integral over R. Let $0 \neq b \in I$ satisfy polynomial $r_n b^n + r_{n-1} b^{n-1} + \cdots + r_1 b + r_0$ over R of minimal degree. Since T is a domain $r_0 \neq 0$. But $r_0 \in R \cap I$.

1.19. — Let $\psi : R \to T$, $\eta : T \to S$ be local homs of rings with $\hat{T}$ a domain, R a subspace of S and T a finite R module, then T is a subspace of S.

*Proof.* — By 1.12 and 1.14, T is subspace of S if and only if $\hat{T}$ injects into $\hat{S}$. Suppose $p = \ker \hat{\eta} \neq 0$. Now $\hat{T}$

is a finite $\hat{R}$ module and $\hat{T}$ is a domain so by 1.18, $0 \neq p \cap \hat{R} = \ker \hat{\eta} \circ \hat{\psi}$, which contradicts the fact that $\hat{R}$ injects into $\hat{S}$.

1.20. — If R, S, T are excellent Henselian rings (such as analytic rings), $\psi : R \to T$, $\eta : T \to S$ are local, S is a domain, $\eta \circ \psi$ is open, and T is a finite R module, then $\eta$ is open.

*Proof.* — Passing to the completion, we have a commutative diagram:

$$\begin{tikzcd} \hat{R} \arrow[r] & \hat{T} \arrow[r, two heads] \arrow[dr] & \hat{T}/(\ker \eta)^{\wedge} \arrow[r, "\eta'"] \arrow[d, "\sigma"] & \hat{S} \\ & & \hat{T}/\ker \hat{\eta} \arrow[ur] & \end{tikzcd}$$

$\sigma$ is clearly surjective. Now $T/\ker \eta$ injects into the domain S so $T/\ker \eta$ is an excellent Henselian domain. Hence $(T/\ker \eta)^{\wedge}$ is a domain. By 1.11 and 1.16 $\varphi$ is open if and only if $\sigma$ is an isomorphism. Suppose $0 \neq \ker \eta' = \ker \sigma$. By 1.18 applied to $\hat{R} \to \hat{T}/(\ker \eta)^{\wedge}$, $\hat{R} \cap \ker \hat{\eta} \neq 0$. Let $g \in \hat{R}$, $\hat{\psi}(g) \in \ker \hat{\eta}$, but $\hat{\psi}(g) \notin (\ker \eta)^{\wedge}$. Since $\eta\psi$ is open, $g \in (\ker \eta\psi)^{\wedge}$. It follows immediately that $\hat{\psi}(g) \in (\ker \eta)^{\wedge}$, a contradiction.

1.21. — If $0 = \bigcap_{i=1}^{k} q_i$, $q_i$ ideals in S, $S_i = S/q_i$, and each $\varphi_i : R \to S_i$ is open where $\varphi_i = \pi_i\varphi$, then $\varphi$ is open.

*Proof.* — We have $0 = \bigcap_{i=1}^{k} \hat{q}_i$, $\ker \varphi = \cap \ker \varphi_i$, $\ker \varphi = \cap \ker \varphi_i$, and each $(\ker \varphi_i)^{\wedge} = \ker \hat{\varphi}_i$. Hence

$$(\ker \varphi)^{\wedge} = (\cap \ker \varphi_i)^{\wedge} = \cap (\ker \varphi_i)^{\wedge} = \cap \ker \hat{\varphi}_i = \ker \hat{\varphi}.$$

Let X denote the germ at a point of an algebraic subvariety of $\mathbf{C}^n$ and $\mathscr{A}(X)$ the ring of germs of algebraic on X, i.e. the localization of the affine ring at the given point; $\mathcal{O}(X)$ the ring of germs of holomorphic functions on X; $\hat{\mathcal{O}}(X)$ the completion of $\mathscr{A}(X)$ in the $m(X)$ adic topology, where $m(X)$ denotes the maximal ideal of $\mathscr{A}(X)$; $\tilde{m}(X)$ the maximal ideal of $\mathcal{O}(X)$; $\hat{m}(X)$ the maximal ideal of $\hat{\mathcal{O}}(X)$. It is well known that $\mathscr{A}(\mathrm{X})$ and $\mathcal{O}(\mathrm{X})$ have the same completions. If Y is a second germ of an algebraic variety and $f\colon \mathrm{X} \to \mathrm{Y}$ is an algebraic morphism, let $f_*\colon \mathrm{A}(\mathrm{Y}) \to \mathrm{A}(\mathrm{X})$, $\tilde{f}_*\colon \mathcal{O}(\mathrm{Y}) \to \mathcal{O}(\mathrm{X})$, $\hat{f}_*\colon \hat{\mathcal{O}}(\mathrm{Y}) \to \hat{\mathcal{O}}(\mathrm{X})$, be the homomorphisms of **C** algebras induced by $f$. We will also denote $\hat{\mathcal{O}}(\mathrm{X})$ by $\mathscr{F}(\mathrm{X})$.

THEOREM A. — *$\tilde{f}_*\colon \mathcal{O}(\mathrm{Y}) \to \mathcal{O}(\mathrm{X})$ is open, if $\mathscr{A}(\mathrm{X})$ is reduced.*

*Proof.* — All the rings in sight are analytically unramified so by 1.21 it is clearly sufficient to consider just the case where $\mathcal{O}(\mathrm{X})$ is a domain. Now $f_*$ might not be injective but we can factor it through an injection:

$$\mathscr{A}(\mathrm{Y}) \overset{\psi}{\twoheadrightarrow} \mathscr{A}(\mathrm{Y}_a) \overset{\eta}{\rightarrowtail} \mathscr{A}(\mathrm{X}).$$

If we knew $\eta$ was open, then by 1.6, $\eta\psi$ is open. Hence we also assume that $f_*$ is injective.

