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ON THE THEOREM OF IVASEV-MUSATOV I

by T.W. KORNER

1. Introduction.

If μ is a measure on the circle $T = R/2\pi Z$ and $\sum_{-\infty}^{\infty} |\hat{\mu}(n)|^2$ converges, then μ must be an L^2 function and so have for support a set of Lebesgue measure greater than zero.

If $\sum_{n=1}^{\infty} |\phi(n)|^2$ diverges, can we always find a measure $\mu \neq 0$ with support of Lebesgue measure zero such that $|\phi(|n|)| \geqslant |\hat{\mu}(n)|$ $[n \neq 0]$? Standard results on measures with $\hat{\mu}(n) = 0$ except when $|n| = 3^m$ $[m \geqslant 1]$ show that the answer is no ([5] Vol. I., p. 202). However, in a series of remarkable papers ([1], [2], [3]) Ivašev-Musatov obtained the following result.

THEOREM 1.1'. – *If $\phi(n)$ is a decreasing positive sequence such that $\sum_{n=1}^{\infty} (\phi(n))^2$ diverges and if the behaviour of $\phi(n)$ is sufficiently regular, then we can find a positive measure $\mu \neq 0$ with support of Lebesgue measure zero yet with*

$$|\hat{\mu}(n)| = O(\phi(|n|)) \quad \text{as} \quad |n| \to \infty$$

More precisely, he proved the following theorem (The reader may check that our statement is equivalent to that of Theorem 1 of [3]) :

THEOREM 1.1. – *Suppose $\phi(n)$ is a decreasing positive sequence such that*

1) $\sum_{n=1}^{\infty} (\phi(n))^2$ *diverges*

2) $n(\phi(n))^2 \to 0$ *as* $n \to \infty$

3) $n^{1+\epsilon}(\phi(n))^2 \to \infty$ *as* $n \to \infty$ *for all* $\epsilon > 0$

4) *We can find an* m *such that* $n^m \phi(n)$ *is an increasing sequence.*

Then we can find a positive measure μ *with support of Lebesgue measure* 0 *yet with*

$$\hat{\mu}(n) = 0(\phi(|n|)) \quad \text{as} \quad |n| \to \infty .$$

Although this theorem is deservedly famous, the proof has a reputation for difficulty which has deterred many people from trying to understand it.

In this paper I shall try to simplify and rewrite the proof in the (mathematical) language of Kahane and Salem [4]. I hope that the approach that I have adopted will be found easier and more transparent but, even if this hope is groundless, the existence of two accounts must be helpful.

It turns out that the result we prove is rather stronger and simpler than the original one of Ivašev-Musatov.

THEOREM 1.2. – *Suppose* $\phi(n)$ *is a positive sequence such that*

(A) $\sum_{n=1}^{\infty} (\phi(n))^2$ *diverges*

(B) *There exists a* $K > 1$ *such that for all* $n \geqslant 1$ *we have*

$$K^{-1}\phi(n) \leqslant \phi(r) \leqslant K\phi(n) \quad \textit{whenever} \quad n \leqslant r \leqslant 2n$$

(C) $n\phi(n) \to \infty$ *as* $n \to \infty$.

Then we can find a positive measure $\mu \neq 0$ *with support* E *of Lebesgue measure zero yet with*

$$|\hat{\mu}(n)| = 0(\phi(|n|)) \quad \text{as} \quad |n| \to \infty .$$

By making various modifications, it is possible to weaken condition (C). However, I want to give the proof in its simplest form, and so I shall defer discussion of this to a second paper. (For example, it can be shown that if $\phi(n)$ is decreasing, then condition (C) can be dropped altogether).

The reputation for difficulty of the original proof of Ivašev-Musatov of which this is an adaptation, is also the reason why I give my argument in a more detailed and leisurely manner than is customary.

Before starting on the argument proper, we make two conventions. Note first that if $\phi(n) > 0$ has properties (A), (B) and (C), so does $\min(1, \phi(n))$. Thus, without loss of generality, we may suppose for the rest of the paper that $\phi(n)$ is a fixed positive sequence having properties (A), (B), (C) and

(D) $\phi(n) \leqslant 1$ for all $n \geqslant 1$.

We shall write $|E|$ for the Lebesgue measure of a closed set E.

2. The standard part of the construction.

In this Section we show that the construction of μ is quite easy once we have the following lemma.

LEMMA 2.1. – *Given* $\epsilon, \eta > 0$ *we can find an* $f \in C(T)$ *such that*

i) $f(t) \geqslant 0$ *for all* $t \in T$

ii) $|\operatorname{supp} f| \leqslant \eta$

iii) $\frac{1}{2\pi} \int_T f(t)\, dt = 1$

iv) $|\hat{f}(r)| \leqslant \epsilon\phi(|r|)$ *for all* $r \neq 0$

v) f *is infinitely differentiable.*

We start by proving the following consequence of Lemma 2.1 :

LEMMA 2.2. – *We can find a sequence of functions*

$$g_1, g_2, g_3, \ldots, \in C(T)$$

such that

$(1)_n$ $g_n(t) \geqslant 0$ *for all* $t \in T$

$(2)_n$ $|\operatorname{supp} g_n| \leqslant 2^{-n}$

$(3)_n$ $2 + 2^{-n} \geqslant \frac{1}{2\pi} \int_T g_n(t)\, dt \geqslant 2^{-1} - 2^{-n}$

$(4)_n$ $|\hat{g}_n(r)| \leqslant (1 - 2^{-n})\,\phi(|r|)$ *for all* $r \neq 0$

$(5)_n$ g_n *is infinitely differentiable and, when* $n \geqslant 2$,

$(6)_n$ $\operatorname{supp} g_n \subseteq \operatorname{supp} g_{n-1}$.

