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## EXOTIC CHARACTERISTIC CLASSES AND SUBFOLIATIONS

by Luis A. CORDERO and P.M. GADEA (\*)

### 1. Introduction.

The development in the last years of the study of topological invariants associated to a foliated structure on a differentiable manifold(\*\*) (usually called exotic characteristic classes of the foliation) has been well known.

Within the general context of this study, the following problem appears in a canonical way : let  $M$  be a differentiable manifold on which two foliations  $F_1$  and  $F_2$  are defined, and such that  $F_1 \subset F_2$ , that is, every leaf of  $F_2$  is, itself, foliated by leaves of  $F_1$  ; briefly,  $F_1$  is said to be subfoliation of  $F_2$  ; in fact, this geometrical structure on  $M$  can be described as a special type of multifoliate structure (in the sense of Kodaira-Spencer ({8})) ; now, we present two questions : 1) does a relation exist between exotic classes of  $F_1$  and  $F_2$  ?, and 2) : is it possible to give a topological obstruction to the existence of such a structure on  $M$  ?.

In this paper we give the answer to these questions, by studying the problem through a more general situation and using Lehmann's techniques ({9}, {10}). For this purpose, we consider the following situation : let  $Q_i, i = 1, 2$ , be two  $G_i$ -principal fibre bundles over  $M$ , and let  $\Pi : Q_1 \rightarrow Q_2$  be a morphism of principal fibre bundles (over the identity of  $M$ ) ; by an appropriate choice of connections on these fibre bundles we point out a relation between the images of Lehmann's exoticism associated to those connections (Theorem 4.5) ; in the special case of  $F_1$  and  $F_2$ , two foliations as above, that relation gives the answer to our questions : every exotic characteristic class of  $F_2$  is also an

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(\*\* ) Always manifolds will mean paracompact differentiable manifold of class  $C^\infty$ .

exotic characteristic class of  $F_1$  ; in fact, this result can be expressed as a topological obstruction  $F_1$  to be a subfoliation of  $F_2$ .

## 2. Notations and basic concepts.

Let  $M$  be a differentiable manifold. We shall denote  $\mathfrak{X}(M)$  the Lie algebra of vector fields and  $A^*(M)$  the exterior algebra of differential forms on  $M$ .

Given a  $G$ -principal fibre bundle  $E \rightarrow M$ ,  $G$  being the structural Lie group,  $\omega$  indistinctly denotes an (infinitesimal) connection on the bundle or the 1-form of that connection ;  $I(G)$  is the algebra of invariant polynomials over the Lie algebra  $\underline{G}$  of  $G$  ;  $I(G)$  is a graded algebra,  $I(G) = \bigoplus_{k \geq 0} I^k(G)$  and  $I^+(G)$  denotes its maximal ideal

$$I^+(G) = \bigoplus_{k \geq 1} I^k(G).$$

Denote by  $\lambda_\omega : I(G) \rightarrow A^*(M)$  the Chern-Weil homomorphism, defined by  $\lambda_\omega(f) = f(\Omega)$ , for  $f \in I(G)$  and  $\Omega$  being the curvature form of  $\omega$ . If  $I = [0, 1]$  is the unit interval,  $\int_0^1 : A^k(M \times I) \rightarrow A^{k-1}(M)$  denotes the integration along the fibre of  $M \times I \rightarrow M$ . If  $\omega'$  is another connection on  $E$ , we write  $[\overrightarrow{\omega}, \overrightarrow{\omega}']$  the connection on  $E \times I \rightarrow M \times I$  defined by

$$[\overrightarrow{\omega}, \overrightarrow{\omega}'] \left( \frac{\partial}{\partial t} \right) = 0, [\overrightarrow{\omega}, \overrightarrow{\omega}']|_{E \times \{t\}} = t \omega' + (1-t) \omega$$

and by  $\Delta_{\omega, \omega'} : I^k(G) \rightarrow A^{2k-1}(M)$  the composition  $\int_0^1 \cdot \lambda_{[\overrightarrow{\omega}, \overrightarrow{\omega}']}$ .