Now the injection $\mathscr{A}(\mathrm{Y}) \to \mathscr{A}(\mathrm{X})$ usually does *not* give rise to an injection $\mathcal{O}(\mathrm{Y}) \to \mathcal{O}(\mathrm{X})$, but we can factor the map: $\mathcal{O}(\mathrm{Y}) \rightarrowtail \mathcal{O}(\mathrm{Y}_h) \rightarrowtail \mathcal{O}(\mathrm{X})$. Now the affine ring corresponding to $\mathscr{A}(\mathrm{Y})$ can be written as a finite extension of a regular affine ring. Upon localization we have an injection $\mathscr{A}_n \to \mathscr{A}(\mathrm{Y})$ which might not be a finite extension. However the associated maps $\mathcal{O}_n \to \mathcal{O}(\mathrm{Y})$, $\mathscr{F}_n \to \mathscr{F}(\mathrm{Y})$ are both injections and finite integral extensions. We have a commutative diagram:

$$\begin{tikzcd}
\mathscr{A}(\mathrm{X}) \arrow[dd] & & \mathscr{A}(\mathrm{Y}) \arrow[ll, tail] \arrow[dd] & \mathscr{A}_n \arrow[l, tail] \arrow[dd] \\
& & & \\
\mathcal{O}(\mathrm{X}) \arrow[dd] & & \mathcal{O}(\mathrm{Y}) \arrow[ll, "f_*"'] \arrow[dl] \arrow[dd] & \mathcal{O}_n \arrow[l, tail] \arrow[dd] \\
& \mathcal{O}(\mathrm{Y}_n) \arrow[ul] \arrow[dd] & & \\
\mathscr{F}(\mathrm{X}) & & \mathscr{F}(\mathrm{Y}) \arrow[ll] \arrow[dl] & \mathscr{F}_n \arrow[l, tail] \\
& \mathscr{F}(\mathrm{Y}_h) \arrow[ul] & &
\end{tikzcd}$$

Note that $\mathcal{O}(Y_h)$ injects into the domain $\mathcal{O}(X)$. It follows that its completion $\mathcal{F}(Y_h)$ is a domain.

Now $\mathcal{A}(Y)$ is a spot over $\mathcal{A}_n$ and $\mathcal{A}(X)$ is a spot over $\mathcal{A}(Y)$ so $\mathcal{A}(X)$ is a spot over $\mathcal{A}_n$ and $\mathcal{A}_n$ is analytically irreducible so by ZST, $\mathcal{A}_n$ is a subspace of $\mathcal{A}(X)$. By 1.12 and 1.14, $\mathcal{F}_n$ is a subspace of $\mathcal{F}(X)$. By 1.10 and 1.7, $\mathcal{O}_n$ is a subspace of $\mathcal{F}(X)$. By 1.8 applied to $\mathcal{O}_n \to \mathcal{O}(X) \to \mathcal{F}(X)$, $\mathcal{O}_n$ is a subspace of $\mathcal{O}(X)$. By 1.19 applied to

$$\mathcal{O}_n \to \mathcal{O}(Y_h) \to \mathcal{O}(X),$$

$\mathcal{O}(Y_h)$ is a subspace of $\mathcal{O}(X)$. By 1.4 and 1.5 applied to $\mathcal{O}(Y) \to \mathcal{O}(Y_h) \to \mathcal{O}(X)$, $\tilde{f}_*$ is open. QED.

*Remark* 1. — By considering the map $\mathbf{C} \to \mathbf{C}^2$, given by $t \to (t^2 - 1,\ t^3 - t)$ and which has image $Y^2 = X^3 + X$, one sees that it is not true that $f_* : A(Y) \to A(X)$ is open.

*Remark* 2. — Let $N_n$ be the local ring of nash functions in $n$ variables over the complex numbers, i.e., the algebraic closure of $\mathbf{C}[z_1, \ldots, z_n]$ in $\mathbf{C}[[z_1, \ldots, z_n]]$. Then $\mathcal{A}_n \subset N_n \subset \mathcal{O}_n$, and $N_n$ is an excellent Henselian ring. Similarly if $X$ is an algebraic variety over $\mathbf{C}$, we can define $N(X)$ as the algebraic closure of $\mathcal{A}(X)$ in $\mathcal{F}(X)$. (One usually speaks of the Henselization of a ring of finite type over a field.) Theorem A extends to nash functions as follows: If $\varphi : N(Y) \to N(X)$ is a local ring homomorphism and both rings are reduced, then $\varphi$ is open and the induced map $\mathcal{O}(Y) \to \mathcal{O}(X)$ is also open. The proof given here is applicable without change as in ZST we have $N(X)$ is of finite transcendence degree over $N(Y)$.

In fact passing from $A(X)$ to $N(X)$ seems the nicest way to prove theorem A because the analytic variety $Y_h$ arising in the proof is not necessarily algebraic but is a nash variety. Details will be left to the reader.

## 2. Closed maps of Krull topologies.

Let $\varphi : R \to S$ be a local homomorphism of local Noetherian commutative rings with unit, $\varphi(R)$ is also such a ring and endowed with two possible topologies, the natural $\varphi(m_{\mathrm{R}}^{k})$, and the induced $m_{\mathrm{S}} \cap \varphi(\mathrm{R})$. These topologies are equivalent if and only if $\varphi$ is open. We will consider the completions $\hat{\varphi}(\hat{\mathrm{R}})$ and $\varphi(\mathrm{R})^{\wedge}$ in the natural and induced topologies respectively. There is a natural induced diagram

$$\begin{tikzcd}
 & \hat{\varphi}(\hat{\mathrm{R}}) \arrow[dr] \arrow[dd, dash, "\sigma"] & \\
\varphi(\mathrm{R}) \arrow[ur] \arrow[dr] & & \hat{\mathrm{S}}. \\
 & \varphi(\mathrm{R})^{\wedge} \arrow[ur] &
\end{tikzcd}$$