Proof. – We construct $g_1, g_2, g_3, \ldots$ inductively. The existence of a suitable g_1 follows directly from Lemma 2.1 on setting a $\epsilon = \eta = 1/2$.

Now suppose g_n has been constructed. Since g_n is infinitely differentiable, it follows that $|\hat{g}_n(r)| \leqslant Ar^{-2}$ for all $r \neq 0$ and some constant A. Choose f as in Lemma 2.1 with $\eta = 2^{-n-1}$ and $\epsilon > 0$ to be determined. We claim that, provided only that ϵ is small enough, $g_{n+1} = fg_n$ will satisfy condition $(1)_{n+1}$ to $(6)_{n+1}$.

Indeed, conditions $(1)_{n+1}$, $(2)_{n+1}$, $(5)_{n+1}$, $(6)_{n+1}$ are trivial consequences of $(1)_n$ and (i), (ii), $(5)_n$ and (v), and the relation $\operatorname{supp} g_{n+1} \subseteq \operatorname{supp} f \cap \operatorname{supp} g_n$ respectively. To prove $(3)_{n+1}$ and $(4)_{n+1}$ we note first that

$$
\begin{aligned}
|\hat{g}_{n+1}(r) - \hat{g}_n(r)| &= |(g_n f)\hat{\ }(r) - \hat{g}_n(r)\hat{f}(0)| \\
&= \Big| \sum_{m \neq r} \hat{g}_n(m)\hat{f}(r-m) \Big| \\
&\leqslant \sum_{m \neq r} |\hat{g}_n(m)|\,|\hat{f}(r-m)| \\
&= \sum_{|m| \leqslant r/2} |\hat{g}_n(m)|\,|\hat{f}(r-m)| \\
&\qquad + \sum_{|m| \geqslant r/2 \ m \neq r} |\hat{g}_n(m)|\,|\hat{f}(r-m)| \\
&\leqslant \epsilon \sum_{|m| \leqslant r/2} |\hat{g}_n(m)|\,|\phi(|r-m|)| \\
&\qquad + \epsilon \sum_{|m| \geqslant r/2 \ m \neq r} |\hat{g}_n(m)|\,\phi(|r-m|).
\end{aligned}
$$

If $r = 0$ we observe that, by condition (D), $\phi(|r-m|) \leqslant 1$ for $m \neq r$ and so

$$\left|\frac{1}{2\pi}\int_{\mathrm{T}} g_{n+1}(t)\,dt - \frac{1}{2\pi}\int_{\mathrm{T}} g_n(t)\,dt\right| = |\hat{g}_{n+1}(0) - \hat{g}_n(0)|$$
$$\leqslant \epsilon \sum_{m\neq 0} |\hat{g}_n(m)|$$
$$\leqslant \epsilon \mathrm{A} \sum_{m\neq 0} m^{-2}$$
$$\leqslant 10\epsilon \mathrm{A}$$

so, provided only that $\epsilon \leqslant 2^{-n-1}/10\,\mathrm{A}$, condition $(3)_{n+1}$ follows from $(3)_n$.

If $r \neq 0$, we also use condition (B) to tell us that

$$\phi(|r-m|) \leqslant \mathrm{K}\phi(|r|)$$

for all $|m| < r/2$ and thus

$$|\hat{g}_{n+1}(r) - \hat{g}_n(r)| \leqslant \epsilon \mathrm{K}\phi(|r|) \sum_{|m|\leqslant r/2} |\hat{g}_n(m)| + \epsilon \sum_{|m|\geqslant r/2} |\hat{g}_n(m)|$$
$$\leqslant \epsilon \mathrm{K}\phi(|r|) \sum_{-\infty}^{\infty} |\hat{g}_n(m)| + \epsilon \mathrm{A} \sum_{|m|\geqslant r/2} m^{-2}$$
$$\leqslant \epsilon \mathrm{K}\phi(|r|)(10\mathrm{A}+2) + 10\epsilon \mathrm{A}|r|^{-1}$$
$$\leqslant \epsilon\phi(|r|)\left((10\mathrm{A}+2)\mathrm{K} + 10\mathrm{A} \sup_{k=1\,.} (k\phi(k))^{-1}\right)$$

so, taking $\epsilon \leqslant 2^{-n-1}/((10\mathrm{A}+2)\mathrm{K} + 10\mathrm{A}\sup_{k\geqslant 1}(k\phi(k))^{-1})$, we can make condition $(4)_{n+1}$ follow from $(4)_n$. The induction can now be restarted.

□

Remark. – Note that condition (A) was not used in the argument above and that, by tightening the inequalities, we could have used very much weaker conditions than (B) and (C). The key to the proof does not lie in this Section.

Using Lemma 2.2. it is easy to prove Theorem 1.2.