As it is well known,  $\lambda_\omega$  induces an homomorphism  $\lambda :$

$$I(G) \rightarrow H^{\text{even}}(M, \mathbb{R})$$

which is independent of  $\omega$ .

Let  $J \subset I(G)$  be a homogeneous ideal ;  $\omega$  is said a  $J$ -connection if  $\lambda_\omega(f) = 0$  for every  $f \in J$ . If  $P$  denotes a property of the degree of elements of  $I(G)$ ,  $J(P)$  denotes the homogeneous ideal generated by the elements satisfying the property  $P$ . For example, if  $\dim M = n$ , every connection on  $E$  is a  $J \left( > \left[ \frac{n}{2} \right] \right)$ -connection.

If  $G = \text{Gl}(q, \mathbb{R})$ , it is  $I(G) = \mathbb{R}[c_1, \dots, c_q]$ , where  $c_1, \dots, c_q$  are the usual generators given by

$$\det(I + tA) = 1 + \sum_{i=1}^q c_i(A)t^i, \text{ for every } A \in \text{gl}(q, \mathbb{R})$$

If  $Q \rightarrow M$  is a vector bundle,  $\nabla$  denotes the derivation law of a linear connection on  $Q$ . Thus, every metric connection on  $Q$  is a  $J(\text{odd})$ -connection.

If  $Q \rightarrow M$  is the normal bundle of a foliation on  $M$ , of codimension  $q$ , and  $\nabla$  is a basic connection on  $Q$  (in the sense of Bott ( $\{1\}$ )), then  $\nabla$  is a  $J(> q)$ -connection.

**3. The Lehmann's exoticism ( $\{9\}$ ), ( $\{10\}$ )).**

Let  $E$  be a  $G$ -principal fibre bundle on  $M$ . Consider  $J, J'$  homogeneous ideals of  $I(G)$ ; if  $f \in I^k(G)$ , we write

$$\bar{f} = f \pmod{J}, \quad \overline{\overline{f}} = f \pmod{J'}$$

and introduce a graduation on the quotient algebras  $I(G)/J, I(G)/J'$  by  $\deg \bar{f} = \deg \overline{\overline{f}} = 2k$ , for every  $f \in I^k(G)$ ; also, we shall denote  $\Lambda(I^+(G))$  the exterior algebra over  $\mathbb{R}$  generated by the elements of  $I^+(G)$  and define a graduation on  $\Lambda(I^+(G))$  by  $\deg f = 2k - 1$ , for every  $f \in I^k(G), k > 0$ . Then, consider the graded algebra

$$\hat{W}(J, J') = I(G)/J \otimes_{\mathbb{R}} I(G)/J' \otimes_{\mathbb{R}} \Lambda(I^+(G))$$

and  $I(G)/J, I(G)/J', \Lambda(I^+(G))$  are canonically identified to subalgebras of  $\hat{W}(J, J')$ ;  $I^+(G)$  can be identified to one part of  $\Lambda(I^+(G)) \subset \hat{W}(J, J')$  by the isomorphism

$$h : I^+(G) \rightarrow \Lambda^1(I^+(G))$$

and, if  $G = \text{Gl}(q, \mathbb{R})$ , we write  $h_i = h(c_i)$ .

$\hat{W}(J, J')$  is endowed with a structure of graded differential algebra by defining a differential (of degree 1)

$$\begin{aligned} d(\bar{f}) &= d(\overline{\overline{f}}) = 0, \quad \text{for } f \in I(G) \\ d(f) &= \bar{f} - \overline{\overline{f}}, \quad \text{for } f \in I^+(G) \end{aligned}$$

and, clearly,  $d^2 = 0$ .

