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CONICAL MEASURES AND VECTOR MEASURES

by Igor KLUVÁNEK

One of the most important objects related to a family $\{f_\iota : \iota \in \mathrm{I}\}$ of (real-valued) random variables is the joint distribution of this family. It is a mesure on the cylindrical σ-algebra in the space $\mathbf{R}^{\mathrm{I}}$. Indeed, all properties of this family (as a family of random variables) are reflected by its distribution. It can happen that neither the family of random variables itself nor the probability measure on the underlying space are available but we are given the indefinite integrals of all random variables f_ι, $\iota \in \mathrm{I}$ (assuming them integrable). In that case the joint distribution cannot be determined. However, the conical measure on $\mathbf{R}^{\mathrm{I}}$ determined by the joint distribution still can be reconstructed. It is clear that there remain many properties of the family of random variables which can be inferred from this conical measure.

Noticing the obvious fact that the family of indefinite integrals of the functions f_ι, $\iota \in \mathrm{I}$, can be interpreted as a measure with values in the space $\mathbf{R}^{\mathrm{I}}$, we propose to investigate the indicated situation in a more general setting. To every abstract measure with values in a locally convex topological vector space E, there corresponds a conical measure on E and also on the weak completion E'^* of E. There are many properties of the vector measure which can be determined from the corresponding conical measure. Also conversely, the vector measure can be made a tool for studying the conical measure.

In the first Section, it is proved that every conical measure on a weak complete space E is represented (not uniquely) as integration with respect to a non-negative measure λ on the cylindrical σ-algebra in E. And λ can even be extended on a larger σ-algebra so as to become a direct sum of finite measures.

This result and the connection between measures with values in E and conical measures on E is used in Sections 2 and 3 to obtain information on the general structure of abstract E-valued measures. In the final Section are stated some results concerning conical measures. In particular, some conditions are given guaranteeing that a conical measure on E be expressible as integration with respect to a finite (σ-additive) measure on the cylindrical σ-algebra in E.

1. Representation of conical measures.

Let E be a real locally convex topological vector space with topological dual E′ and algebraic dual E*.

The vector lattice of functions on E generated by E′ is denoted by h (E). Every element f of h (E) can be written as

$$f(x) = \sup \{f_i(x) : 1 \leqslant i \leqslant l\} - \sup \{f_i(x) : l + 1 \leqslant i \leqslant k\}, x \in E,$$

where $f_i \in E'$, for $1 \leqslant i \leqslant k$, $1 \leqslant l < k$.

A non-negative linear form on h (E) is termed a *conical measure* on E (see Choquet [3] or [4]).

The interest in this Section is focused on weak complete spaces. If E is a weak complete topological vector space then there exists an index set I such that, up to a topological vector space isomorphism,

$$E = \mathbf{R}^I ;$$

the structure of the topological vector space on $\mathbf{R}^I$ being that of the product of one-dimensional ones.

Without a loss of generality it will be assumed that I is well-ordered and, indeed, that I is an interval of ordinal numbers with 0 as the least element.

For every $\kappa \in I$, let p_κ be the projection of $E = \mathbf{R}^I$ onto its κ-th co-ordinate ; that is, $p_\kappa(x) = x_\kappa$, for every element $x = (x_\iota)_{\iota \in I}$ of E.

It is known that $\{p_\iota : \iota \in I\}$ forms a Hamel basis for E'. Also h (E) is the vector lattice generated by $\{p_\iota : \iota \in I\}$.

For every $\kappa \in I$, define

$$T_\kappa = \{x \in E : p_\iota(x) = 0 \text{ for } \iota < \kappa \text{ and } |p_\kappa(x)| = 1\}.$$

Furthermore, let

$$T = \bigcup_{\kappa \in I} T_\kappa .$$

Given a function f on E, by $f \mid T$ is denoted its restriction to T and by $f \mid T_\kappa$ its restriction to T_κ ; $h(E) \mid T_\kappa = \{f \mid T_\kappa : f \in h(E)\}$, $\kappa \in I$; $h(E) \mid T = \{f \mid T : f \in h(E)\}$.

Let S_κ be the σ-algebra of subsets of T_κ generated by $h(E) \mid T_\kappa$, i.e. the associated σ-algebra such that all functions in $h(E) \mid T_\kappa$ are S_κ-mesurable, $\kappa \in I$. Let S be the direct sum of σ-algebras S_κ, $\kappa \in I$. That is, S consists of all sets $X \subset T$ such that $X \cap T_\kappa \in S_\kappa$, for every $\kappa \in I$.

If λ_κ is a measure on S_κ, for each $\kappa \in I$, a measure λ on S is termed the direct sum of the measures λ_κ, $\kappa \in I$, if

$$\lambda(X) = \sum_{\kappa \in I} \lambda_\kappa (X \cap T_\kappa),\ X \in S.$$

We say also, in this case, that the measure space (T, S, λ) is the direct sum of the measure spaces $(T_\kappa, S_\kappa, \lambda_\kappa)$, $\kappa \in I$.

THEOREM 1.– *Let u be a conical measure on the weak complete space* $E = \mathbf{R}^I$. *Then, for every $\kappa \in I$, there exists a unique finite measure λ_κ on $\underline{S}_k$ such that, if (T, S, λ) is the direct sum of the measure spaces $(T_\kappa, S_\kappa, \lambda_\kappa)$, $\kappa \in I$, then*

$$u(f) = \int_T (f \mid T)\, d\lambda \tag{1}$$

for every $f \in h(E)$. The vector lattice $h(E) \mid T$ is dense in $L^1(\lambda)$.

The main part of this Section is devoted to the proof of Theorem 1.

According to Choquet [3, Théorème 17] or [4, Theorem 38.3], u is a Daniell integral on h (E). Let L be a linear lattice of functions on E containing h (E) and such that

i) there is a Daniell integral on L which is an extension of u and which will be denoted by u, again ;

ii) the Dominated convergence theorem (Lebesgue theorem) holds for this extension of u on L ;

iii) for every $f \in L$ and $\epsilon > 0$, there exists $g \in h$ (E) such that $u(|f - g|) < \epsilon$;

iv) all function in L are real-valued (the values $\pm \infty$ are excluded) and belong to the least class of functions containing h (E) which is closed with respect to the point-wise convergence of sequences.

The general theory of Daniell integrals guarantees that such L and the corresponding extension of u exist.

We put, as customary, $L^+ = \{f \in L : f \geqslant 0\}$.