Proof of Theorem 1.2. – Let $d\mu_n(t) = f_n(t)\dfrac{dt}{2\pi}$ so that μ_n is a measure on T. By $(3)_n$, $\|\mu_n\| \leqslant 2$ and so the sequence μ_n must have a a weak $*$ limit point μ. (In fact it is easy to see without appealing to general theorems but with a little extra work that the μ_n constructed in the proof of Lemma 2.2 actually have a weak $*$ limit μ).

By $(1)_n$, $\mu_n \in M^+(T)$ for all $n \geqslant 1$, so $\mu \in M^+(T)$. By $(6)_n$, $\operatorname{supp} \mu \subseteq \operatorname{supp} \mu_m$ for all $m \geqslant 1$ so that, using $(2)_n$ we have

$$|\operatorname{supp} \mu| \leqslant 2^{-m}$$

for all $m \geqslant 1$. Thus supp μ has Lebesgue measure 0. By $(4)_n$,

$$|\hat{\mu}_n(r)| \leqslant \phi(|r|)$$

for all $n \geqslant 1$ and so $|\hat{\mu}(r)| \leqslant \phi(|r|)$ for all $r \neq 0$. Finally, by $(3)_n$, $\hat{\mu}_n(0) \geqslant 2^{-1}$ for all $n \geqslant 1$ so that $\hat{\mu}(0) \geqslant 2^{-1}$ and μ is a non zero measure.

□

3. Van der Corput's lemma.

It is my opinion that in order to understand the original or the present proof of the theorem of Ivašev-Musatov, it is necessary not merely to know but also to understand the lemma of Van der Corput.

If the reader is clear in his own mind how this lemma works he should skip at once to the next Section, pausing only to convince himself of the truth of Lemma 3.3 (the constant $\pi/2$ is not essential and can be replaced by any other). If not, we shall give a leisurely account which should be compared with the standard, much faster, proof in e.g. ([5] Vol. I., p. 197).

LEMMA 3.1. – (i) *(Dirichlet.) If* $t_0 \geqslant t_1 \geqslant t_2 \ldots \geqslant t_n \geqslant 0$, *then* $0 \leqslant \sum_{r=0}^{n} (-1)^r tr \leqslant t_0$.

– (ii) *If* $n \geqslant m$ *and* $t_0 \geqslant t_1 \geqslant \cdots \geqslant t_n \geqslant 0$, *then*

$$\left| \sum_{r=0}^{n} t_r \exp(ir\pi/m) \right| \leqslant \sum_{r=0}^{m-1} t_r .$$

Proof. — (i) $t_0 \geqslant t_0 - (t_2 - t_3) - (t_4 - t_5) - \cdots$

$$= \sum_{r=0}^{n} (-1)^r t_r$$

$$\geqslant (t_0 - t_1) + (t_2 - t_3) + \cdots$$

$$\geqslant 0.$$

(ii) $$\left| \sum_{r=0}^{n} t_r \exp(ir\pi/m) \right| \leqslant \sum_{k=0}^{m-1} \left| \sum_{0 \leqslant k+qm \leqslant n} t_{k+qm} \exp(i(q + k/m)\pi) \right|$$

$$= \sum_{k=0}^{m-1} \left| \sum_{0 \leqslant k+qm \leqslant n} (-1)^q t_{k+qm} \right|$$

$$\leqslant \sum_{k=0}^{m-1} t_k$$

using (i).

LEMMA 3.2. *(Van der Corput).* — (i) *If* $h : [a, b] \to \mathrm{R}$ *has increasing derivative* h' *and there exists a* $\lambda > 0$ *such that* $h'(u) \geqslant \lambda$ *for all* $u \in (a, b)$ [*or* $h'(u) \leqslant -\lambda$ *for all* $u \in (a, b)$] *then*

$$\left| \int_a^b \exp(ih(u))\, du \right| \leqslant \frac{\pi}{\lambda} \cdot$$

Proof. — We do the case $h'(u) \geqslant \lambda$. Choose an m and let t_r be the time spent by h between the values $h(a) + \frac{r\pi}{m}$ and $h(a) + \frac{r+1}{m}\pi$. (More formally, let $t_r = h^{-1}\left(a + \frac{r+1}{m}\pi\right) - h^{-1}\left(a + \frac{r\pi}{m}\right)$ when $b \geqslant a + \frac{r+1}{m}\pi$, $t_n = b - h^{-1}\left(a + \frac{n\pi}{m}\right)$ when

$$a + \frac{n+1}{m}\pi \geqslant b \geqslant a + \frac{n\pi}{m}$$

and $t_r = 0$ otherwise). Then, since $h'(u)$ is increasing, it follows that $t_0 \geqslant t_1 \geqslant t_2 \geqslant \cdots$ and so by Lemma 3.1. (ii)

$$\left|\sum t_r \exp(ir\pi/m)\right| \leqslant \sum_{r=0}^{m-1} t_r \,.$$

But since $h'(u) \geqslant \lambda$, $t_r \leqslant \lambda^{-1}\ \pi/m$, so

$$\left|\sum t_r \exp(ir\pi/m)\right| \leqslant \pi\lambda^{-1} \,.$$

Letting $m \to \infty$ this gives

$$\left|\int_a^b \exp(ih(u))\,du\right| \leqslant \frac{\pi}{\lambda} \cdot$$

□

LEMMA 3.2. *(Van der Corput).* – (ii) *If* $h : [a, b] \to \mathrm{R}$ *is twice differentiable and there exists a* $\mu > 0$ *such that* $h''(u) \geqslant \mu$ *for all* $u \in (a, b)$, *then*

$$\left|\int_a^b \exp(ih(u))\,du\right| \leqslant 4\left(\frac{\pi}{\rho}\right)^{1/2} \cdot$$