$\sigma$ is onto because $\hat{\varphi}(\hat{\mathrm{R}})$ and $\varphi(\mathrm{R})^{\wedge}$ are both complete local rings with the same elements as generators for the maximal ideal (as in [7, 30.6]). Hence $\hat{\varphi}(\hat{\mathrm{R}}) \cap \mathrm{S} = \varphi(\mathrm{R})$ if and only if $\varphi(\mathrm{R})^{\wedge} \cap \mathrm{S} = \varphi(\mathrm{R})$. We say that $\varphi$ is closed if $\hat{\varphi}(\hat{\mathrm{R}}) \cap \mathrm{S} = \varphi(\mathrm{R})$. We say that $\varphi$ is strongly injective if $\hat{\varphi}^{-1}(\mathrm{S}) = \mathrm{R}$, i.e. the induced homomorphism of abelian groups $\hat{\mathrm{R}}/\mathrm{R} \to \hat{\mathrm{S}}/\mathrm{S}$ is injective. The following are obvious:

2.1. — If $\varphi$ is strongly injective, then $\varphi$ is closed.

2.2. — If $\varphi$ is closed and $\hat{\varphi}$ is injective, then $\varphi$ is strongly injective.

2.3. — If $\varphi$ is injective and strongly injective, then $\varphi$ is injective, and hence R is a subspace of S.

2.4. — Mappings of complete rings are always strongly injective.

2.5. — If $\varphi$ is surjective, then $\varphi$ is closed.

2.6. — If $\psi : \mathrm{R} \to \mathrm{T}$, $\eta : \mathrm{T} \to \mathrm{S}$ are local homs and strongly injective, then $\eta \circ \psi$ is strongly injective.

2.7. — If $\psi : \mathrm{R} \to \mathrm{T}$ is closed and $\eta : \mathrm{T} \to \mathrm{S}$ is strongly injective, then $\eta\psi$ is closed.

2.8. — If $\psi : \mathrm{R} \to \mathrm{T}$ is surjective and $\eta : \mathrm{T} \to \mathrm{S}$ is closed, then $\eta\psi$ is closed.

2.9. — If $\varphi$ is closed and open, then $\mathrm{R}/\ker \varphi \to \mathrm{S}$ is strongly injective.

2.10. — If $R/\ker\varphi \to S$ is closed then $\varphi$ is closed.

2.11. — If $\psi : R \to T$, $\eta : T \to S$, and $\eta(\psi)$ is strongly injective then $\psi$ is strongly injective.

2.12. — If $q_i$ are ideals in $R$, $\bigcap_{i=1}^{k} q_i = 0$, $\pi_i : R \to R_i = R/q_i$ the natural projections, $f \in \hat{R}$ and $\hat{\pi}_i(f) \in R_i$ for each $i$, then $f \in R$.

*Proof.* — The map $R \to R_1 \oplus R_2 \oplus \cdots \oplus R_k = M$ is an injection of finite $R$ modules. Hence $\hat{R} \cap M = R$ by the Artin Reese lemma.

It follows immediately from (2.12) that:

2.13. — Let $\{p_i\}_{i=1}^{k}$ be the minimal primes of zero in $S$, $S_i = S/p_i$, $q_i = \varphi^{-1}(p_i)$, $R_i = R/q_i$, and $\varphi_i : R_i \to S_i$ the natural induced maps. If each $\varphi_i$ is strongly injective, then $\varphi$ is strongly injective.

THEOREM B. — *Let* $X$ *and* $Y$ *be reduced algebraic varieties and* $f : X \to Y$ *an algebraic map, then the induced map* $\tilde{f} : \mathcal{O}(Y) \to \mathcal{O}(X)$ *is closed.*

*Outline of Proof.* — By applying 2.13, it follows easily that it suffices to show the maps $\mathcal{O}(Y) \to \mathcal{O}(X_i)$ are closed where $X_i$ are the irreducible components of $X$. By 2.10 applied to $\tilde{f}$ it suffices to consider the case where $\tilde{f}$ is injective and show $\tilde{f}$ is strongly injective. Hence we also assume $\mathcal{O}(Y)$ is a domain. Now as in [8, theoreme B′, lemme 2.2, lemme 2.3] it suffices to prove conjecture 1.9.

DEFINITION. — *Let* $\mathcal{O}(V)$, $\mathcal{F}(V)$ *be the local analytic ring of a complex analytic variety (a reduced complete local Noetherian ring whose residue field is the complex numbers), respectively.*

By a curve $\eta$ on $V$ we mean a nonzero local algebra homomorphism $\mathcal{O}(V) \to \mathcal{O}_0(\mathbf{C})$, $(\mathcal{F}(V) \to \mathbf{C}[[t]])$, respectively. In each case the curve is given by power series $(\eta_1(t), \ldots, \eta_m(t))$. If $\eta$ is a curve on $V$ and $\nu$ an integer, $\nu > 0$ then $C_\nu(\eta, V)$ is the set of curves $\eta'$ on $V$ such that for each $i$, $\operatorname{ord}_t(\eta_i'(t) - \eta_i(t)) \geqslant \nu$. By the ideal of $\eta$ in

$\mathcal{F}(V)$ we mean the set of $f \in \mathcal{F}(V)$ such that $f(\eta(t)) = 0$. Also $I(C_v(\eta, V)) = \bigcap_{\eta' \in C_v(\eta, V)} I(\eta')$ is an ideal in $\mathcal{F}(V)$.

Definition. — *Let* $V$ *be a complex analytic variety,* $\eta(t)$ *be the normalization of a curve* $C$ *on* $V$, $\pi : \mathcal{F}(V) \to \mathcal{F}(C)$ *be the natural restriction, and* $g \in \mathcal{F}(V)$, *we say* $g$ *is analytic on* $C$ *if* $\pi(g) \in \mathcal{O}(C)$. *We say* $g$ *is weakly holomorphic on* $C$ *if* $g(\eta(t)) \in \mathcal{O}_1$. *Note that* $g$ *is analytic on* $C$ *if and only if it is weakly holomorphic on* $C$, *because* $\tilde{\mathcal{O}}(C) \cap \hat{\mathcal{O}}(C) = \mathcal{O}(C)$.