The property (iv) implies that every function f in L is positive-homogeneous ($\alpha > 0$ and $x \in$ E imply that $f(\alpha x) = \alpha f(x)$). The positive homogeneity in turn implies that u can be considered a Daniell integral on the vector lattice $L \mid \mathrm{T} = \{f \mid \mathrm{T} : f \in L\}$. Specifically, if $f_n \in L$, $n = 1, 2, \ldots$, and if $f_n(t) \downarrow 0$ for every $t \in$ T, then $f_n(x) \downarrow 0$ for every $x \in$ E and $u(f_n) \to 0$, $n \to \infty$.

Let u_ι and v_ι be functionals defined on L inductively for every $\iota \in$ I as follows.

Let $\kappa \in$ I. Assume that u_ι is already defined for each $\iota < \kappa$. Denote

$$v_\kappa(f) = u(f) - \sum_{\iota < \kappa} u_\iota(f), \tag{2}$$

for every $f \in L$, and put

$$u_\kappa(f) = \sup \{v_\kappa(f \wedge \alpha \mid p_\kappa \mid) : \alpha > 0\} \tag{3}$$

for every $f \in L^+$; furthermore $u_\kappa(f) = u_\kappa(f^+) - u_\kappa(f^-)$ for every $f \in L$.

LEMMA 2. – *The functionals u_ι, v_ι are well-defined Daniell integrals on and L and $0 \leqslant u_\iota \leqslant v_\iota \leqslant u$, for every $\iota \in$ I.*

Proof. – Let $\kappa \in I$. Assume that u_ι, v_ι are Daniell integrals and $0 \leqslant u_\iota \leqslant v_\iota \leqslant u$, for every $\iota < \kappa$.

Given $f \in L^+$, we have

$$0 \leqslant v_\nu(f) = u(f) - \sum_{\iota<\nu} u_\iota(f)$$

for every $\nu < \kappa$, Hence

$$0 < \sum_{\iota<\kappa} u_\iota(f) = \sup \{\sum_{\iota<\nu} u_\iota(f) : \nu < \kappa\} \leqslant u(f).$$

i.e. $0 \leqslant v_\kappa(f) \leqslant u(f)$. In particular, v_κ is well-defined by (2). It is then clear also that v_κ is a linear form on L. The inequality

$$0 \leqslant v_\kappa(f) \leqslant u(f).$$

for $f \in L^+$, implies that v_κ is a Daniell integral.

Now, u_κ is well-defined by (3) and $0 \leqslant u_\kappa(f) \leqslant v_\kappa(f)$ for every $f \in L^+$. The definition then extends on the whole of L. We show that u_κ is a linear form.

Let $f, g \in L^+$. Since $(f+g) \wedge \alpha \,|p_\kappa| \leqslant (f \wedge \alpha\,|p_\kappa| + (g \wedge \alpha\,|p_\kappa|)$ for every $\alpha > 0$, the definition (3) gives that $u_\kappa(f+g) \leqslant u_\kappa(f) + u_\kappa(g)$. On the other hand, for every $\epsilon > 0$ there is $\alpha > 0$ and $\beta > 0$ such that

$$u_\kappa(f) - \epsilon < v_\kappa(f \wedge \alpha\,|p_\kappa|)\ ,\ u_\kappa(g) - \epsilon < v_\kappa(g \wedge \beta\,|p_\kappa|).$$

Consequently,

$$u_\kappa(f) + u_\kappa(g) - 2\,\epsilon < v_\kappa(f \wedge \alpha\,|p_\kappa|) + v_\kappa(g \wedge \beta\,|p_\kappa|)$$

$$\leqslant v_\kappa((f+g) \wedge (\alpha+\beta)\,|p_\kappa|) \leqslant u_\kappa(f+g).$$

So, $u_\kappa(f+g) = u_\kappa(f) + u_\kappa(g)$.

For every $f \in L^+$ and $c > 0$,

$$u_\kappa(cf) = \sup\{v_\kappa(cf \wedge \alpha\,|p_\kappa|) : \alpha > 0\} =$$

$$= c \sup\{v_\kappa(f \wedge (\alpha/c)\,|p_\kappa|) : \alpha > 0\} = c\,u_\kappa(f).$$

The linearity of u_κ follows.

The inequality $0 \leqslant u_\kappa(f) \leqslant v_\kappa(f)$, for each $f \in L^+$, implies that u_κ is a Daniell integral.

LEMMA 3. – *Let $f \in L^+$. Then*

$$\sum_{\iota \leqslant \kappa} u_\iota(f) = \sup \{u(f \wedge \varphi) : \varphi \in L, \varphi(t) = 0 \text{ for } t \in \bigcup_{\kappa < \iota} T_\iota\} \quad (4)$$

and

$$\sum_{\iota < \kappa} u_\iota(f) = \sup \{u(f \wedge \varphi) : \varphi \in L, \varphi(t) = 0 \text{ for } t \in \bigcup_{\kappa \leqslant \iota} T_\iota\} \quad (5)$$

for every $\kappa \in I$.

Proof. – If $\kappa = 0$, the statement (4) is true by the definition of u_0 and (5) is true vacuously.

Let $\kappa \in \mathrm{I}$, $\kappa > 0$. Assume that (4) holds if κ is replaced by any $\nu \in \mathrm{I}$, $\nu < \kappa$. Then (5) follows. An inductive proof will be proving (4).

Let φ_n be functions vanishing on every T_ι with $\kappa \leqslant \iota$ such that $\varphi_n \leqslant \varphi_{n+1}$, $n = 1,2,\ldots$, and $u(f \wedge \varphi_n) \to \sum_{\iota < \kappa} u_\iota(f)$. Let

$$g = \lim f \wedge \varphi_n = \sup f \wedge \varphi_n.$$

Then

$$u(g) = u(f \wedge g) = \sum_{\iota < \kappa} u_\iota(f). \quad (6)$$

Moreover, if $\theta \in L^+$ is a function vanishing on every T_ι with $\kappa \leqslant \iota$, then

$$u((f - g) \wedge \theta) = 0. \quad (7)$$

Indeed, if (7) were not true then

$$u(f \wedge (g + \theta)) = u(f \wedge g) + u((f - g) \wedge \theta) > u(g)$$

would follow, contradicting (6) and (5).

Also, if $f' \leqslant f$, $f' \in L^+$, then $\sum_{\iota < \kappa} u_\iota(f') = u(f' \wedge g)$. This because, for any $\varphi \in L^+$ vanishing on every T_ι with $\kappa \leqslant \iota$, we have

$$u(f' \wedge \varphi) \leqslant u(f' \wedge g) + u((f' - f' \wedge g) \wedge (\varphi - \varphi \wedge g)) \leqslant u(f' \wedge g) + $$
$$+ u((f - g) \wedge (\varphi - \varphi \wedge g)) = u(f' \wedge g),$$

choosing $0 = \varphi - \varphi \wedge g$ in (7).

We have, in fact, proved that $v_\kappa(f') = u(f') - u(f' \wedge g)$ for every $f' \in L^+$, $f' \leqslant f$.