Proof. – For each $\mathrm{K} > 0$ we can split (a, b) into 3 (possibly empty) intervals I_1, I_2, I_3 such that $h'(u) \leqslant -\mathrm{K}$ for $u \in \mathrm{I}_1$, $-\mathrm{K} < h'(u) < \mathrm{K}$ for $u \in \mathrm{I}_2$ and $h'(u) \geqslant \mathrm{K}$ for $u \in \mathrm{I}_3$. Since $h''(u) \geqslant \mu$ it follows that $|\mathrm{I}_2| \leqslant 2\mathrm{K}\mu^{-1}$, so that

$$\left|\int_{\mathrm{I}_2} \exp(ih(u))\,du\right| \leqslant |\mathrm{I}_2| \leqslant \frac{2\mathrm{K}}{\mu}$$

whilst, by Lemma 3.2. (i), $\left|\int_{\mathrm{I}_j} \exp(ih(u))\,du\right| \leqslant \frac{\pi}{\mathrm{K}}$ for $j = 1$ or 3. Thus $\left|\int_a^b \exp(ih(u))\,du\right| \leqslant \frac{2\pi}{\mathrm{K}} + \frac{2\mathrm{K}}{\mu}$ and, setting

$$\mathrm{K} = \pi^{1/2}\,\mu^{-1/2}\,,$$

we have the stated result.

□

In exactly the same spirit we prove the following lemma.

LEMMA 3.3. – *If* $h : (0, 2\pi) \to \mathrm{T}$ *has increasing derivative* h' *and there exists a* $\lambda > 0$ *such that* $h'(u) \geqslant \lambda$ *for all* $u \in (0, 2\pi)$, *then, for each interval* I *in* T,

$$\| h^{-1}(\mathrm{I})| - |\mathrm{I}\| \leqslant \frac{\pi}{2\lambda} \cdot$$

Proof. – We may suppose $|\mathrm{I}| \neq 0, 2\pi$. Write CI for the complement $\mathrm{T} \backslash \mathrm{I}$ of I. Suppose $0 \in h^{-1}(\mathrm{I})$. Then we can find

$$0 = t_0 \leqslant s_0 < t_1 < s_1 < t_2 < \cdots$$

such that $h(t) \in \mathrm{I}$ for $t_j < t < s_j$, $h(t) \geqslant \mathrm{I}$ for $s_j < t < t_{j+1}$. Since h' is increasing $\frac{t_1 - s_0}{|\mathrm{CI}|} \geqslant \frac{s_1 - t_1}{|\mathrm{I}|} \geqslant \frac{t_2 - s_1}{|\mathrm{CI}|} \geqslant \frac{s_2 - t_2}{|\mathrm{I}|} \geqslant \cdots$ and

so $\frac{t_1 - s_0}{|\mathrm{CI}|} \geqslant \frac{t_1 - s_0}{|\mathrm{CI}|} - \frac{s_1 - t_1}{|\mathrm{I}|} + \frac{t_2 - s_1}{|\mathrm{CI}|} - \frac{s_2 - t_2}{|\mathrm{I}|} \cdots \geqslant 0$

whence

$$\frac{t_2 - s_0}{|\mathrm{CI}|} \geqslant - \frac{t_0 - s_0}{|\mathrm{I}|} + \frac{t_1 - s_0}{|\mathrm{CI}|} - \frac{s_1 - t_1}{|\mathrm{I}|} + \frac{t_2 - s_1}{|\mathrm{CI}|} - \cdots \geqslant - \frac{t_0 - s_0}{|\mathrm{I}|} .$$

In other words,

$$\frac{t_1 - s_0}{|\mathrm{CI}|} \geqslant - \frac{|h^{-1}(\mathrm{I})|}{|\mathrm{I}|} + \frac{|h^{-1}(\mathrm{CI})|}{|\mathrm{CI}|} \geqslant - \frac{t_0 - s_0}{|\mathrm{I}|} .$$

But $h'(u) \geqslant \lambda$ for all $u \in (0, 2\pi)$, so $\lambda \geqslant \frac{t_1 - s_0}{|\mathrm{CI}|}, \frac{t_0 - s_0}{|\mathrm{I}|}$ and

$$\left| \frac{|h^{-1}(\mathrm{I})|}{|\mathrm{I}|} - \frac{|h^{-1}(\mathrm{CI})|}{|\mathrm{CI}|} \right| \leqslant \frac{1}{\lambda} .$$

Since the formula is symmetric in I and CI, it also holds when $0 \notin \mathrm{I}$ (i.e. when $0 \in \mathrm{CI}$).