If  $\omega$  is a J-connection and  $\omega'$  is a  $J'$ -connection on E, a homomorphism of graded algebras  $\rho_{\omega\omega'} : \hat{W}(J, J') \rightarrow A^*(M)$  is defined by

$$\rho_{\omega\omega'}(\bar{f}) = \lambda_{\omega}(f)$$

$$\rho_{\omega\omega'}(\bar{\bar{f}}) = \lambda_{\omega'}(f)$$

$$\rho_{\omega\omega'}(f_1 \wedge \dots \wedge f_r) = \Delta_{\omega, \omega'}(f_1) \wedge \dots \wedge \Delta_{\omega, \omega'}(f_r), \text{ for } f_i \in I^+(G)$$

and, in cohomology,  $\rho_{\omega\omega'}$  induces a homomorphism of graded algebras

$$\rho_{\omega\omega'}^* : H^*(\hat{W}(J, J')) \rightarrow H^*(M, \mathbb{R})$$

The elements of  $\text{Im } \rho_{\omega\omega'}^*$  are said to be the exotic characteristic classes associated to J,  $J'$ ,  $\omega$  and  $\omega'$ .

Let  $J \subset I(G)$  be a homogeneous ideal and  $\omega_0, \omega_1$  two J-connections on E ;  $\omega_0$  and  $\omega_1$  are said to be differentiably J-homotopic if there does exist a J-connection  $\tilde{\omega}$  on  $E \times I \rightarrow M \times I$  such that

$$\tilde{\omega}|_{E \times \{0\}} = \omega_0, \quad \tilde{\omega}|_{E \times \{1\}} = \omega_1$$

and, in a more general form,  $\omega_0$  and  $\omega_1$  are said to be J-homotopic if there does exist a finite sequence  $\omega_0 = \omega_{s_0}, \omega_{s_1}, \dots, \omega_{s_k} = \omega_1$  of J-connections such that, for every  $i = 0, 1, \dots, k-1$ ,  $\omega_{s_i}$  and  $\omega_{s_{i+1}}$  are differentiably J-homotopic. A set C of connections on E is said to be J-connected if it is not-empty and any two connections in C are J-homotopic.

**PROPOSITION 3.1.** — *Im  $\rho_{\omega\omega'}^*$  depends only on the J-connected component of  $\omega$  and the  $J'$ -connected component of  $\omega'$ .*

In particular, if C is the set of basic connections on the transversal bundle Q of a q-codimensional foliation on M and  $C'$  is the set of metric connections on Q, Lehmann shows that C is  $J(> q)$ -connected and  $C'$  is J(odd)-connected ; moreover, in this case  $\hat{W}(J(> q), J(\text{odd}))$  has the same cohomology that its subalgebra

$$WO_q = \mathbb{R} [c_1, \dots, c_q] / J(> q) \otimes_{\mathbb{R}} \Lambda(h_1, h_3, \dots, h_{(q)})$$

where  $(q)$  denotes the largest odd integer  $\leq q$  and  $h_i = h(c_i)$ . Therefore

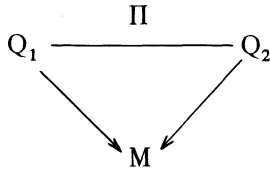
PROPOSITION 3.2. — *The homomorphism  $\rho_{\nabla\nabla'}^*$  :*

$$H^*(W O_q) \rightarrow H^*(M, \mathbb{R})$$

*does not depend on the choice of  $\nabla \in C$  and  $\nabla' \in C'$ .*

**4. Homomorphism of principal fibre bundles and the Lehmann's exoticism.**

In this paragraph we shall consider the following situation : let  $Q_i \rightarrow M$  be a  $G_i$ -principal fibre bundle ( $i = 1,2$ ) and let



a homomorphism of principal fibre bundles ; also, denote

$$\Pi : G_1 \longrightarrow G_2$$

the corresponding homomorphism of Lie groups and assume that  $\Pi$  is surjective but not a submersion in general, e.g.