Let $h = \lim_{n\to\infty} f \wedge n\,|p_\kappa|$. Hence $u_\kappa(f) = v_\kappa(h) = v_\kappa(f \wedge h)$. Now

$$\sum_{\iota \leqslant \kappa} u_\iota(f) = \sum_{\iota < \kappa} u_\iota(f) + u_\kappa(f) = u(f \wedge g) + v_\kappa(f \wedge h) = u(f \wedge g) +$$

$$+ \; u(f \wedge h) - u(f \wedge h \wedge g) = u(f \wedge (g \vee h)).$$

Suppose that $\varphi \in L^+$ and that φ vanishes on every T_ι such that $\kappa < \iota$. Then

$$u(f \wedge \varphi) \leqslant u(f \wedge (g \vee h) \wedge \varphi) + u((f - g \vee h) \wedge \varphi). \qquad (8)$$

The function $\theta = (f - g \vee h) \wedge \varphi$ vanishes on every T_ι with $\kappa \leqslant \iota$ and $\theta = (f - g) \wedge \theta$, hence, by (7), $u((f - g \vee h) \wedge \varphi) = 0$. So (8) gives that $u(f \wedge \varphi) \leqslant u(f \wedge (g \vee h))$, proving (4).

LEMMA 4. – *For every $f \in L$ vanishing on every T_ι such that $\kappa \leqslant \iota$, the relation*

$$\sum_{\iota < \kappa} u_\iota(f) = u(f) \qquad (9)$$

holds. The equality

$$\sum_{\iota \in I} u_\iota(f) = u(f) \qquad (10)$$

holds for every $f \in L$.

Proof. – The first statement (9) follows immediately form (4). This implies (10) for every $f \in h(E)$ since each $f \in h(E)$ vanishes on every T_ι, $\kappa < \iota$, for some $\kappa \in I$.

Let now $f \in L$. For every $\epsilon > 0$ there exists $g \in h(E)$ such that $u(|f - g|) < \epsilon$. Consequently,

$$|u(f) - \sum_{\iota \in I} u_\iota(f)| \leqslant |u(f) - u(g)| + |u(g) - \sum_{\iota \in I} u_\iota(g)| +$$

$$+ |\sum_{\iota \in I} u_\iota(g) - \sum_{\iota \in I} u_\iota(f)| \leqslant u(|f - g|) + |u(g) - \sum_{\iota \in I} u_\iota(g)| +$$

$$+ \sum_{\iota \in I} u_\iota(|f - g|) \leqslant 2\,u(|f - g|) < 2\,\epsilon.$$

LEMMA 5. – *Let* $\kappa \in \mathrm{I}$. *Let* $f_n \in L$, $n = 1,2,\ldots$, *and let* $f_n(t) \downarrow 0$, *for every* $t \in \mathrm{T}_\kappa$. *Then* $u_\kappa(f_n) \to 0$.

Proof. – If $g \in L$ and $g(t) = 0$ for every $t \in \mathrm{T}_\kappa$, then

$$g(t) \wedge \alpha \,|\, p_\kappa(t) \,| = 0$$

for every $t \in \bigcup_{\kappa \leqslant \iota} \mathrm{T}_\iota$. Hence, by Lemma 3, $v_\kappa(g \wedge \alpha \,|\, p_\kappa \,|) = 0$ for each $\alpha > 0$. So, $u_\kappa(g) = 0$.

Now put $g_1 = f_1$ and $g_n = g_{n-1} \wedge f_n$, $n = 2,3,\ldots$, Then

$$g_n(t) \downarrow g(t),$$

for every $t \in \mathrm{T}$, where g is a function in L such that $g(t) = 0$ for all $t \in \mathrm{T}_\kappa$. Hence $u_\kappa(f_n) = u_\kappa(g_n) \to u_\kappa(g) = 0$.

Proof of theorem 1. – By Lemma 5, the functional

$$f \,|\, \mathrm{T}_\kappa \to u_\kappa(f), f \in L$$

is a Daniell integral on the vector lattice $\{f \,|\, \mathrm{T}_\kappa : f \in L\}$, for every $\kappa \in \mathrm{I}$. Since this vector lattice, or even $h(\mathrm{E}) \,|\, \mathrm{T}_\kappa$, contains non-zero constants, there exists a unique measure λ_κ on S_κ such that

$$u_\kappa(f) = \int_{\mathrm{T}_\kappa} (f \,|\, \mathrm{T}_\kappa)\, d\lambda_\kappa$$

for every $f \in L$. The measure λ_κ is finite, $\lambda_\kappa(\mathrm{T}_\kappa) < \infty$, and $h(\mathrm{E}) \,|\, \mathrm{T}_\kappa$ is dense in $L^1(\lambda_k)$.

If λ is the direct sum of measures λ_κ, $\kappa \in \mathrm{I}$, then (1) holds for every $f \in L$ and $h(\mathrm{E}) \,|\, \mathrm{T}$ becomes dense in $L^1(\lambda)$.

The cylindrical σ-algebra $C = C(\mathrm{E})$ in the space E is by definition the smallest σ-algebra of subsets of E such that every continuous linear form on E is C-measurable ; it is the smallest σ-algebra such that every function in $h(\mathrm{E})$ is C-measurable.

COROLLARY 6. – *If* E *is a weak complete space then, for every conical measure* u *on* E, *there exists a* σ*-additive measure* λ *on the cylindrical* σ*-algebra* C *such that*

$$u(f) = \int_{\mathrm{E}} f\, d\lambda \qquad (11)$$

for every $f \in h(\mathrm{E})$. *Moreover there exists a* σ*-algebra* D *of subsets of* E *such that* $C \subset D$ *and for every conical measure* u *on* F *there is a* σ*-additive measure* λ *on* D *which is a direct sum of finite measures such that* (10) *holds.*

Proof. – Define D to be the family of all sets $\mathrm{X} \subset \mathrm{E}$ such that $\mathrm{X} \cap \mathrm{T}_\kappa \in S_\kappa$, for each $\kappa \in \mathrm{I}$. Given a conical measure u, define the measure λ on D by $\lambda(\mathrm{X}) = \sum_{\kappa \in \mathrm{I}} \lambda(\mathrm{X} \cap \mathrm{T}_\kappa)$, $\mathrm{X} \in D$. Then, clearly, (11) holds and λ is a direct sum of finite measures.

It is clear that $C \subset D$ and that if λ is restricted to C then (11) remains valid.