Multiplying up, we obtain (since $|\mathrm{CI}| = 2\pi - |\mathrm{I}|$)

$$\begin{aligned} \| h^{-1}(\mathrm{I})| - |\mathrm{I}\| &= (2\pi)^{-1} |\mathrm{I}| \, |\mathrm{CI}| \left| \frac{|h^{-1}(\mathrm{I})|}{|\mathrm{I}|} - \frac{|h^{-1}(\mathrm{CI})|}{|\mathrm{CI}|} \right| \\ &\leqslant (2\pi)^{-1} |\mathrm{I}| \, (2\pi - |\mathrm{I}|) \\ &\leqslant \frac{\pi}{2\lambda} \end{aligned}$$

as stated.

□

4. A Simple function with small Fourier transform.

How are we going to construct the function f of Lemma 2.1. ? The answer is suprisingly simple.

Let g be a smooth positive function of small support. Then, if h is any smooth function with sufficiently enormous acceleration, $g \circ h(t) = g(h(t))$ will (apart from a few technical details) be a suitable f. Properties (i), (ii), (iii) are (subject to slight technical changes) inherited by $f = g \circ h$ form g. Property (v) is inherited by f from g and h. Finally property (iv) is inherited from h.

How are we going to prove this last statement ? Observe that $g(t) = \sum_{q=-\infty}^{\infty} a_q \exp(iqt)$ where $|a_q| \to 0$ quite fast. Thus

$$f(t) = g \circ h(t) = \sum_{q=-\infty}^{\infty} a_q h_q(t),$$

where $h_q(t) = \exp(iqh(t))$, and so $\hat{f}(n) = \sum_{q=-\infty}^{\infty} a_q \hat{h}_q(n)$. If we can show that $\hat{h}_q(n) \to 0$ very rapidly, it will follow that $\hat{f}(n) \to 0$ very rapidly and so property (iv) holds. (It has been pointed out to me that this procedure is reminiscent of that employed by Wiener and Wintner in their original work on this subject ([5] Vol. II., p. 146).)

In Lemma 4.2 we show that the $\hat{h}_q(n)$ can indeed be made small under hypotheses very closely related to conditions (A), (B) and (C). To indicate how closely, we make the following trivial observations.

LEMMA 4.1. – (i) *Suppose* $\psi : [1, \infty) \to \mathrm{R}$ *is a positive function. Suppose that for some* $\mathrm{L} \geqslant 1$ *we have* $\mathrm{L}^{-1} \psi(s) \leqslant \psi(t)$ *for all* $1 \leqslant s \leqslant t \leqslant 2s$. *Then we can find an integer* $m \geqslant 1$ *such that* $s^m \psi(s) \leqslant \mathrm{L} t^m \psi(t)$ *for all* $t \geqslant s \geqslant 1$.

– (ii) *If* $t\psi(t) \to \infty$ *as* $t \to \infty$ *then we can find a* t_0 *such that, if* $t \geqslant t_0$ *and* $t \geqslant s > 1$, *then* $\psi(s) \geqslant t^{-1}$.

Proof. – Obvious.

LEMMA 4.2. – *Suppose* $\psi : [1, \infty) \to \mathrm{R}$ *is a positive function such that*

(a) $\int_u^v (\psi(t))^2 \, dt = 2\pi$ *for some* $v > u \geqslant 4$

(b) *There exist an* $L \geqslant 1$ *and an integer* $m \geqslant 1$ *such that*

$$L^{-1} \psi(s) \leqslant \psi(t) \leqslant L \psi(s)$$

for all $1 \leqslant s \leqslant t \leqslant 2s$ *and* $s^m \psi(s) \leqslant L t^m \psi(t)$ *for all* $t \geqslant s \geqslant 1$.

$(c)_1$ $t \psi(t) \geqslant 1$ *for all* $t \geqslant u$

$(c)_2$ $\psi(t) \geqslant u^{-1}$ *for all* $1 \leqslant t \leqslant u$.

Write $\Psi(s) = \int_u^s (\psi(t))^2 \, dt$

$h(x) = \int_0^x \Psi^{-1}(y) \, dy$

and $h_q(x) = \exp(iqh(x))$ $[0 \leqslant x < 2\pi]$.

Then $|\hat{h}_q(n)| \leqslant 10 \pi L^2 |q|^m \psi(|n|)$ *for all* $n, q \neq 0$

and $|\hat{h}_q(0)| \leqslant 2\pi/u$ *for all* $q \neq 0$.

Proof. – We wish to bound

$$|\hat{h}_q(n)| = \left| \int_0^{2\pi} \exp i \left(q \int_0^x \Psi^{-1}(y) \, dy - nx \right) dx \right| .$$

Since $|\hat{h}_{-q}(n)| = |\hat{h}_q(-n)|$ we may suppose $q \geqslant 1$. If $n \leqslant -1$ or $0 \leqslant n \leqslant u/2$, the estimation is very simple.