$$d\Pi : \underline{G}_1 \longrightarrow \underline{G}_2$$

is not of maximal rank in general ; the linear mapping  $d\Pi$  permits to define :

DEFINITION 4.1. — *If  $f \in I^k(G_2)$ ,  $i(f)$  is defined by*

$$i(f) (X_1 \otimes \dots \otimes X_k) = f(d\Pi(X_1) \otimes \dots \otimes d\Pi(X_k)),$$

*for every  $X_j \in \underline{G}_1$ ,  $j = 1,2,\dots,k$*

A direct application of this definition shows

PROPOSITION 4.2. — *For every  $f \in I(G_2)$ ,  $i(f) \in I(G_1)$  and*

$$i : I(G_2) \rightarrow I(G_1)$$

*is a homomorphism of graded algebras. Moreover, if  $d\Pi$  is of maximal rank, then  $i$  is injective.*

Let  $J_2 \subset I(G_2)$  be an homogeneous ideal and  $J_1$  an arbitrary homogeneous ideal of  $I(G_1)$ , such that  $J_1 \supseteq i(J_2)$  (in particular,  $J_1$  could be thought as the homogeneous ideal generated by the elements of  $i(J_2)$ ).

**THEOREM 4.3.** — *Let  $\omega_1$  be a connection in  $Q_1$ , and  $\Omega_1$  its curvature form. Then :*

- a) there is a unique connection  $\omega_2$  in  $Q_2$  such that the horizontal subspaces of  $\omega_1$  are mapped into horizontal subspaces of  $\omega_2$  by  $\Pi$ .*
- b) if  $\Omega_2$  is the curvature form of  $\omega_2$ , then*

$$\Pi^*\omega_2 = d\Pi \cdot \omega_1$$

$$\Pi^*\Omega_2 = d\Pi \cdot \Omega_1$$

- c) if  $\omega_1$  is a  $J_1$ -connection, then  $\omega_2$  is a  $J_2$ -connection.*

*Proof.* — a) and b) are well-known results (see Kobayashi-Nomizu, vol I ({7}), p. 79).

In order to prove c), we have to show that, if  $f \in J_2$  with  $\deg f = k$ , then  $\lambda_{\omega_2}(f) = 0$ , e.g.

$$f(\Omega_2)(Y_1 \otimes \dots \otimes Y_{2k}) = 0, \text{ for } Y_1, \dots, Y_{2k} \in \mathfrak{X}(Q_2)$$

But it suffices to show that when  $Y_i, i = 1, \dots, 2k$ , is horizontal with respect to  $\omega_2$  and, in this case, there exist  $X_1, \dots, X_{2k} \in \mathfrak{X}(Q_1)$  such that  $d\Pi(X_i) = Y_i$  for every  $i = 1, 2, \dots, 2k$ . But  $i(f) \in J_1$ , then

$$\begin{aligned} 0 &= i(f)(\Omega_1)(X_1 \otimes \dots \otimes X_{2k}) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} i(f)(\Omega_1(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \dots \otimes \Omega_1(X_{\sigma(2k-1)}, X_{\sigma(2k)})) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(d\Pi(\Omega_1(X_{\sigma(1)}, X_{\sigma(2)})) \otimes \dots \otimes d\Pi(\Omega_1(X_{\sigma(2k-1)}, X_{\sigma(2k)}))) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f((\Pi^*\Omega_2)(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \dots \otimes (\Pi^*\Omega_2)(X_{\sigma(2k-1)}, X_{\sigma(2k)})) = \\ &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(\Omega_2(d\Pi(X_{\sigma(1)}), d\Pi(X_{\sigma(2)})) \otimes \dots \otimes \Omega_2(d\Pi(X_{\sigma(2k-1)}), d\Pi(X_{\sigma(2k)}))) = \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(\Omega_2(Y_{\sigma(1)}, Y_{\sigma(2)}) \otimes \dots \otimes \Omega_2(Y_{\sigma(2k-1)}, Y_{\sigma(2k)})) = \\
 &= f(\Omega_2)(Y_1 \otimes \dots \otimes Y_{2k})
 \end{aligned}$$

*Remark.* – Note that, if  $\bar{J}_2$  is another homogeneous ideal of  $I(G_2)$  with  $\bar{J}_2 \supset J_2$ , it might happen that  $\omega_2$  be, in fact, a  $\bar{J}_2$ -connection.

