Biholomorphic maps determined on the boundary


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BIHOLOMORPHIC MAPS DETERMINED ON THE BOUNDARY
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Let $X$, $Y$ be complex manifolds of pure dimension $n$ where the holomorphic functions on $X$ separate points; let $D$ be a relatively compact open subset of $X$, and $\hat{D}$ a neighborhood of $D$. Let $f : \hat{D} \to Y$ be a holomorphic map. The object of the present note is to show under certain conditions that if $f$ is one-to-one when restricted to the boundary $bD$ of $D$, then $f : D \to f(D)$ is biholomorphic. In case $X = Y =$ the complex plane, if $bD$ is a rectifiable Jordan curve, then $f(D)$ is the domain surrounded by the curve $f(bD)$ and $f : D \to f(D)$ is conformal. A corollary is deduced to extend this theorem to the case of higher dimensions. We begin with a lemma which will be stated in a form a little more general than actually needed.

**Lemma.** Let $f : \hat{D} \to Y$ be a holomorphic map. If $f$ has finite fibres on $bD$, then so does $f$ on $D$.

**Proof.** Let $F = \{p \in \hat{D} | f(p) = q_0\}$, $q_0 \in f(D)$, and suppose that $F \cap D$ is noncompact. Then $F \cap bD \neq \emptyset$; this constitutes a finite set of points $\{p_1, p_2, \ldots, p_s\}$. There exists a point $p_i$ such that $F \cap D \cap U \neq \emptyset$ for every neighborhood $U$ of $p_i$. We take mutually disjoint open neighborhoods $U_i$ of $p_i$ in $\hat{D}$, $i = 1, 2, \ldots, s$, for which $F \cap U_i = V_i \cup V_i^2 \cup \ldots \cup V_i^{m_i}$ is the decomposition of $F$ into irreductible branches at $p_i$, and the sets $R(V_i^{m_i})$ of regular points of $V_i^{m_i}$ are connected manifolds which are dense in $V_i^{m_i}$. There are a point $p_j$ and a branch $V_j^{m_i}$ such that $V_j^{m_i} \cap D \neq \emptyset$ and $V_j^{m_i} - \hat{D} \neq \emptyset$, because, if this is not the case, then $F \cap D$ and all the branches contained
in $\tilde{D}$ constitute a compact subvariety of $D \cup \bigcup_{i=1}^{s} U_i$, so that $F \cap D$ becomes a finite set of points. The dimension of such a variety $V^m_j$ at $p_j$ is positive. We choose $p'_1 \in \mathbb{R} (V^m_j) \cap D$ and $p'_2 \in \mathbb{R} (V^m_j) - \overline{D}$. Then there is a curve in the connected manifold $\mathbb{R} (V^m_j) - \{p_j\}$ which joins $p'_1$ to $p'_2$, and this must intersect $bD$. But this is impossible, and the proof is completed.

In what follows, differentiability will mean that of $C^\infty$. We denote by $\partial D$ the totality of regular points of $bD$; that is, $p_0 \in \partial D$ if and only if $p_0 \in bD$ and there exist a neighborhood $U$ of $p_0$ and a differentiable coordinate system $\phi = (x_1, x_2, \ldots, x_{2n}) : U \rightarrow \Delta (0 ; \varepsilon)$, the $\varepsilon$-cube in $\mathbb{R}^{2n}$ centered at the origin 0, such that

$$\phi (p_0) = 0, \quad \overline{D} \cap U = \{p \in U \mid x_{2n} (p) \geq 0\}.$$  

**Theorem.** — Let $D$ be a relatively compact open subset of $X$ such that $\partial D \neq \emptyset$. If $f$ is one-to-one on $bD$ and $f(D) - f(bD)$ is connected, then $f : D \rightarrow f(D)$ is biholomorphic.