2. Families of scalar measures.

In this Section a theorem is proved establishing an isometry and vector lattice isomorphism of any vector lattice of finitely additive real measures onto a dense linear sub-lattice of a space $L^1(\lambda)$ with a suitable measure λ which is direct sum of finite measures. This result can be, of course, obtained using Kakutani's concrete representation theorem of abstract L-spaces. The aim is, however, to base the proof on Theorem 1 so as to show its implications and to achieve a unity of method. This could be of interest by offering a new and, possibly, more direct approach to Kakutani's theorem. The proved theorem will be used as a lemma in the subsequent Section concerning the structure of vector measures.

Let T be an abstract space and let R be an algebra of subsets of T. By ba(R) is denoted the set of all bounded finitely additive real-valued functions (measures) on R. The set ba(R) carries several structures. It is a vector lattice with respect to the set-wise linear operations and order. If $\mu, \nu \in \mathrm{ba}(R)$, then $\mu \vee \nu$ is the least element in ba(R) majoring both μ and ν and $\mu \wedge \nu$ is the greatest element majored by μ and ν ; $|\mu| = \mu \vee (-\mu)$ for every $\mu \in \mathrm{ba}(R)$. The norm $\mu \mapsto \|\mu\| = |\mu|(\mathrm{T})$, $\mu \in \mathrm{ba}(R)$, makes of ba(R) a Banach space.

The set of all σ-additive elements of ba(R) is denoted by ca(R). It is a closed normed subspace of ba(R), hence also a Banach space ; also it is a vector sub-lattice of ba(R).

Let $H \subset ba(R)$. Two sets $X_1, X_2 \in R$ are declared H-equivalent if $|\mu|(X_1 \Delta X_2) = 0$, for every $\mu \in H$. The H-equivalence class of a set X is denoted $[X]_H$ whenever the distinction is material ; otherwise we do not distinguish between a set in R and its H-equivalence class. The set of all H-equivalence classes is denoted by R (H). It is a Boolean algebra with respect to the induced operations ; i.e. R (H) equals to R modulo the ideal of sets H-equivalent to ϕ.

Every measure $\mu \in H$ defines the Nikodym pseudo-metric d_μ on R and on R (H) ; namely $d_\mu(X, Y) = |\mu|(X \Delta Y)$, for $X, Y \in R$. By τ (H) is denoted the topology and also the uniform structure on R and on R (H) determined by the system $\{d_\mu : \mu \in H\}$ of pseudo-metrics.

In the situation when R is a σ-algebra and λ is a σ-additive measure on R having possibly infinite values, $R(\lambda)$ and $\tau(\lambda)$ are interpreted as $R(L^1(\lambda))$ and $\tau(L^1(\lambda))$, respectively, where $L^1(\lambda)$ in its turn is interpreted as the family of indefinite integrals of λ-integrable functions, i.e. as a subset of $ca(R) \subset ba(R)$. In the case when λ is localizable in the sense of Segal [10], and this is the case of our main interest, we can say that $\tau(\lambda)$ stands for the topology τ (H) where H is the set of all finite measures absolutely continuous with respect to λ. In this situation, a net X_i of sets in R tends to $X \in R$ in the topology $\tau(\lambda)$ if and only if $\lambda(X_i \Delta X) \to 0$.

Let $H \subset ba(R)$ be a vector sub-lattice. Let $\hat{E} = H^*$ be the algebraic dual of the vector space H.

For every $\mu \in H$ and $x \in \hat{E}$, denote $\mu^*(x) = x(\mu)$. If we give $\hat{E}$ its $\sigma(\hat{E}, H)$-topology, then $\hat{E}'$ will be identified with H by the star-mapping. This is to say, μ^* belongs to $\hat{E}'$, for every $\mu \in \hat{H}'$. and, conversely, every element of $\hat{E}'$ is equal to μ^* for some $\mu \in H$. It follows that every function $f \in h(\hat{E})$ can be expressed as

$$f(x) = \sup\{x(\mu_i) : 1 \leqslant i \leqslant l\} - \sup\{x(\mu_i) : l+1 \leqslant i \leqslant k\} =$$

$$= \sup\{\mu_i^*(x) : 1 \leqslant i \leqslant l\} - \sup\{\mu_i^*(x) : l+1 \leqslant i \leqslant k\}, x \in \hat{E}, \qquad (12)$$

for some $\mu_i \in H$, $1 \leqslant i \leqslant k$, $1 \leqslant l < k$.

LEMMA 7. — *There exists a unique vector lattice homomorphism* $\Phi : h(\hat{E}) \to H$ *such that* $\Phi(\mu^*) = \mu$, *for* $\mu \in H$. *For every* $f \in h(\hat{E})$, *written in the form* (12), *the value* $\Phi(f)$ *is given by*

$$\Phi(f) = \bigvee_{i=1}^{l} \mu_i - \bigvee_{i=l+1}^{k} \mu_i, \tag{13}$$

where the lattice operations are those of H *or* ba (R).

Proof. – If such a homomorphism exists then, clearly, (13) holds. Hence it suffices to show that (13) defines unambiguously a homomorphism $\Phi : h(\hat{E}) \to H$.

Suppose that $f \in h(\hat{E})$ is given by (12) and that

$g(x) = \sup \{x(\nu_i) : 1 \leqslant i \leqslant l'\} - \sup \{x(\nu_i) : l' + 1 \leqslant i \leqslant k'\}$, $x \in \hat{E}$, with $\nu_i \in H$, $1 \leqslant i \leqslant k'$, $1 \leqslant l' < k'$.

If $f = g$, i.e. $f(x) = g(x)$, for every $x \in \hat{E}$, then, in particular,

$$\sup \{\mu_i(X) : 1 \leqslant i \leqslant l\} - \sup \{\mu_i(X) : l + 1 \leqslant i \leqslant k\} =$$

$$= \sup \{\nu_i(X) : 1 \leqslant i \leqslant l'\} - \sup \{\nu_i(X) : l' + 1 \leqslant i < k'\},$$

for every $X \in R$. It then follows that

$$\bigvee_{i=1}^{l} \mu_i - \bigvee_{i=l+1}^{k} \mu_i = \bigvee_{i=1}^{l'} \nu_i - \bigvee_{i=l'+1}^{k'} \nu_i$$

Consequently, $\Phi(f) = \Phi(g)$.

Hence the mapping $\Phi : h(\hat{E}) \to H$ is defined unambigously by (13). The formula (13) now implies easily that Φ is a vector lattice homomorphism.

Let $\hat{T}$ be the subset of $\hat{E}$ and $\hat{S}$ the σ-algebra of subsets of $\hat{T}$ as in Theorem 1 ; this theorem is applicable since $\hat{E}$ is a weak complete space. It means that every conical measure u on $\hat{E}$ is represented as

$$u(f) = \int_{\hat{T}} f(t)\, d\lambda(t), f \in h(\hat{E}), \tag{14}$$

where λ is a measure on $\hat{S}$, a direct sum of finite measures. We say that λ represents u.