Now, let  $J_2, J'_2$  (respect.  $J_1, J'_1$ ) homogeneous ideales of  $I(G_2)$  (respect.  $I(G_1)$ ) such that

$$J_1 \supseteq i(J_2), J'_1 \supseteq i(J'_2)$$

By virtue of Theorem 4.3, given  $\omega_1$  a  $J_1$ -connection and  $\omega'_1$  a  $J'_1$ -connection, there exist  $\omega_2$  a  $J_2$ -connection and  $\omega'_2$  a  $J'_2$ -connection satisfying the condition b) in the Theorem. Then, consider the graded differential algebras

$$\hat{W}_1(J_1, J'_1) = I(G_1)/J_1 \otimes_{\mathbb{R}} I(G_1)/J'_1 \otimes_{\mathbb{R}} \Lambda(I^+(G_1))$$

$$\hat{W}_2(J_2, J'_2) = I(G_2)/J_2 \otimes_{\mathbb{R}} I(G_2)/J'_2 \otimes_{\mathbb{R}} \Lambda(I^+(G_2))$$

The homomorphism  $i : I(G_2) \rightarrow I(G_1)$  induces canonically a new homomorphism of graded algebras

$$\bar{i} : \hat{W}_2(J_2, J'_2) \rightarrow \hat{W}_1(J_1, J'_1)$$

PROPOSITION 4.4. – *The following diagram is commutative*

$$\begin{array}{ccc}
 \hat{W}_1(J_1, J'_1) & \xleftarrow{\bar{i}} & \hat{W}_2(J_2, J'_2) \\
 \rho_{\omega_1, \omega'_1} \searrow & & \swarrow \rho_{\omega_2, \omega'_2} \\
 & A^*(M) &
 \end{array} \tag{4.1}$$

*Proof.* – It suffices to prove the commutativity for  $\bar{f} = f(\text{mod } J_2)$ ,  $\bar{f}' = f(\text{mod } J'_2)$  with  $f \in I(G_2)$ , and  $\Delta_{\omega_2, \omega'_2} = \Delta_{\omega_1, \omega'_1} \cdot \bar{i}$ .

If  $\bar{i} : I(G_2)/J_2 \rightarrow I(G_1)/J_1$  denotes, once more, the mapping given by  $\bar{i}(\bar{f}) = \bar{i}(f)$ , we have

$$\rho_{\omega_2, \omega'_2}(\bar{f}) = \lambda_{\omega_2}(f) = f(\Omega_2)$$

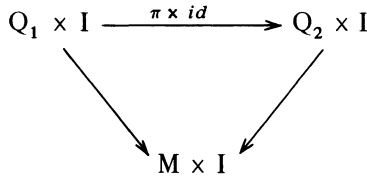


and

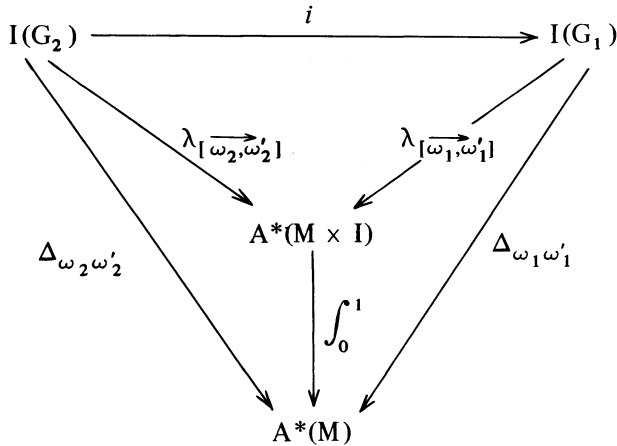
$$\rho_{\omega_1, \omega'_1}(\overline{i(\overline{f})}) = \rho_{\omega_1, \omega'_1}(\overline{i(f)}) = i(f) (\Omega_1)$$