*Conjecture* 1.9. — Let $\zeta(t)$ be an analytic curve on an irreducible complex analytic $V$ with local parametrization $\pi : V \to \mathbf{C}^r$, and $\Delta(\zeta) \neq 0$, where $\Delta$ is the discriminant of $P(z', z_{r+1})$ the minimal polynomial of $z_{r+1}$ in $\mathcal{O}(V)$ over $\mathcal{O}_n$, and $z_{r+1}$ is a primitive element for the induced field extension. If $\varphi \in \hat{\mathcal{O}}(X)$ is analytic on each $\zeta' \in C_v(\zeta, V)$, then $\varphi \in \mathcal{O}(X)$.

*Outline of proof of conjecture.* — By Hironaka's proof of the resolution of singularities, there is a finite sequence of quadratic transforms of $V$, each of which is a blowing up of a nonsingular variety or an analytic isomorphism, which resolves the singularity of $V$, and so that the resolution $g : M \to V$ is an isomorphism outside the singular locus of $V$. Since $\Delta(\zeta) \neq 0$, the curve $\zeta$ is not contained in $\mathrm{SgV}$; since $g$ is onto, $\zeta$ has a unique lifting to curve $\rho(t)$ on $M$ such that $g(\rho(t)) = \zeta(t)$. (We don't need fractional power series here.) Clearly $g(C_v(\rho, M)) \subset C_v(\zeta, V)$, and $\varphi(g)$ is analytic on $\rho' \in C_v(\rho, M)$ if and only if $\varphi$ is analytic on $g(\rho') \in C_v(\zeta, V)$. By the regular case [8, Lemme 1.4], $\varphi(g)$ is analytic. It suffices to show that each of the quadratic transforms used to form $M$ induce a strongly injective map of local rings. The local ring of the blowup at a point of the fiber over $\mathrm{SgV}$ might not be analytically irreducible even though the original variety was analytically irreducible. The curve $\rho(t)$ intersects the fiber in a unique point $p$ and the curve must be in one of the irreducible components of the germ of the blowup at $p$. In section 3 we will prove that the induced map of $R$ into the local ring of each component is strongly injective. (Actually we will only consider the blowing up of the maximal ideal as the proof for other nonsingular varieties is similar and will be omitted.)

*Remark* 3. — In the above it is not necessary to invoke the resolution of singularities, actually we only need local uniformization, a far weaker result.

From [5] we have the following:

DEFINITION. — *Let* $f\colon (X, x_0) \to (Y, y_0)$ *be a germ of an analytic map,* X *will be irreducible at* $x_0$, *and* $\varphi = f_*\colon \mathcal{O}(Y, y_0) \to \mathcal{O}(X, x_0)$ *the induced map of local rings.*

*Let* $r_1 =$ *geometric rank of the map* $f$

$= $ *sup over points* $x \in X - \operatorname{Sg}X$ *near* $x_0$ *of the jacobian rank of the map* $f$

$= $ *rank of the matrix* $\left(\dfrac{\partial f_i}{\partial x_j}\right)$ *over the quotient field of the local ring* $\mathcal{O}(X, x_0)$.

$r_2 = \dim \mathcal{O}(Y, y_0)^\wedge/\ker \hat{\varphi}$

$r_3 = \dim \mathcal{O}(Y, y_0)/\ker \varphi$.

Since completion preserves krull dim and $\mathcal{O}(Y, y_0)^\wedge/(\ker \varphi)^\wedge$ surjects onto $\mathcal{O}(Y, y_0)^\wedge/\ker \hat{\varphi}$, it is clear that $r_2 \leqslant r_3$. If X is nonsingular at $x_0$, it follows easily by differentiating all the formal relations of the map $f$, that $r_1 \leqslant r_2$. (But this will not be used here.) By [5, theorem 5.5] if $r_1 = r_2$, then $\varphi$ is closed.

Since a quadratic transform is birational it is clear that for the maps employed in the proof of the proposition, we have $r_1 = r_3$. In section 3, we show that $r_2 = r_3$ for these maps, i.e., the map is open. Furthermore these maps are clearly injective so by 1.2, 1.12 and 2.2, strongly injective.

Before starting on section 3, we have a few interesting comments.

*Remark* 4. — It is an elementary calculation using the formula for the radius of convergence, to show the map $\mathbf{C}\langle\langle Y_1, \ldots, Y_n\rangle\rangle \to \mathbf{C}\langle\langle Z_1, \ldots, Z_n\rangle\rangle$ defined by $Y_1 \to Z_1$, $Y_i \to Z_1 Z_i$ for $i \geqslant 2$, is strongly injective. Recall a power series $f(x) = \Sigma a_\alpha X^\alpha$ is convergent iff there exist real numbers $r$ and $b$ such that for all $\alpha$, $|a_\alpha| r^{|\alpha|} \leqslant b$. Now let

$f(Y) = \Sigma a_\alpha Y^\alpha$ be a formal power series and the induced power series $f(Z) = f(Y_1, Y_1Y_2, \ldots, Y_1Y_n) = \Sigma a_\alpha Y_1^{|\alpha|}(Y)^{\alpha'}$, where $Y' = (Y_2, \ldots, Y_n)$ and $\alpha' = (\alpha_2, \ldots, \alpha_n)$. Note that $|\alpha|$ and $\alpha'$ uniquely determine $\alpha$ so there can be no cancellation of terms in the new power series. Hence there exist $r < 1$ and $b$ such that for every $\alpha$, $|a_\alpha|\, r^{|\alpha|+|\alpha'|} \leqslant b$. Since $|\alpha'| \leqslant |\alpha|$ and $r < 1$, we have $r^{|\alpha|+|\alpha'|} \leqslant r^{2|\alpha|}$. Hence for every $\alpha$, $|a_\alpha|(r^2)^{|\alpha|} \leqslant b$. So $f(Y)$ is convergent.