THEOREM 8. – *Let* $u(f) = \Phi(f)(T)$, *for every* $f \in h(\hat{E})$. *Then* u *is a conical measure on* $\hat{E}$; *let* λ *be the measure on* $\hat{S}$ *representing* u.

The mapping $f \mid \hat{T} \to \Phi(f)$, $f \in h(\hat{E})$, *is a vector lattice isomorphism and an isometry of the dense subset* $h(\hat{E}) \mid \hat{T}$ *of* $L^1(\lambda)$ *onto* H.

There exists an injective mapping $\gamma : R(H) \to \hat{S}(\lambda)$ which is a *Boolean algebra isomorphism of* R (H) *onto* $\gamma(R(H))$ *and also a homeomorphism of* R (H) *in its* τ (H)-*topology onto* $\gamma(R(H))$ *in the relative* $\tau(\lambda)$-*topology and* $\gamma(R(H))$ *is* $\tau(\lambda)$-*dense in* $\hat{S}(\lambda)$.

Proof. – Clearly, u is a conical measure.

Given $X \in R$, let

$$v_X(f) = \Phi(f)(X), f \in h(\hat{E}).$$

Then v_X is a conical measure on $\hat{E}$ and $v_X \leqslant u = v_T$. Let λ_X be the measure on $\hat{S}$ which represents v_X. Then $\lambda_X \leqslant \lambda = \lambda_T$. Since λ is a direct sum of finite measures, the Radon-Nikodym theorem is applicable [10, Theorem 5.1]. Hence there exists an $\hat{S}$-measurable function g_X on $\hat{T}$ such that $0 \leqslant g_X \leqslant 1$,

$$\lambda_X(\hat{X}) = \int_X g_X \, d\lambda$$

for every $\hat{X} \in \hat{S}$, and

$$v_X(f) = \int_{\hat{T}} f(t) \, g_X(t) \, d\lambda(t).$$

for every $f \in h(\hat{E})$. The relations $v_{T-X} + v_X = u$ and $v_X \wedge v_{T-X} = 0$ imply that $g_{T-X}(t) + g_X(t) = 1$ and $\min \{g_{T-X}(t), g_X(t)\} = 0$, λ-almost everywhere on $\hat{T}$. It follows that g_X is λ-equivalent to the characteristic function of a set in $\hat{S}$; denote it and also its λ-equivalence class by $\gamma(X)$. It is clear that $\gamma(X)$ depends on the H-equivalence class $[X]_H$ only and not on its individual representative X. Hence γ is well-defined on R (H).

Accepting these conventions it is clear that $\gamma : R(H) \to \hat{S}(\lambda)$ is injective and that it is a Boolean algebra isomorphism of R (H) onto $\gamma(R(H))$.

These definitions also give that

$$\mu(X) = \Phi(\mu^*)(X) = v_X(\mu^*) = \int_T \mu^*(t) \, g_X(t) \, d\lambda(t) =$$

$$= \int_{\gamma(X)} \mu^*(t) \, d\lambda(t),$$

for every $\mu \in \mathrm{H}$ and $\mathrm{X} \in R$. More generally,

$$\Phi(f)(\mathrm{X}) = \int_{\gamma(\mathrm{X})} f(t)\, d\lambda(t),$$

for every $f \in h(\hat{\mathrm{E}})$ and $\mathrm{X} \in R$. Since Φ is a vector lattice isomorphism, the variation $|\Phi(f)|$ of $\Phi(f)$ is equal to $\Phi(|f|)$, where $|f|$ is the absolute value of f. It then follows that

$$\|\Phi(f)\| = |\Phi(f)|(\mathrm{T}) = \Phi(|f|)(\mathrm{T}) = \int_{\gamma(\mathrm{T})} |f(t)|\, d\lambda(t) = \\ = \int_{\hat{\mathrm{T}}} |f|\hat{\mathrm{T}}|\, d\lambda = \|f \mid \hat{\mathrm{T}}\|_1.$$

That is, the norm of $\Phi(f)$ as an element of ba (R) is equal to the L^1-norm of $f \mid \hat{\mathrm{T}}$. This establishes the isometry of $h(\hat{\mathrm{E}}) \mid \hat{\mathrm{T}}$ and H.

Moreover, since

$$\|\mu\| = |\mu|(\mathrm{T}) = \sup\{|\mu(\mathrm{X})| + |\mu(\mathrm{Y})| : \mathrm{X}, \mathrm{Y} \in R, \mathrm{X} \cap \mathrm{Y} = \phi\},$$

we have also

$$\|\varphi\|_1 = \sup\{|\int_{\gamma(\mathrm{X})} \varphi\, d\lambda| + |\int_{\gamma(\mathrm{Y})} \varphi\, d\lambda| : \mathrm{X}, \mathrm{Y} \in R, \mathrm{X} \cap \mathrm{Y} = \phi\}, \quad (15)$$

for every $\varphi \in h(\hat{\mathrm{E}}) \mid \hat{\mathrm{T}}$. Now, (15) holds also for every $\varphi \in L^1(\lambda)$, since $h(\hat{\mathrm{E}}) \mid \hat{\mathrm{T}}$ is dense in $L^1(\lambda)$. It follows that, if $\varphi \in L^1(\lambda)$ and if the integral of φ vanishes on every set $\gamma(\mathrm{X})$, with $\mathrm{X} \in R$, then φ is λ-equivalent to 0. This implies that $\gamma(R)$ is $\tau(\lambda)$-dense in $\hat{S}(\hat{\lambda})$.

We say that the algebra R separates the points of the space T if, for every $t_1\ t_2$ in T, $t_1 \neq t_2$, there exists $\mathrm{X} \in R$ such that $t_1 \in \mathrm{X}$ and $t_2 \notin \mathrm{X}$.

There is no loss of generality in assuming that R indeed separates points of T. If R does not separate points of T we introduce the equivalence relation in T by declaring $t_1 \equiv t_2$ if and only if $t_1 \in \mathrm{X}$, implies $t_2 \in \mathrm{X}$, for every $\mathrm{X} \in R$. Then, if we replace T by so obtained set $\widetilde{\mathrm{T}}$ of equivalence classes and R by its image $\widetilde{R}$ under the natural mapping of T into $\widetilde{\mathrm{T}}$, we obtain a situation where the algebra separates the points of the space $\widetilde{\mathrm{T}}$ and ba $(\widetilde{R})$ coincides with ba (R) both as a vector lattice and as a Banach space.

We say that the vector lattice $\mathrm{H} \subset$ ba (R) separates the algebra R, if the equivalence classes $[\mathrm{X}]_{\mathrm{H}}$, $\mathrm{X} \in R$, are reduced to individual sets ; i.e. there is not another element in R but X which is equivalent to X, for every $X \in R$. In this case $R(H) = R$ as sets and also as Boolean algebras.