and it is clear that  $f(\Omega_2)$  and  $i(f) (\Omega_1)$  define the same element of  $A^*(M)$ . In a similar way, the commutativity is proved for  $\overline{f}$ .

Now, we consider



where  $\overrightarrow{[\omega_2, \omega'_2]}$  is the unique connection in  $Q_2 \times I$  which might be obtained from  $\overrightarrow{[\omega_1, \omega'_1]}$  in  $Q_1 \times I$  through Theorem 4.3 ; hence, the following diagram is commutative



*Remark.* – If  $\overline{J}_2$  and  $\overline{J}'_2$  are homogeneous ideales of  $I(G_2)$  such that  $\overline{J}_2 \supset J_2, \overline{J}'_2 \supset J'_2$ , and the connections  $\omega_2, \omega'_2$  are not only  $J_2$ - and  $J'_2$ -connections but  $\overline{J}_2$ - and  $\overline{J}'_2$ -connections, respectively, and if

$$\eta : \hat{W}_2(J_2, J'_2) \rightarrow \hat{W}_2(\overline{J}_2, \overline{J}'_2)$$

is the canonical projection, (4.1) can be enlarged to a new commutative diagram

$$\begin{array}{ccc}
 \hat{W}_1(J_1, J'_1) & \xleftarrow{\bar{i}} & \hat{W}_2(J_2, J'_2) \\
 \downarrow \rho_{\omega_1 \omega'_1} & \nearrow \rho_{\omega_2 \omega'_2} & \downarrow \eta \\
 A^*(M) & \xleftarrow{\rho_{\omega_2 \omega'_2}} & \hat{W}_2(\bar{J}_2, \bar{J}'_2)
 \end{array} \tag{4.2}$$

THEOREM 4.5. — *Diagram (4.1) induces, in cohomology, a new commutative diagram*

$$\begin{array}{ccc}
 H^*(\hat{W}_1(J_1, J'_1)) & \xleftarrow{\bar{i}^*} & H^*(\hat{W}_2(J'_2, J_2)) \\
 \searrow \rho_{\omega_1 \omega'_1}^* & & \swarrow \rho_{\omega_2 \omega'_2}^* \\
 & H^*(M, \mathbb{R}) &
 \end{array}$$

Hence

$$\text{Im } \rho_{\omega_2 \omega'_2}^* \subset \text{Im } \rho_{\omega_1 \omega'_1}^* \tag{4.3}$$

Moreover,  $\text{Im } \rho_{\omega_2 \omega'_2}^*$  does not change when  $\omega_1$  (respect.  $\omega'_1$ ) runs over its  $J_1$ -connected component (respect.  $J'_1$ -connected component).

*Proof.* — The commutativity of this diagram is evident from that of (4.1), and this fact implies trivially (4.3).

In order to prove the last assertion, it suffices to show that if  $\omega_1$  (respect.  $\omega'_1$ ) runs over its  $J_1$ -connected (respect.  $J'_1$ -connected) component, then  $\omega_2$  (respect.  $\omega'_2$ ) does it over its  $J_2$ -connected (respect.  $J'_2$ -connected) component.