CASE 1, $n \leqslant -1$. Then, by $(c)_1$ and $(c)_2$, $\dfrac{1}{u + |n|} \leqslant \psi(|n|)$. Since Ψ^{-1} is increasing, so is $q\Psi^{-1}(x) - n$. Moreover, for $0 \leqslant x < 2\pi$,

$$\frac{d}{dx} \left(q \int_0^x \Psi^{-1}(y) \, dy - nx \right) = q\Psi^{-1}(x) - n = q\Psi^{-1}(0) - n$$
$$= qu + |n| .$$

Thus, by Van der Corput's Lemma (Lemma 3.2 (i))

$$|\hat{h}_q(n)| \leqslant \frac{\tau}{qu + |n|} \leqslant \pi \psi'(n) \leqslant 10 \pi L^2 |q|^m \psi(|n|) .$$

CASE 2, $0 \leqslant n \leqslant u/2$. We have

$$\frac{d}{dx}\left(q \int_0^x \Psi^{-1}(y)\, dy - nx\right) = q\Psi^{-1}(x) - n \geqslant q\Psi^{-1}(0) - n$$
$$= qu - n \geqslant u/2 .$$

Thus, using Lemma 3.2 (i),

$$|\hat{h}_q(n)| \leqslant \frac{2\pi}{u} \leqslant 2\pi\psi(|n|) \leqslant 10\,\pi L^2 |q|^m \psi(|n|)$$

if $n \neq 0$, and $|\hat{h}_q(0)| \leqslant \frac{2\pi}{u}$.

The remaining case requires a little more work. The basis of the calculation may become clearer if the reader concentrates on the estimation when $q = 1$ and $F, G, H \neq \phi$.

CASE 3, $n \geqslant u/2$. Note that $2n \geqslant u$ and so, using (b) and $(c)_1$,

$$2L\psi(n) \geqslant 2\psi(2n) \geqslant 2(2n)^{-1} = n^{-1}.$$

We have

$$\hat{h}_q(n) = \int_F + \int_G + \int_H \exp i\left(q \int_0^x \Psi^{-1}(y)\, dy - nx\right) dx$$

where
$$F = \{0 \leqslant x < 2\pi : \Psi^{-1}(x) < n/2q\}$$
$$G = \{0 \leqslant x < 2\pi : n/2q \leqslant \Psi^{-1}(x) \leqslant 2n/q\}$$
$$H = \{0 \leqslant x < 2\pi : 2n/q \leqslant \Psi^{-1}(x)\}$$

are (possibly empty) intervals.

The estimation of the integrals over F and H follows the pattern above.

$$\frac{d}{dx}\left(q \int_0^x \Psi^{-1}(y)\, dy - nx\right) = q\Psi^{-1}(x) - n$$

is an increasing function bounded above by $q(n/2q) - n = -n/2$ on F. Thus by Van der Corput's Lemma (Lemma 3.2 (i))

$$\left|\int_F \exp i\left(q \int_p^x \Psi^{-1}(y)\, dy - nx\right) dx\right| \leqslant \frac{2\pi}{n}.$$

Similarly

$$\frac{d}{dx}\left(q\int_0^x \Psi^{-1}(y)\,dy - nx\right) = q\Psi^{-1}(x) - n$$

is an increasing function bounded below by $q(2n/q) - n = n$ on H and so

$$\left|\int_H \exp i\left(q\int_0^x \Psi^{-1}(y)\,dy - nx\right)dx\right| \leqslant \frac{\pi}{n}.$$

To estimate the integral over G when $G \neq \phi$, we use the second version of Van der Corput's Lemma (Lemma 3.2 (ii)). We have, for all $x \in G$,

$$\begin{aligned}\frac{d^2}{dx^2}\left(q\int_0^x \Psi^{-1}(y)\,dy - nx\right) &= q\,\frac{d}{dx}\,(q\Psi^{-1}(x) - n)\\ &= \frac{q}{(\psi(\Psi^{-1}(x)))^2}\\ &\geqslant q\,(\sup_{t\in G}\psi(\Psi^{-1}(t)))^{-2}\\ &\geqslant q\,(\sup_{n/2q\leqslant s\leqslant 2n/q}\psi(s))^{-2}\\ &\geqslant q\,(L\,\psi(n/q))^{-2}\\ &\geqslant q\,(L^2 q^m\,\psi(n))^{-2}.\end{aligned}$$

(The last two inequalities use (b). Note that $n/q \geqslant 1$, since $G \neq \phi$.) Thus, by Lemma 3.2 (ii),

$$\left|\int_G \exp i\left(q\int_0^x \Psi^{-1}(y)\,dy - nx\right)dx\right| \leqslant 4\pi^{1/2}\,L^2\,q^{m-1/2}\,\psi(n).$$

Collecting terms and using the fact stated above that

$$\psi(n) \geqslant (2L)^{-1}\,n^{-1},$$

we see that

$$\begin{aligned}|\hat{h}_q(n)| &\leqslant \frac{2\pi}{n} + 4\pi^{1/2}\,L^2\,q^{m-1/2}\,\psi(n) + \frac{\pi}{n}\\ &\leqslant 6\,L\pi\psi(n) + 4\pi^{1/2}\,L^2\,q^{m-1/2}\,\psi(n)\\ &\leqslant 10\pi L^2\,|q|^m\,\psi(|n|).\end{aligned}$$

□

Remark. – The definition of h and the argument giving a bound on $|\hat{h}_q(n)|$ are, to all intents and purposes, due to Ivašev-Musatov. Our h corresponds to his f (p. 108 of the English translation of [2]) and our interval G to his interval Δ (p. 119 ibid.).