**Proof.** — We may assume that $X$ and $Y$ have countable bases for open sets. Note that $f : D \rightarrow Y$ is an open map by the above lemma. Let $G = f(D)$, $G_0 = G - f(bD)$, and $D_0 = D - f^{-1} (f(bD))$. $G_0$ is dense in $G$, since $f : bD \rightarrow f(bD)$ is a homeomorphism. Let $S = \{p \in \tilde{D} \mid \text{rank}_p f < n\}$. By Sard’s theorem, $D \cap S$ is a nowhere dense analytic subvariety of $D$, so it can be assumed, by shrinking $\tilde{D}$ if necessary, that $S$ is nowhere dense in $\tilde{D}$. The restricted map $f : D_0 \rightarrow G_0$ is proper, and

$$f_0 : D_0 - f^{-1} (f(D_0 \cap S)) \rightarrow G_0 - f(D_0 \cap S)$$  

is a finitely sheeted covering map. $G_0 - f(D_0 \cap S)$ is dense in $G$; it follows that if $f_0$ is one-to-one, then so is $f : D \rightarrow G$. For the differentiable map $f : D_0 \rightarrow G_0$, the connectedness of $G_0$ guarantees the existence of a constant $\delta$, the degree of $f$, such that if $\omega$ is a 2n-form of compact support in $G_0$ then

$$\int_{D_0} f^* \omega = \delta \int_{G_0} \omega ;$$
this $\delta$ coincides with the number of sheets of the covering map $f_0 ([1])$. Thus, we have only to show that $\delta = 1$.

We shall show that $f' (\partial D - S) \subset \partial G$, where it should be noted that $\partial D \not\subset S$ since $\partial D$ is a real $(2n - 1)$-dimensional manifold. Let $p_0 \in \partial D - S$, $q_0 = f'(p_0)$. We take an open neighborhood $U'$ of $p_0$ in $\tilde{D}$ such that $f' = f | U' : U' \to V'$ is biholomorphic where $V'$ is a neighborhood of $q_0$. We assume that

$$
\phi = (x_1, x_2, \ldots, x_{2n}) : U' \to \Delta (0 ; \epsilon)
$$

is a coordinate system for which

$$
\phi (p_0) = 0, \quad \tilde{D} \cap U' = \{ p \in U' : x_{2n} (p) > 0 \}.
$$

Let $y_i = x_i \circ f'^{-1}$, $i = 1, 2, \ldots, 2n$, then

$$
\psi = (y_1, y_2, \ldots, y_{2n}) : V' \to \Delta (0 ; \epsilon)
$$

is a coordinate system for $V'$. Suppose that $q_0 \in G$ and $V' \subset G$. Since $q_0 \notin f (bD - U')$, we can find $V = \psi^{-1} (\Delta (0 ; \rho))$, $0 < \rho < \epsilon$, so that $V \cap f (bD - U') = \emptyset$. Put $U = f'^{-1} (V)$. Let $\omega$ be a 2n-form:

$$
\omega = g \, dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{2n}
$$

where $g$ is a differentiable function of compact support in $V$. Let $\{ \rho_k \}$, $\{ \rho'_k \}$ be sequences of positive numbers such that

$$
\rho_1 < \rho_2 < \ldots < \rho, \quad \rho_k \to \rho ; \rho_1 > \rho_1' > \rho_2' > \ldots, \rho'_k \to 0,
$$

and let

$$
Q_k = \{ q \in V | \left| y_i (q) \right| < \rho_k, 1 \leq i \leq 2n - 1 ; \rho'_k < \left| y_{2n} (q) \right| < \rho_k \},
$$

$k = 1, 2, \ldots$. Note that $Q_k \subset G_0$. We choose differentiable functions $g_k$ with the property that

$$
g_k (q) = \begin{cases} 
g (q), & q \in \overline{Q_k} \\ 0, & q \in Y - Q_{k+1} \end{cases}
$$

and $|g_k (q)| \leq \text{const.}$ for all $q \in Y$ and $k$. Putting $\omega_k = g_k \, dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{2n}$, we have

$$
\int_D f^* \omega_k = \delta \int_G \omega_k , \quad k = 1, 2, \ldots.
$$
Let $H = D - U$, then $(\text{supp } f^* \omega) \cap \overline{D} \subset H \cup (\overline{D} \cap U)$. The set $E = \{ q \in V \mid y_{2n} (q) = 0 \}$ is of measure zero in $Y$ and, since $f$ is locally biholomorphic on $\overline{D} - S$,

$$f^{-1} (E) \cap \overline{D} = (f^{-1} (E) \cap \overline{D} \cap S) \cup (f^{-1} (E) \cap (\overline{D} - S))$$

is also of measure zero. Therefore, $g_k \rightarrow g$, a.e., on $\overline{G}$ and $g_k \circ f \rightarrow g \circ f$, a.e., on $H \cup (\overline{D} \cap U)$. Thus, we obtain

$$I = \lim_{k \to \infty} \int_D f^* \omega_k = \int_H f^* \omega + \int_{D \cap U} f^* \omega, \quad I = \delta \int_G \omega.$$ 