For that, let  $\bar{\omega}_1$  be a connection in  $Q_1$  differentially  $J_1$ -homotopic to  $\omega_1$  and let  $\bar{\omega}_2$  be the connection in  $Q_2$  corresponding to  $\bar{\omega}_1$  through Theorem 4.3 ;  $\bar{\omega}_2$  is a  $J_2$ -connection. Now, consider the connection  $\tilde{\omega}$  in  $Q_1 \times I \rightarrow M \times I$  which defines the  $J_1$ -homotopy between  $\omega_1$  and  $\bar{\omega}_1$  ;  $\tilde{\omega}$  is also a  $J_1$ -connection and its corresponding connection in  $Q_2 \times I$  through Theorem 4.3 is a  $J_2$ -connection which

defines a  $J_2$ -homotopy between  $\omega_2$  and  $\bar{\omega}_2$ . All these facts can be easily checked by a direct calculation.

**5. Application to subfoliations.**

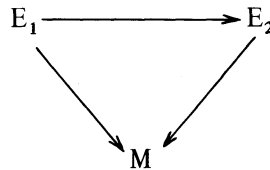
The geometric situation which we have described in § 1 is a particular case of multifoliate structure on the manifold  $M$  and is defined as follows : consider the set  $P = \{1, 2, 3\}$  with the usual order,  $1 < 2 < 3$ , and suppose  $\dim M = n$ . Now, we define a mapping

$$\alpha = \{1, 2, \dots, n\} \rightarrow P$$

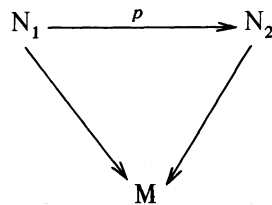
and, thus,  $\{\alpha\}$  is  $P$ -multifoliate and we have determined the subgroup  $G_P \subset Gl(n, \mathbb{R})$  of matrices

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \begin{matrix} \text{---} P_1 \\ \text{---} P_2 \end{matrix}$$

Let us suppose given an integrable  $G_P$ -structure on  $M$  ; then, on  $M$ , there exist two foliations  $F_1, F_2$  of dimensions  $p_1, p_2$ , respectively, and such that every leaf of  $F_2$  is, itself, foliated by leaves of  $F_1$ . This fact is equivalent to the existence of two vector subbundles  $E_i \subset TM, i = 1, 2$ , and an injective morphism



If  $N_i = TM/E_i, i = 1, 2$ , is the normal bundle of  $F_i$ , there is canonically defined a surjective morphism



Denote  $q_i = n - p_i = \text{codim } F_i, i = 1, 2$  ; it is possible to choose a covering  $\{U\}$  of  $M$  which trivializes simultaneously  $N_1$  and  $N_2$ , and a local basis of sections of  $N_1$

$$\omega^1, \dots, \omega^{q_2}, \omega^{q_2+1}, \dots, \omega^{q_1}$$

in such form that  $\omega^1, \dots, \omega^{q_2}$  is a local basis of sections of  $N_2$  ; it is clear that this choice can be done compatibly with  $p : N_1 \rightarrow N_2$ . Moreover, as  $E_1$  and  $E_2$  are completely integrable

$$d\omega^i = \theta_j^i \wedge \omega^j, \quad i, j = 1, 2, \dots, q_2$$

$$d\omega^a = \theta_j^a \wedge \omega^j + \theta_b^a \wedge \omega^b, \quad a, b = q_2 + 1, \dots, q_1$$

and the matrix of 1-forms

$$\theta = \begin{pmatrix} \theta_j^i & 0 \\ \theta_j^a & \theta_b^a \end{pmatrix}$$

is the 1-form of a connection in  $N_1$ , which is basic with respect to  $F_1$ , and

$$\theta' = (\theta_j^i)$$

is the 1-form of a connection in  $N_2$ , basic with respect to  $F_2$ . If  $\nabla$  (respect.  $\nabla'$ ) denotes to derivation law associated to  $\theta$  (respect.  $\theta'$ ), the following diagram commutes

$$\begin{CD} \Gamma(N_1) @>\nabla>> \Gamma(T^*M \otimes N_1) \\ @VpVV @VV1 \otimes pV \\ \Gamma(N_2) @>\nabla'>> \Gamma(T^*M \otimes N_2) \end{CD} \tag{5.1}$$

Similarly, if we consider a weakly-compatible Riemannian metric (see Vaisman ({11})) on the multifoliate manifold  $M$ , it is possible to define two metric connections  $\tilde{\nabla}$  and  $\nabla'$  on  $N_1$  and  $N_2$  respectively, which permit to write a new commutative diagram like (5.1) (in particular, by using the techniques introduced in ({4}), it is possible to write the global expression of these connections).