At first sight it looks as though h has been pulled out of a hat, but, once the form of the argument has been grasped, it is possible to see why h was chosen. Let us give a heuristic argument. We want $\int \exp(i(h(t) - \lambda t))\, dt$ to be small. If $\frac{d}{dt}(h(t) - \lambda t)$ is large, then the first form of Van der Corput's lemma deals with the problem. Hence we only have to worry when $h'(t)$ is close to λ. We then want the acceleration

$$\frac{d^2}{dt^2}(h(t) - \lambda t) = h''(t)$$

to be as large as $\frac{1}{(\psi(\lambda))^2}$. Simplifying slightly, we want

$$h''(t) = \frac{1}{(\psi(h(t)))^2},$$

where $h'(t) = \lambda$, i.e. we want to solve $h''(t) = \frac{1}{(\psi(h(t)))^2}$. Multiplying up, we have $h''(t)(\psi(h(t)))^2 = 1$ and integrating gives

$$\int_0^{h'(x)} (\psi(t))^2\, dt = x$$

i.e. $h'(x) = \Psi^{-1}(x)$. The idea is simple, but one can only admire the mathematical clearheadedness of the man who found it.

Looking at the proof of Lemma 3.2 (ii) one might ask whether a better choice of G could be found. But to improve our estimates over F and H we would need to take G larger, and if we did this we would no longer know that the acceleration $(\psi(\Psi^{-1}(x)))^{-2}$ was close to $(\psi(n))^{-2}$ on G.

5. Conclusion of the construction.

It is easy to find an infinitely differentiable positive function $\phi_0 : [1, \infty) \to \mathrm{R}$ such that $\phi_0(r) = \phi(r)$ and $\phi_q(y)$ lies between $\phi(r)$ and $\phi(r+1)$ whenever $r \leqslant y \leqslant r+1$ [$r \geqslant 1$ integral]. Trivial estimates then show that (setting $\mathrm{L} = 2\mathrm{K}^2$, say)

(A)′ $\int_1^\infty (\phi_0(t))^2 \, dt$ diverges

(B)′ $\mathrm{L}^{-1} \phi_0(s) \leqslant \phi_0(t) \leqslant \mathrm{L}\phi_0(s)$ for all $1 \leqslant s \leqslant t \leqslant 2s$

(C)′ $t \, \phi_0(t) \to 0$ as $t \to \infty$.

By Lemma 4.1 (i) we can find an integer $m \geqslant 1$ such that

(B)$'_1$ $s^m \phi_0(s) \leqslant \mathrm{L} t^m \phi_0(t)$ for all $1 \leqslant s \leqslant t$.

In the next lemma we shall take ϕ_0, L and m as fixed once and for all.

LEMMA 5.1. – *Given $\gamma > 0$ and $\lambda \geqslant 1$ we can find a continuous function $h : [0, 2\pi] \to \mathrm{R}$ such that*

i) *h is infinitely differentiable on* $(0, 2\pi)$

ii) $h(0) = 0$, $h(2\pi) = 2p\pi$ *for some* $p \in \mathrm{Z}$

iii) $h'(t) \geqslant \lambda$ *for all* $t \in (0, 2\pi)$

iv) *Writing* $h_q(t) = \exp(1qh(t))$ $[0 \leqslant t \leqslant 2\pi]$ *we have* $|\hat{h}_q(n)| \leqslant \gamma |q|^m \phi(|n|)$ *for all n, $q \neq 0$ and*

v) $|\hat{h}_q(0)| \leqslant \gamma$ *for all* $q < 0$.

Proof. – Set $\epsilon_0 = \min((100\pi)^{-1}, \gamma(20\pi \mathrm{L}^2)^{-1})$. As in Lemma 4.1. (ii) We can find a u such that

(c)$'_1$ $\epsilon_0 t \phi_0(t) \geqslant 1$ for all $t \geqslant u$

(c)$'_2$ $\epsilon_0 \phi_0(t) \geqslant u^{-1}$ for all $1 \leqslant t \leqslant u$

and (e) $2\pi u^{-1} \leqslant \gamma$, $\lambda \leqslant u$.

Let
$$\Psi_\alpha(s) = \int_u^s (\alpha\phi_0(t))^2\, dt$$
$$w(\alpha) = \int_0^{2\pi} \Psi_\alpha^{-1}(y)\, dy$$

Now w is a continuous function of α and, since $0 \leqslant \epsilon_0\phi_0(t) \leqslant (100\,\pi)^{-1}$, $w(2\epsilon_0) - w(\epsilon_0) \geqslant 2\pi$. Thus we can find an $2\epsilon_0 \geqslant \epsilon \geqslant \epsilon_0$ such that $w(\epsilon) = 2p\pi$ for some integer p.
Thus we can find an $2\epsilon_0 \geqslant \epsilon \geqslant \epsilon_0$ such that $w(\epsilon) = 2p\pi$ for some integer p.