However it is necessary to extend this calculation to power series rings modulo an ideal. This is nontrivial.

*Remark* 5. — It is interesting to note that the hypothesis in conjecture 1.9 that $\Delta(\zeta) \neq 0$ is unnecessary; in fact it is vacuous because if $\Delta(\zeta) = 0$, there exist a curve $\tilde{\zeta} \in C_\nu(\zeta, V)$ with $\Delta(\tilde{\zeta}) \neq 0$. Clearly $C_\nu(\tilde{\zeta}, V) = C_\nu(\zeta, V)$ so we reduced to the previous hypothesis. To prove the existence of $\tilde{\zeta}$, we need some preparation.

LEMMA 2.14. — *If* $\mathcal{O}(V)$ *is regular and* $\eta$ *is a curve on* V, *and* $\nu > 0$, *then* $I(C_\nu(\eta, V)) = 0$.

*Proof.* — Let $\eta(t) = (\eta_1(t), \ldots, \eta_n(t))$ and $\varphi \in \mathcal{F}_n$, $\varphi \neq 0$. We will find an $\eta' \in C_\nu(\eta, \mathbf{C}^n)$ with $\varphi(\eta') \neq 0$. If $\varphi(\eta) \neq 0$ we just pick any $\eta'$ such that $\operatorname{ord}(\eta' - \eta) > \max(\nu, \operatorname{ord} \varphi(\eta))$. If $\varphi(\eta) = 0$, let the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ have the lexographic ordering and $\alpha$ be the smallest index such that $(D^\alpha\varphi)\eta \neq 0$, where $D^\alpha$ denotes $\dfrac{\partial^{\alpha_1}}{\partial z_1}\dfrac{\partial^{\alpha_2}}{\partial z_2}\cdots\dfrac{\partial^{\alpha_n}}{\partial z_n}$. (There exist some $\alpha$ with $(D^\alpha\varphi)\eta \neq 0$ because there exist an $\alpha$ with $D^\alpha\varphi$ a unit). Now by Taylor expansion

$$\varphi(\eta + \zeta) = \varphi(\eta) + \sum_\alpha \zeta^\alpha (D^\alpha\varphi)\eta.$$

Let $\eta' = \eta + \zeta$, where $\zeta \neq 0$, and

$$\operatorname{ord} \zeta > \max\{\nu, \operatorname{ord}(D^\alpha\varphi)\eta\}. \qquad \text{QED.}$$

To construct the desired $\tilde{\zeta}$ of the remark, let $g$: $M \to V$ be a resolution of V, $C =$ the image of $\zeta$ in V and $N = g^{-1}(C)$.

It is easy to show there exists a curve $C_0$ in N such that the restriction of $g : C_0 \to C$ is onto. Let $\rho(t)$ be the normalization of $C_0$. The mapping $g|C_0$ induces an analytic map $\tilde{g} : \mathbf{C} \to \mathbf{C}$ on the normalizations. By appropriate choice of $\rho$, we may take $\tilde{g}(t) = t^n$ for some $n > 0$. Then $g(\rho(t)) = \zeta(t^n)$. By lemma 2.14 there exist $\tilde{\rho}(t) \in C_\nu(\rho(t), M)$ such that $\Delta \circ g(\tilde{\rho}) \neq 0$. Then $\tilde{\zeta}(t) = g(\tilde{\rho}(t)) \in C_\nu(\zeta(t^n), V)$. This proves Remark 5.

*Remark* 6. — In [8, lemme 1.6], it was shown that if V is an analytic variety of pure dim $r$ in $\mathbf{C}^n$, $\pi : \mathbf{C}^n \to \mathbf{C}^r$ a projection which induces a finite extension $\mathcal{O}_r \to \mathcal{O}(V)$, $z_{r+1}$ is a primitive element of the induced field extension,

$$P(t) = t^p + a_1 t^{p-1} + \cdots + a_p$$

is the minimal polynomial for $z_{r+1}$ in $\mathcal{O}(V)$ over $\mathcal{O}_r$, $\Delta$ the discriminant of $P(t)$, i.e. the resultant of $P(z_{r+1})$ and $\frac{\partial P}{\partial t}(z_{r+1})$ and $\zeta$ is a nontrivial analytic curve on V, $\eta = \pi(\zeta)$, with $\Delta(\eta) \neq 0$, then for every integer $\nu \geqslant 1$, there is an integer $\mu \geqslant 1$ such that $C_\mu(\eta, \mathbf{C}^r) \subset \pi(C_\nu^{p!}(\zeta, V))$. That is every curve $\eta'$ near $\eta$ in $\mathbf{C}^r$ can be lifted to a curve $\zeta'$ near $\zeta$ in V by using fractional power series. We now show that one can lift $\eta'$ to a curve $\zeta'$ which is a power series with integer exponents.

Proposition 2.15. — *With the above notation, let $\zeta(t)$ be an analytic curve on* V, $\eta = \pi(\zeta)$, *and* $\Delta(\eta) \neq 0$, *then for every integer* $\nu \geqslant 1$, *there exists an integer* $\mu \geqslant 1$ *such that* $C_\mu(\eta, \mathbf{C}^r) \subset \pi(C_\nu(\zeta, V))$.