Let $h$ be a nonnegative differentiable function of compact support in $V$ such that $h \left( q^0 \right) > 0$ and let $\theta = h \, dy_1 \wedge \ldots \wedge dy_{2n-1}$. The support of $f^* \theta$ in $H$ is compact, so we get from the preceding formula applied to $\omega = d\theta$

$$I = \delta \int_G d\theta = 0,$$

a contradiction. Thus, we have proved $f(\partial D - S) \subset bG$. Now take $U'$, $V'$ as in the above. Since $f(\partial D \cap U') \subset bG \subset f(\partial D)$ where $f(\partial D \cap U')$ is open in $f(\partial D)$, we can find a neighborhood $W$ of $q_0$ in $V'$ so that $bG \cap W \subset f(\partial D \cap U')$. Take $V = \psi^{-1} (\Delta (0 ; \rho))$ in $W$ such that $V \cap f(bD - U') = \phi$, and let $U = f^{-1} (V)$. We see that $bG \cap V = f(\partial D \cap U)$. $V$ is decomposed as follows:

$$V = f(D \cap U) \cup f(\partial D \cap U) \cup f(U - D) = (G \cap V) \cup (bG \cap V) \cup (V - G),$$

where $f(D \cap U) \subset G \cap V$, $V - \overline{G} \subset f(U - D)$. Suppose that

$$V - \overline{G} = \phi.$$

Then, from $V - f(\partial D \cap U) \subset G$, we can deduce a contradiction just as in the above. Thus, $V - \overline{G} \neq \phi$ and, from the connectedness of $f(U - D)$, we see that $f(U - D) \cap G \cap V = \phi$, which implies that $f(D \cap U) = G \cap V$. It follows that

$$G \cap V = \{ q \in V \mid y_{2n} (q) > 0 \}$$
and \( \partial G \cap V = \partial G \cap V \). In the present situation, let
\[
Q_k = \{ q \in V | |y_i(q)| < \rho_k, \rho_k' < y_{2n}(q) < \rho_k \}, \quad k = 1, 2, \ldots ,
\]
and choose \( g_k \) as before for \( \omega = g dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{2n} \). For \( \omega = d \theta \), we have
\[
I = \int_{\partial D \cap U} f^* \theta = \int_{\partial G \cap V} \theta, \quad I = \delta \int_G d \theta = \delta \int_{\partial G \cap V} \theta ;
\]
these yield \( \delta = 1 \). This completes the proof.

As a typical example in which the condition of Theorem is satisfied, we deal with the following case.

**Corollary.** — *Let \( D \) be a bounded open subset of the complex \( n \)-space \( \mathbb{C}^n \) such that \( \partial D \) is topologically a \((2n-1)\)-dimensional sphere in \( \mathbb{R}^{2n} \) with \( \partial D \neq \emptyset \), and let \( f : \tilde{D} \rightarrow \mathbb{C}^n \) be holomorphic. If \( f \) is one-to-one on \( \partial D \), then \( f : D \rightarrow f(D) \) is biholomorphic where \( f(D) \) is the domain surrounded by the sphere \( f(\partial D) \).*

**Proof.** — \( f(\partial D) \) is a \((2n-1)\)-sphere in \( \mathbb{C}^n \), so that \( \mathbb{C}^n - f(\partial D) \) is decomposed into two components \( G \) and \( G' \) with \( f(\partial D) = bG = bG' \). Let \( G \) be the bounded component. Let \( f(D) \cap G' \neq \emptyset \). If \( G' \not\subset f(D) \), then \( bf(D) \cap G' \neq \emptyset \), which contradicts \( bf(D) \subset f(\partial D) \); hence we have \( G' \subset f(D) \), which contradicts the boundedness of \( f(D) \). Thus, \( f(D) \subset G \). It follows from the same reasoning that \( f(D) = G \). We have \( f(\partial D) = bf(\partial D) \), and the proof is completed.

**BIBLIOGRAPHY**