Set $\psi(t) = \epsilon\phi_0(t)$ and take v to be the solution of
$$\int_u^v (\epsilon\,\phi_0(t))^2\, dt = 2\pi$$
(such a solution exists since $\int_u^\infty (\phi_0(t))^2\, dt$ diverges). The function ψ satisfies all the conditions of Lemma 4.2 and so, defining h just as in that lemma, we have

iv) $$|\hat{h}_q(n)| \leqslant \epsilon\, 10\,\pi\, L^2\, |q|^m\, \phi_0(|n|) \leqslant \gamma\, |q|^m\, \phi_0(|n|) = \gamma\, |q|^m\, \phi(|n|)$$

for all $n,\ q \neq 0$, whilst (using (e))

v) $$|\hat{h}_q(0)| \leqslant 2\pi u^{-1} \leqslant \gamma \quad \text{for all} \quad q \neq 0\,.$$

The truth of condition iii) is an immediate consequence of the definition and the fact that $w(\epsilon) = 2p\pi$. Again, it follows from the definition that h is differentiable on $(0\,,2\,\pi)$ with

iii) $$h'(t) \geqslant \Psi^{-1}(t) \geqslant u \geqslant \lambda \quad \text{for all} \quad t \in (0\,,2\pi)\,.$$

Finally, since ψ is infinitely differentiable on $(0\,,2\,\pi)$, so is Ψ^{-1} $\left(\text{observe that } \frac{d^2}{dt^2}\Psi^{-1}(t) = \frac{1}{\psi(\Psi^{-1}(t))^2}\right)$.

□

We have now got the estimates which will enable us to carry through the program outlined at the begining of Section 4, prove Lemma 2.1 and so complete the proof of Theorem 1.2.

Proof of Lemma 2.1. – Note first that it suffices to prove Lemma 2.1 with iii) replaced by iii)′ $\frac{1}{2\pi}\int_{T} f(t)\, dt \geqslant 1/2$

and that, without loss of generality, we may suppose $0 < \eta,\ \epsilon < 1$.

Choose a $g \in C(T)$ such that

1) g is infinitely differentiable

2) $g(t) \geqslant 0$ for all $t \in T$

3) $\text{supp}\, g \subset [\pi - \eta/4,\ \pi + \eta/4]$

4) $\frac{1}{2\pi}\int_{T} g(t)\, dt = 1$.

Since g is infinitely differentiable, it follows on integrating by parts $m + 2$ times, that

5) $|\hat{g}(q)| \leqslant A\, |q|^{-m-2}$ $[q \neq 0]$

for some constant $A \geqslant 1$ depending on g alone.

Set $\gamma = \epsilon A^{-1}/100$ and $\lambda = \pi\eta^{-1}$ and construct h as in Lemma 5.1. We claim that $f(t) = g(h(t))$ $[t \in T]$ defines a continuous function f satisfying the conclusion of Lemma 2.1. with iii) replaced by iii)′. (Note that condition ii) of Lemma 5.1 is needed to make the definition of $f(0)$ unambiguous.)

Condition i) of the lemma follows from (2). Since g is infinitely differentiable everywhere and h is infinitely differentiable on $T \setminus \{0\}$ it follows that f is infinitely differentiable except possibly at 0. But, by condition ii) of Lemma 5.1, $h(0) = 0$ and by (3) g is constant in an open interval containing 0, so f is infinitely differentiable at 0 also, and condition v) is thus verified.

To prove condition ii), we note that

$$\text{supp}\, f \subseteq \{x : h(x) \in [\pi - \eta/4,\ \eta + \eta/4]\}$$
$$= h^{-1}([\pi - \eta/4,\ \eta + \eta/4])$$

and that by Lemma 3.3 and condition iii) of Lemma 5.1

$$\| h^{-1}([\pi - \eta/4,\ \pi + \eta/4])| - \eta/2| \leqslant \pi/(2\lambda) = \eta/2$$

so $|h^{-1}([\eta - \eta/4,\ \pi + \eta/4])| \leqslant \eta$ and $|\text{supp}\, f| \leqslant \eta$ as required.

Next we observe that, since $g \in A(T)$, we have

$$6) \quad g(t) = \sum_{q=-\infty}^{\infty} \hat{g}(q) \exp iqt$$

and

$$7) \quad f(t) = 1 + \sum_{q \neq 0} \hat{g}(q) \exp(iqh(t)),$$

the convergence being uniform. Thus, in the notation of Lemma 5.1,

$$8) \quad \hat{f}(r) = \sum_{q \neq 0} \hat{g}(q)\, \hat{h}_q(r) \quad [r \neq 0]$$

$$8)' \quad \frac{1}{2\pi} \int_T f(t)\, dt = 1 + \sum_{q \neq 0} |\hat{g}(q)\, \hat{h}_q(0)\,.$$

From 8), 5) and condition iv) of Lemma 5.1 we obtain

$$|\hat{f}(r)| \leqslant \sum_{q \neq 0} |\hat{g}(q)|\ |\hat{h}_q(r)|$$

$$\leqslant A\gamma \sum_{q \neq 0} |q|^{-2} \phi(|r|)$$

$$\leqslant \epsilon\phi(|r|) \quad [r \neq 0]$$

so condition iv) holds. Similarly 8)′, 5) and condition v) of Lemma 5.1 give

$$\frac{1}{2\pi} \int_T f(t)\, dt \geqslant 1 - \sum_{q \neq 0} |\hat{g}(q)|\ |\hat{h}_q(0)|$$

$$\geqslant 1 - \gamma A \sum_{q \neq 0} |q|^{-2-k}$$

$$\geqslant 1/2$$

so condition iii)′ holds.

□

The proof of our theorem is complete.

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