In particular, H separates R if H contains all Dirac measures carried by points of T.

COROLLARY 9. – *If the algebra R separates the points of the space* T *and if the vector lattice* H *separates R, then there exists an injection* $\delta : T \to \hat{T}$ *such that* $\delta(X) = \gamma(X) \cap \delta(T)$, *for every* $X \in R$.

Proof. – Since λ is a direct sum of finite measures, $\hat{S}(\lambda)$ is a complete Boolean algebra. For every $t \in T$, let

$$[\hat{X}_t]_\lambda = \bigcup_{t \in X \in R} \gamma(X)$$

The element $[\hat{X}_t]_\lambda$ is not equal to $[\phi]_\lambda$. To prove that it suffice to notice that, since λ is a direct sum of finite measures, the dual of the Banach space $L^1(\lambda)$ is $L^\infty(\lambda)$; further each $\gamma(X)$ represents an element of the unit ball in $L^\infty(\lambda)$, so it suffices to use the weak-star compactness of the unit ball and notice that the finite intersections are all of norm 1.

Since H separates R and R separates T, the elements $[\hat{X}_{t_1}]_\lambda$ and $[X_{t_2}]_\lambda$ are disjoint in $S(\lambda)$ if $t_1 \neq t_2$. Now we use again the fact that λ is a direct sum of finite measure to deduce the existence of lifting in $L^\infty(\lambda)$ or in $\hat{S}(\lambda)$ and, hence, the possibility to choose the representations $\hat{X}_t$ of $[\hat{X}_t]_\lambda$ in such a way that they are disjoint as sets (have no points in common) if they correspond to different elements of T.

Let $\delta(t)$ be an arbitrary element of $\hat{X}_t$, for each $t \in T$. Then the mapping $\delta : T \to \hat{T}$ has the claimed properties.

COROLLARY 10. – *If $R(H)$ is $\tau(H)$-complete then $\gamma : R(H) \to \hat{S}(\lambda)$ is a Boolean algebra isomorphism of $R(H)$ onto $\hat{S}(\lambda)$ and, hence, $R(H)$ is a complete Boolean algebra.*

3. Representation of vector measures.

Let E be a locally convex topological vector space.

Let T be an abstract space and let R be an algebra of subsets of T.

An additive and bounded mapping $m : R \to E$ is termed a *pre-measure.* The boundedness means that the range

$$m(R) = \{m(X) : X \in R\}$$

is a bounded subset of E.

If R is σ-algebra and m is σ-additive then it is termed a vector measure (shortly a measure). In this case the boundedness is a consequence of other assumptions.

Given a pre-measure $m : R \to E$, declare two elements X_1 and X_2 of R to be m-equivalent if $m(X) = 0$ for every $X \subset X_1 \Delta X_2$, $X \in R$. The m-equivalence class of an element $X \in R$ is denoted by $[X]_m$ whenever the distinction between a set and its equivalence class is essential. The family of all m-equivalence classes is denoted by $R(m)$. Again, $R(m)$ is considered a Boolean algebra equal to R modulo the ideal of sets m-equivalent to ϕ.

Let U be a convex symmetric neighborhood of 0 in E and let U° be its polar. Define the pseudo-metric d_U on R and on $R(m)$ by putting

$$d_U(X, Y) = \sup \{|x' \circ m| (X \Delta Y) : x' \in U^\circ\}.$$

for every X , Y in R.

The system of pseudo-metrics d_U, for all symmetric convex neighbourhoods U, defines a topology and a uniform structure on R and on $R(m)$; this topology and the uniformity will be denoted by $\tau(m)$. On $R(m)$, $\tau(m)$ is separated (Hausdorff).

If $R(m)$ is a complete space in the uniform structure $\tau(m)$, the pre-measure m is termed *closed* [5].

Let S be σ-algebra of subsets of T and let λ be a non-negative σ-additive mesure on S (admitting also value ∞). A function $f : T \to E$ is said to be integrable if, for every $x' \in E'$, the real-valued function $x' \circ f$ is integrable and if, for every $X \in S$, there exists an element $x_X \in E$ such that

$$x'(x_X) = \int_X x' \circ f \, d\lambda = \lambda_X (x' \circ f).$$

We write

$$x_X = \int_X f \, d\lambda = \lambda_X (f), \; X \in S.$$

Also $\lambda_T (f) = \lambda (f)$.

It is clear that if E is weakly complete then f is integrable if and only if $x' \circ f$ is integrable for every $x' \in E'$.

If $f : T \to E$ is integrable with respect to λ, the mapping $m : S \to E$ defined by $m(X) = \lambda_X (f)$, $X \in S$, is a vector measure. This is in virtue of the Orlicz-Pettis lemma. This measure m is called the indefinite integral of f with respect to λ.

LEMMA 11. – *Let* T *be a space,* S *σ-algebra of subsets of* T *and* λ *a measure on* S. *Let* E *be a locally convex topological vector space with the weak completion* E'^*. *Let* $f : T \to E'^*$ *be a* λ*-integrable function such that* $\lambda_X (f) \in E$, *for every* $X \in S$. *Let m be the indefinite integral of f with respect to* λ. *Then the topology* $\tau(m)$ *is weaker than* $\tau(\lambda)$. *If* λ *is localizable then m is a closed vector measure.*

Proof. – To show that $\tau(m)$ is a weaker topology than $\tau(\lambda)$ amounts to showing that if $\lambda_X (|\varphi|) \to 0$, $X \in S$, for every $\varphi \in L^1(\lambda)$, then $x' \circ m(X) \to 0$ uniformly for $x' \in U^\circ$ whenever U is a convex symmetric neigbourhood of 0 in E.

If U is such a neighbourhood, there exists, by the Rybakov's theorem [9], an element $x'_0 \in U^\circ$ such that the measures $x' \circ m$, $x' \in U^\circ$, are uniformly absolutely continuous with respect to $x'_0 \circ m$. Hence, if $\lambda_X (|x'_0 \circ f|) \to 0$, $X \in S$, then $x' \circ m(X) \to 0$ uniformly with respect to $x' \in U^\circ$.

The statement that m is a closed measure if λ is localizable is the content of Theorem IV.7.3 in [8]. An alternative proof can be obtained as follows.