We first recall the famous lemma of Tougeron [3, Lemma 2.8]: Let $f_1, \ldots, f_q$ be convergent power series in $x$, $y$, let J be the square matrix. $J = \left(\frac{\partial f_i}{\partial y_j}\right) i, j = 1, \ldots, q$ and $\delta = \det J$. Suppose $y^0(x) = (y_1^0(x), \ldots, y_N^0(x))$ are convergent power series in $x$ without constant term such that for $i = 1, \ldots, r$ we have $f_i(x, y^0(x)) \equiv 0 \mod \delta^2(x, y^0(x))m^c$ and $\delta(x, y^0(x)) \neq 0$. Then there exist convergent series

$y(x) = (y_1(x), \ldots, y_N(x))$ with

$$y(x) \equiv y^0(x) \bmod \delta^2(x, y^0(x))m^c$$

and

$$f_i(x, y(x)) = 0 \qquad \text{for} \qquad i = 1, \ldots, q.$$

Since $\Delta$ is the resultant of P and P′, there exist A, B $\in {}_r\mathcal{O}[z_{r+1}]$ such that $\Delta = \mathrm{AP} + \mathrm{BP}'$. So $\Delta(\zeta) \neq 0$ and $\mathrm{P}(\zeta) = 0$ implies $\mathrm{P}'(\zeta) \neq 0$. It is well known that for $r + 2 \leqslant i \leqslant n$, there exist integer $\mathrm{N} > 0$ and $\mathrm{Q}_i \in {}_r\mathcal{O}[z_{r+1}]$ such that $z_i\mathrm{P}' - \mathrm{Q}_i \in \mathrm{I(V)}$ and $f \in \mathrm{I(V)}$ if and only if $(\mathrm{P}')^{\mathrm{N}}f$ lies in the ideal in $\mathcal{O}_n$ generated by

$$(\mathrm{P},\ z_{r+2}\mathrm{P}' - \mathrm{Q}_{r+2},\ \ldots,\ z_n\mathrm{P}' - \mathrm{Q}_n).$$

Hence any curve $\zeta$ satisfying P and each $z_i\mathrm{P}' - \mathrm{Q}_i$ and with $\mathrm{P}'(\zeta) \neq 0$ must lie on V.

Let $\mu > \nu + 2(n - r) \operatorname{ord} \mathrm{P}'(\zeta)$ where $\mathrm{P}'(\zeta)$ is a power series in one variable. Let

$$\zeta(t) = (\zeta_1, \ldots, \zeta_n) = (\eta, \zeta_{r+1}, \ldots, \zeta_n),$$

$\bar{\eta}(t) \in \mathrm{C}_\mu(\eta, \mathbf{C}^r)$, and $\bar{\zeta}(t) = (\bar{\eta}, \zeta_{r+1}, \ldots, \zeta_n)$. Let $f_1, \ldots, f_{n-r}$ denote $\mathrm{P}(\bar{\eta}(t), z_{r+1})$, $\mathrm{P}'(\bar{\eta}(t), z_{r+1})z_{r+i} - \mathrm{Q}_i(\bar{\eta}(t), z_{r+1})$ respectively and consider $f_1, \ldots, f_{n-r}$ as power series equations with unknowns $z_{r+1}(t), \ldots, z_n(t)$. Now $\bar{\eta} \in \mathrm{C}_\mu(\eta, \mathbf{C}^r)$ implies $\operatorname{ord} \mathrm{P}'(\bar{\zeta}) = \operatorname{ord} \mathrm{P}'(\zeta)$ and each

$$\operatorname{ord} f_i(\bar{\zeta}) > \nu + 2(n - r) \operatorname{ord} \mathrm{P}'(\bar{\zeta}).$$

Since these are power series in one variable,

$$f_i(\bar{\zeta}) \equiv 0 \bmod \mathrm{P}'(\bar{\zeta})^{2(n-r)}m^\nu.$$

Clearly the jacobian determinant of $\dfrac{\partial(f_1, \ldots, f_{n-r})}{\partial(z_{r+1}, \ldots, z_n)} = (\mathrm{P}')^{n-r}$. Hence there exist convergent power series $\tilde{\zeta}_{r+1}(t), \ldots, \tilde{\zeta}_n(t)$ so that each $\tilde{\zeta}_i \equiv \zeta_i \bmod \mathrm{P}'(\zeta)m^\nu$ and each $f_i(\tilde{\zeta}) = 0$, where $\tilde{\zeta} = (\bar{\eta}, \tilde{\zeta}_{r+1}, \ldots, \tilde{\zeta}_n)$. It follows that $\mathrm{P}'(\tilde{\zeta}) \neq 0$. $\tilde{\zeta}$ is the required curve. Q.E.D.

## 3. The Krull topology of quadratic transforms.

Let V be a germ of an irreducible complex analytic variety in $\mathbf{C}^n$ of dim $r$, $p =$ the origin be a point of V, and $n =$ the embedding dimension of V at 0. We consider the quadratic transformation B of V with center 0, that is the blowing up of the maximal ideal; B is the closure in $V \times \mathbf{P}^{n-1}$ of the set B′ in $V \times \mathbf{P}^{n-1}$ where

$$B' = \{(y_1, \ldots, y_n, z_1, \ldots, z_n) \in (V - \{p\}) \times \mathbf{P}^{n-1} : \\ y_i z_j = y_j z_i \quad \text{for all} \quad 1 \leqslant i, j \leqslant n\}.$$

It is well known that the natural projection has the following properties:

*a*) $F = \pi^{-1}(p) \subset \mathbf{P}^{n-1}$ is the tangent cone to V at $p$ and so $\dim F = r - 1$.

*b*) $\pi : B - F \rightarrow V - \{p\}$ is a biholomorphism, hence $B - F$ and $B = \overline{B - F}$ are of pure $\dim r$, because the connected manifold $\pi^{-1}(\operatorname{Reg} V)$ is dense in B.

*c*) The germs of B at points of F are not necessarily irreducible, as can easily be seen from the example $y^2 = x^4 + z^4$; letting $y = zv, x = zu$, one gets $v^2 = z^2(1 + u^4)$ which can be factored into power series at each point $(z, v, u)$ except $u = +1, -1, +i, -i$. Also the fiber is not necessarily an irreducible variety; consider the example

$$x^5 - xy^2 - y^2 z = 0.$$

Letting $y = ux$, $z = vx$, one gets $x^2 - u^2(1 + v) = 0$. The fiber is $k[u, v]/(u^2(1 + v))$ which has two components.

*d*) Let $x \in F$ and $B_x = B_1 \cup B_2 \cup \cdots \cup B_n$ be a decomposition into germs of irreducible components. Clearly for each $i$, the induced homomorphism $\mathcal{O}_p(V) \rightarrow \mathcal{O}_x(B_i)$ is an injection since $\pi | B_i - F$ is an open map and V irreducible. It is clear that $m_p(V) = m_x(B_i) \cap \mathcal{O}_p(V)$. The Krull topology of $\mathcal{O}_x(B_i)$ defined by powers of the maximal ideal induces a topology $T_2$ on $\mathcal{O}_p(V)$ which is clearly stronger than the natural topology $T_1$ on $\mathcal{O}_p(V)$.

Lemma 3.1. — $T_1 = T_2$, *that is there is an increasing function* $h : N \to N$ *such that* $m_x(B_i)^{h(j)} \cap \mathcal{O}_p(V) \subset m_p(V)^j$.

*Proof.* — Let $S = \mathcal{O}_x(B)$, $S_i = \mathcal{O}_x(B_i)$, and the $x$ in $B \subset V \times \mathbf{P}^{n-1}$ be $(0, 0, \ldots, 0 : 0, \lambda_2, \ldots, \lambda_n)$. Let

$$\mathcal{O}_p(V) = \mathbf{C}\langle\langle y_1, \ldots, y_n\rangle\rangle,$$

where $y_i =$ residue of $Y_i$ modulo the prime ideal of P. We wish to describe the ring S in terms of $\lambda_2, \ldots, \lambda_n$ and P. If we write $Z_i = \dfrac{Y_i}{Y_1} - \lambda_i = \dfrac{Y_i - \lambda_i Y_1}{Y_1}$ for $i \geqslant 2$ and $Z_1 = Y_1$, then $S = \mathbf{C}\langle\langle Z_1, \ldots, Z_n\rangle\rangle/\tilde{P}$ where $\tilde{P}$ is the ideal obtained from P as follows: For any power series $g(Y_1, \ldots, Y_n) \in P$, write

$$Y_1 = Z_1, \quad Y_2 = Z_1 Z_2 + \lambda_2 Z_1, \ldots, Y_n = Z_1 Z_n + \lambda_n Z_1;$$

then $g(Y_1, Y_2, \ldots, Y_n) = Z_1^s \tilde{g}(Z, Z_2, \ldots, Z_n)$ where $Z_1$ does not divide $\tilde{g}$. The ideal $\tilde{P}$ is the ideal generated by all these power series $\tilde{g}(Z_1, \ldots, Z_n)$. Clearly $Z_1 \notin \tilde{P}$.

First by making a change of coordinates we can assume that $\lambda_2, \lambda_3, \lambda_n = 0$, since

$$C\langle\langle Y_1, \ldots, Y_n\rangle\rangle = C\langle\langle Y_1, Y_2 - \lambda_2 Y_1, \ldots, Y_n - \lambda_n Y_1\rangle\rangle$$

and in the new coordinates $Y_1, Y_2 - \lambda_2 Y_1, \ldots, Y_n - \lambda_n Y_1$, the respective $\lambda_i = 0$ for $i \geqslant 2$.

Now we have that $Y_i = Z_1 Z_i$, $y_i = z_1 z_i$ for $i \geqslant 2$, and $m_x(S) = (z_1, z_2, \ldots, z_n)\mathcal{O}_x(B)$ contains $m_p(V)$.

Let $S' = \mathbf{C}\langle\langle y_1, \ldots, y_n\rangle\rangle[z_2, \ldots, z_n]_{(y_1, z_2, \ldots, z_n)}$. Since R is a domain and $S'$ is contained in the quotient field of R, we have $S'$ is a domain and R injects into $S'$. Clearly there is a map $i : S' \to S$ which is, a priori, not necessarily an injection; we will prove it is an injection.

Let $A_1 = \mathbf{C}\langle\langle Y_1, \ldots, Y_n\rangle\rangle[Z_2, \ldots, Z_n]_m$,

$$A_2 = \mathbf{C}\langle\langle Y_1, \ldots, Y_n, Z_2, \ldots, Z_n\rangle\rangle,$$

$S' = A_1/P_a$, $S = A_2/P_c$ and $R = \mathbf{C}\langle\langle Y_1, \ldots, Y_n\rangle\rangle/P$. Then $A_1 \subset A_2$, $\hat{A}_1 = \hat{A}_2 = \mathbf{C}[[Y_1, \ldots, Y_n, Z_2, \ldots, Z_n]]$, the topology on $A_1$ induced from $A_2$ is the same as the natural topology on $A_1$ $\hat{S}' = \hat{A}_1/\hat{P}_a$, and $\hat{S} = \hat{A}_2/\hat{P}_c$. Clearly $\hat{i}$: $\hat{S}' \to \hat{S}$ is surjective. Note that $P \subset P_a \subset P_c$ and $\tilde{P}^e = P_c$ where upper $e$ denotes extension of an ideal to $A_2$. Note that $P_a = \tilde{P}_a$, that is the required power of $Y_1$ can be factored out in $A_1$: If $f(Y_1, \ldots, Y_n) \in \mathbf{C}\langle\langle Y_1, \ldots, Y_n\rangle\rangle$, ord $f = r$, $f = \Sigma a_\alpha Y^\alpha$, then

$$f(Y_1, \ldots, Y_n) = f(Y_1, Y_2 = Z_1Z_2, \ldots, Y_n = Z_1Z_n, Z_2, \ldots, Z_n)$$
$$= Z_1^r\Big[\sum_{|\alpha|=r} a_\alpha Z_2^{\alpha_2} \ldots Z_n^{\alpha_n} + Z_1 \sum_{|\alpha|>r} Z_2^{\beta_2} \ldots Z_n^{\beta_n} Y_1^{\gamma} \ldots Y_n^{\gamma_n}\Big]$$

where $\alpha = \beta + \gamma$ and $|\beta| = r + 1$.