Take a $\tau(m)$-Cauchy net X_i in S. It represents a net of measurable functions, the characteristic functions of the sets X_i, with values in [0,1] ; in fact, with values only 0 and 1. The set of measurable functions φ with values in [0,1] (more precisely, m-equivalence classes of such functions) is a compact space in the topology, let us denote it σ, making all applications $\varphi \to x' \circ m(\varphi)$ for all $x' \in E'$, continuous; this is because σ is a weaker topology than weak-star topology on $L^\infty(\lambda)$ and λ is localizable, so $L^\infty(\lambda)$ is the dual of $L^1(\lambda)$. Now there exists a function φ with values in [0,1] to which a subnet of characteristic functions of X_i converges in σ. Since $\tau(m)$ is a stronger topology and X_i are $\tau(m)$-Cauchy, they converge in $\tau(m)$ to a function m-equivalent to φ. It must be equivalent to a characteristic function.

THEOREM 12. – *Let* T *be a space, R an algebra of subsets of* T, E *a locally convex topological vector space,* E'^* *its weak completion, and $m : R \to E$ a pre-measure.*

Then there exists a space $\hat{T}$, *a σ-algebra* $\hat{S}$ *of subsets of* $\hat{T}$, *a measure λ on* $\hat{S}$ *which is a direct sum of finite measures and a λ-integrable function* $f : \hat{T} \to E'^*$ *whose indefinite integral is denoted* $\hat{m}$, *with the following properties :*

i) *there is an injection* $\delta : T \to \hat{T}$ *and an injection* $\gamma : R \to \hat{S}$ *such that* $\delta(X) = \{\delta(t) : t \in X\} = \gamma(X) \cap \delta(T)$, *for every* $X \in R$;

ii) $\hat{m}(\gamma(X)) = m(X)$, *for every* $X \in R$;

iii) *the injection γ is a continuous Boolean algebra isomorphism of R with its $\tau(m)$-topology onto* $\gamma(R) = \{\gamma(X) : X \in R\}$ *with its $\tau(\lambda)$-topology and $\gamma(R)$ is $\tau(\hat{m})$-dense in S* ;

iv) $\hat{m}(\hat{S}) = \{\hat{m}(\hat{X}) : \hat{X} \in \hat{S}\}$ *is in the* $\sigma(E'^*, E')$*-closure of* $m(R)$.

Proof. – Let $H \subset ba(R)$ be the least vector lattice containing all measures $x' \circ m$, for $x' \in E'$, and all Dirac measures carried by point of T. Define $\hat{T}$, $\hat{S}$ and λ as in Theorem 8.

Define $\hat{m}(\gamma(X)) = m(X)$, for $X \in R$. Giving $\hat{S}$ its topology $\tau(\lambda)$, $\gamma(R)$ is dense in $\hat{S}$ according to Theorem 8. Also $\hat{m} : \gamma(R) \to E$ is a uniformly continuous mapping if the topology $\sigma(E, E')$ on E is considered. Hence there exists a unique mapping $\hat{m} : \hat{S} \to E'^*$ such that $\hat{m}(\gamma(X)) = m(X)$, $X \in R$, which is continuous if on $\hat{S}$ the topology $\tau(\lambda)$ and on E'^* the topology $\sigma(E'^*, E')$ are considered. The statement (iv) follows.

Clearly, $\hat{m}$ is additive on $\gamma(R)$, hence, by density, it is additive on $\hat{S}$. Its continuity in $\tau(\lambda)$ gives that it is σ-additive under the topology $\sigma(E'^*, E)$. The continuity gives, further, that $x' \circ m$ is absolutely continuous with respect to λ, for every $x' \in E'$. Let $f_{x'}$ be the Randon-Nikodym derivative of $x' \circ m$ with respect to λ, $x' \in E$; notice that λ is a direct sum of finite measures, hence the Radon-Nikodym derivative exists. For every $t \in \hat{T}$, let $f(t) \in E'^*$ be the vector such that $x'(f(t)) = f(t)(x') = f_{x'}(t)$, $x' \in E'$.Then f is integrable and $\hat{m}$ is its indefinite integral.

By Lemma 11, $\tau(\hat{m})$ is a weaker topology on $\hat{S}$ than $\tau(\lambda)$, hence $\gamma(R)$ is also $\tau(\hat{m})$-dense in $\hat{S}$.

COROLLARY 13. – *Let* T *be a space and* S *a* σ*-algebra of subsets of* T. *A vector measure* $m : S \to$ E *is closed if and only if there exists a localizable measure* λ *on* S *and a* λ*-integrable function* f : T $\to$ E'^* *such that* m *is the indefinite integral of* f *with respect to* λ.

Proof. – If such λ and f exist, them m is closed by Lemma 11.

Conversely, if m is closed then $S(m)$ is $\tau(m)$-complete and hence $S(m) = \hat{S}(\hat{m})$, i.e. $\hat{\gamma}$ is a Boolean algebra isomorphism of $S(m)$ onto the whole of $\hat{S}(\hat{m})$. Clearly, the measure λ induces a σ-additive function, denoted by λ again, on $\hat{S}(\hat{m})$. Then $\lambda \circ \gamma^{-1}$ defines a σ-additive function on $S(m)$ which induces a localizable measure on S.

The same method as used for proving Theorem 12 can give more precise results if an extra information is available. As an illustration we formulate some such results in form of two corollaries ; these are, perhaps, corollaries to the proof rather than to theorem itself.

Denote by $\tilde{E}$ the quasi-completion of the space E. Since the space $\tilde{E}$ can be substantially smaller than the weak completion E'^*, the result in the next corollary is really more precise than the Theorem 12.

COROLLARY 14. – *If* R *is a* σ*-algebra and if* m *is* σ*-additive then the values of* $\hat{m}$ *are in the closure of* $m(R)$ *in the space* $\tilde{E}$.

Proof. – Since $\gamma(R)$ is $\tau(\lambda)$-dense in $\hat{S}$, it suffices to prove that $\hat{m} : \gamma(R) \to E$ is uniformly continuous with respect to $\tau(\lambda)$ on $\gamma(R)$. This follows from Rybakov's theorem. In fact, given any symmetric convex neighbourhood U of 0 in E, the Rybakov's theorem [9] guarantees the existence of an element $x'_0 \in U^\circ$ such that $x' \circ m(X) \to 0$ uniformly with respect to $x' \in U^\circ$, whenever $x'_0 \circ m(X) \to 0$, $X \in R$. The uniform continuity of $\hat{m}$ readily follows.

This corollary was stated in form of Theorem 3 in [5]. Possibly the proof there was less economical.

The following corollary implies many theorems concerning the extension of a σ-additive measure from an algebra onto the generated σ-algebra. A more systematic account of such problems can be found in [6]. It is worth noticing, however, that the family H′ in the condition (c) below consists of finitely-additive measures. That the replacement of countably additive measures by finitely additive ones is possible was pointed out for the first time by Uhl in [11]; he deals with the case when $E = \tilde{E}$ is a Banach space and then H′ is reduced to a single finitely additive measure.

COROLLARY 15. – *Assume that the pre-measure m is σ-additive on the algebra R. Then the following conditions are equivalent to each other.*

(a) *The values of $\hat{m}$ belong to* $\tilde{E}$.