Since $f(Y_1, \ldots, Y_n)$ is a convergent power series it is easy to check the required convergence of the above series in the brackets by using the classical formula for the radius of convergence of a power series, which is that the radius of convergence of $\Sigma a_\alpha z^\alpha$ is $\liminf_{|\alpha|\to\infty} |a_\alpha|^{\frac{-1}{|\alpha|}}$.

We will now show $P_a^{\hat{}_1} = P_c^{\hat{}_2}$ where the subscripts on the hats indicate which topology is being used for the completion. We have $P \subset P_a \Longrightarrow \tilde{P} \subset \tilde{P}_a$, but $\tilde{P}_a = P_a$ so

$$\tilde{P} \subset P_a \Longrightarrow \tilde{P}^e \subset P_a^e \subset P_c, \qquad \text{but} \qquad \tilde{P}^e = P_c$$

so

$$P_a^e = P_c \Longrightarrow P_a^{\hat{}_1} = P_a\hat{A}_1 = P_a\hat{A}_2 = P_aA_2\hat{A}_2$$
$$= P_a^e\hat{A}_2 = P_c\hat{A}_2 = P_c^{\hat{}_2}.$$

Hence $\hat{S}' = \hat{A}_1/P_a^{\hat{}_1} = \hat{A}_2P_c^{\hat{}_2} = \hat{S}$. We have a commutative diagram

$$\begin{tikzcd} \hat{S}' \arrow[r, equal] & \hat{S} \\ S' \arrow[u, "\alpha'"] \arrow[r, "i"] & S \arrow[u, "\alpha"] \end{tikzcd} \qquad \text{with } \alpha' \text{ and } \alpha$$

injections so it follows that $i$ is injective. Note that since $S'$ and $S$ have the same completion, they must have the same Krull dimension.

$S'$ is a subspace of $\hat{S}'$, and ZST tells us that R is a subspace of $S'$ so by 1.7, R is a subspace of $\hat{S}'$. Also R injects into $S'$ and $S'$ injects in S so R injects in S. So

$R \subset \hat{S} = \hat{S}'$ so by the above line and 1.8, $R$ is a subspace of $S$.

Now let $q$ be the prime in $S$ of the minimal component of the variety in question and $S_i = S/q$. We must show $R$ is a subspace of $S_i$. Let

$$S'' = \hat{R}[z]_m = C[[y_1, \ldots, y_n]][z_2, \ldots, z_n]_{(y_1, z_2, \ldots, z_n)}.$$

We first show that $\hat{S}' = R[z]^\wedge_m = \hat{R}[z]^\wedge_m = \hat{S}''$: Since $R \subset \hat{R} \subset$ quotient field of $R$, and $z \in$ quotient field of $R$; we have $R[z] \subset \hat{R}[z]$. Since a maximal ideal of $R[z]$ contracts to a maximal ideal of $R[z]$, we have $R[z]_m \subset \hat{R}[z]_m$. This yields a map $\hat{R}[z]^\wedge_m \longrightarrow R[z]^\wedge_m$. On the other hand $R$ is a subspace of $R[z]_m$ so $\hat{R} \subset R[z]^\wedge_m$. Since $z \in R[z]^\wedge_m$, $\hat{R}[z] \subset R[z]^\wedge_m$. Next the maximal ideal of $\hat{R}[z]^\wedge_m$ contracts to a maximal ideal of $\hat{R}[z]$ so $\hat{R}[z]_m \subset R[z]^\wedge_m$. This yields a map $\hat{R}[z]^\wedge_m \longrightarrow R[z]^\wedge_m$. This gives maps in both directions. It is clear that it is all functorial and either double composition is the identity.

We now have the following pretty picture:

$$\begin{tikzcd}
\hat{R} \arrow[r] & \hat{R}[z]_m = S'' \arrow[rr] & & \hat{S}'' \arrow[r] & \hat{S}''/\hat{q} \\
R \arrow[u, phantom, "\cup"] \arrow[r] & R[z]_m = S' \arrow[u, phantom, "\cup"] \arrow[r] & S \arrow[r] \arrow[dr] & \hat{S}' = \hat{S} \arrow[u, equal] \arrow[r] & \hat{S}/\hat{q} \arrow[u, equal] \\
& & & S/q \arrow[ur] &
\end{tikzcd}$$

Now $q$ is a minimal prime of $S$ so $\hat{q}$ is a minimal prime of $\hat{S}''$. Hence $\hat{q} \cap S''$ is a minimal prime of $S''$.

$$(ht(\hat{q} \cap S'') = ht(\hat{q} \cap S'')^\wedge \leqslant ht(\hat{q}) \quad \text{as} \quad (\hat{q} \cap S'') \subset \hat{q}.)$$

Since $S''$ is a domain, $\hat{q} \cap S'' = (0)$. Hence $S''$ injects into $\hat{S}''/\hat{q}$. Hence $\hat{R}$ injects into $\hat{S}''/\hat{q}$. By the Chevalley subspace theorem $\hat{R}$ is a subspace of $\hat{S}''/\hat{q}$. By 1.10 and 1.7, $R$ is a subspace of $\hat{S}''/\hat{q}$. By the lower most line of the diagram $R$ injects in $S/q$ and $R$ is a subspace of $S/q$ by 1.8. Q.E.D.

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