(b) *For every symmetric convex neighbourhood* U *of 0 on* E, *there exists* $\mu_U \in ca(R)$ *such that* $x' \circ m(X) \to 0$ *uniformly with respect to* $x' \in U^\circ$ *whenever* $\mu_U(X) \to 0$, $X \in R$.

(c) *There exists a family* $H' \subset ba(R)$ *such that m is uniformly continuous on R if R is equipped with the topology* $\tau(H')$.

(d) *The values of $\hat{m}$ belong to the closure of $m(R)$ in* $\tilde{E}$.

(e) *$m(R)$ is contained in a weakly compact subset of* $\tilde{E}$.

Proof. – (a) implies (b) by Rybakov's theorem ; (b) implies (c) trivially.

Assuming (c) to hold we choose for H in the proof of Theorem 12 to be the vector lattice generated by H′ and by all Dirac measures on R. Then m will become uniformly continuous on $\gamma(R)$ in the topology $\tau(\lambda)$ and so its extension on $\hat{S}$, $\tau(\lambda)$-closure of $\gamma(R)$, will have values in the closure of $m(R)$. This establishes (d).

The $\sigma(E'^*, E')$-closed convex hull K of $\hat{m}(\hat{S})$ in E'^* is $\sigma(E'^*, E')$-compact. Now assuming (d), we have $K \subset \widetilde{E}$ and so K is weakly compact ; i.e. (e) holds. Finally (e) implies (a) trivially.

4. Conical measure associated with a vector measure.

Let E be a locally convex topological vector space and E'^* its weak completion.

If u is a conical measure on E'^*, by $r(u)$ is denoted the element $x \in E'^*$ such that $u(x') = x'(x) = x(x')$, for every $x' \in E'$. The element $r(u)$ is called the resultant of u. Since E'^* is a weak complete space it, obviously, exists.

Let now u be a conical measure on E'^*. By K_u is denoted the set of all resultants $r(v)$, for all conical measures v on E'^* such that $v \leqslant u$ (see [3] and [4, Chapter 9]). A set $K \subset E'^*$, such that there exists a conical measure u on E'^* with $K = K_u$, is termed a *zonoform.*

Let T be an abstract space, S a σ-algebra of its subsets and $m : S \to E$ a closed vector measure. By Corollary 13 to Theorem 12, there exist a localizable measure λ on S and a λ-integrable function $f : T \to E'^*$ such that m is the indefinite integral of f with respect to λ.

Define

$$u(\varphi) = \int_T \varphi \circ f \, d\lambda, \tag{16}$$

for every $\varphi \in h(E'^*)$.

LEMMA 16. – *The functional u defined by* (16) *is a conical measure on* E'^* *which depends on the vector measure m only and not on the choice of f and* λ.

Proof. – Since λ is localizable the Radon-Nikodym theorem is available [10, Theorem 5.1]. The independence of u on f and λ is a simple consequence. Clearly, u is a conical measure.

The conical measure defined by (16) is denoted by $u = \Delta(m)$.

LEMMA 17. – *If $m : S \to E$ is a closed vector measure and $u = \Delta(m)$ then $K_u = \overline{\text{co}}\, m(S)$.*

If u is a conical measure on E'^* such that $K_u \subset E$, then there exists a space T, a σ-algebra S of subsets of T and a closed vector measure $m : S \to E$ such that $m = \Delta(m)$. In fact if S is a σ-algebra of subsets of E'^* and λ a localizable measure on S representing u and $m : S \to E'^*$ is defined by

$$m(X) = \int_X x \, d\lambda(x), \; X \in S, \tag{17}$$

then $m(X) \in E$, for $X \in S$, and $u = \Delta(m)$.

Proof. – Let $X \in S$. Define

$$v_X(\varphi) = \int_X \varphi \circ f \, d\lambda, \; \varphi \in h(E'^*).$$

Then v_X is a conical measure on E'^* and $v_X \leqslant u$. Moreover, $r(v_X) = m(X)$. Hence $m(S) \subset K_u$. As K_u is a convex weakly compact set, [3] or [4, Theorem 38.2], $\overline{\text{co}}\, m(S) \subset K_u$.

Conversely, let v be a conical measure on E'^* and $v \leqslant u$. Then, by the Radon-Nikodym theorem, there exists an S-measurable function g on T such that

$$v(\varphi) = \int_T g\, \varphi \circ f \, d\lambda, \; \varphi \in h(E'^*),$$

and $0 \leqslant g \leqslant 1$. Then it follows easily that

$$r(v) = \int_T g f \, d\lambda \in \overline{\text{co}}\, m(S).$$

So, $K_u \subset \overline{\text{co}}\, m(S)$.

If the vector measure m is defined by (17) then, clearly, $u = \Delta(m)$.

Before stating the last theorem, let us remind that a point x_0 is called an exposed point of a set $K \subset E'^*$ if $x_0 \in K$ and if there exists $x' \in E'$ such that $x'(x) < x'(x_0)$ whenever $x \in K$ and $x \neq x_0$.

THEOREM 18. – *Let u be a conical measure on E'^*. Each of the following conditions is sufficient for the existence of a finite*

σ-additive measure λ on the cylindrical σ-algebra $C(\mathrm{E}'^*)$ which represents u :

(a) *The zonoform K_u has an exposed point.*

(b) $\mathrm{K}_u \subset \mathrm{E}$ *and* E *admits a metrizable topology consistent with the dualitv of* E *and* E'.

Proof. – Let T be a space, S a σ-algebra of its subsets and $m : S \to \mathrm{E}'^*$ a vector measure such that $u = \Delta(m)$ and, hence, $\mathrm{K}_u = \overline{\mathrm{co}}\ m(S)$. These objects do exist according to Lemma 17.

It was first observed by Anantharaman in [1] that if $\overline{\mathrm{co}}\ m(S)$ has an exposed point and if x' is the exposing functional, then m is absolutely continuous with respect to the finite positive σ-additive measure $\mu = |x' \circ m|$ (see also [8, Theorem VI.3.1] for a formulation in a generality required here). Let $f : \mathrm{T} \to \mathrm{E}'^*$ the function such that m is the indefinite integral of f with respect to μ. We put $\lambda = \mu \circ f^{-1}$ to obtain a measure λ on $C(\mathrm{E}'^*)$ representing μ.

If $\mathrm{K}_u = \overline{\mathrm{co}}\, m(S) \subset \mathrm{E}$ and E is metrizable then there exists a finite measure μ on S such that m is absolutely continous with respect to μ. This is only a slight and known generalization of the classical result of Bartle, Dunford and Schwartz in [2] stated there for E normable. The rest of the proof follows as in case (a).

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