By another part, consider the Lie groups  $G_1$  and  $G_2$  given as follows :  $G_1 \subset Gl(q_1, \mathbb{R})$  is the group of all matrices

$$m = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with  $A \in Gl(q_2, \mathbb{R})$ , and  $G_2 = Gl(q_2, \mathbb{R})$ , and the homomorphism

$$\begin{array}{ccc} G_1 & \longrightarrow & G_2 \\ m & \longmapsto & A \end{array}$$

Next, consider  $I(G_1)$  and  $I(G_2)$  and their homogeneous ideales given by

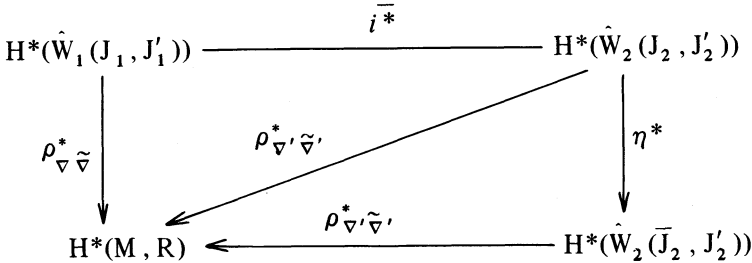
Ideales of  $I(G_1)$  :  $J_1 = J(> q_1)$ ,  $J'_1 = J(\text{odd})$

Ideales of  $I(G_2)$  :  $J_2 = J(> q_1)$ ,  $J'_2 = J(\text{odd})$ ,  $\bar{J}_2 = J(> q_2)$

Clearly,  $\nabla$  (respect.  $\nabla'$ ) is a  $J_1$ -connection (respect.  $J_2$ -connection) and  $\tilde{\nabla}$  (respect.  $\tilde{\nabla}'$ ) is a  $J'_1$ -connection (respect.  $J'_2$ -connection) ; in fact,  $\nabla'$  is a  $\bar{J}_2$ -connection.

Under these assumptions, we can use the results of § 4 and state

PROPOSITION 5.1. – *The following diagram commutes*



Hence,  $Im \rho_{\nabla' \tilde{\nabla}'}^* \subset Im \rho_{\nabla \tilde{\nabla}}^*$ , e.g. the set of exotic classes of  $F_2$  is a subset of the set of exotic classes of  $F_1$ .

This result permits us to give a topological obstruction to  $F_1$  be a subfoliation of  $F_2$ , as follows :

COROLLARY 5.2. – *A necessary condition for  $F_1$  be a subfoliation of  $F_2$  is that every exotic class of  $F_2$  be also an exotic class of  $F_1$ .*

At last, note that if  $F_2$  is given by  $E_2 = TM$ , e.g. if  $F_2$  has the manifold  $M$  as unique leaf, that obstruction is trivial.

## BIBLIOGRAPHY

- [1] R. BOTT, Lectures on characteristic classes and foliations, *Lecture Notes on Math.*, n° 279 (1972), Springer-Verlag, Berlin.
- [2] S.S. CHERN and J. SIMONS, Some cohomology classes in principal fibre bundles and their applications to Riemannian geometry, *Proc. Nat. Acad. Sci. U.S.A.*, 66 (1971), 791-794.
- [3] S.S. CHERN and J. SIMONS, Characteristic forms and transgression, I, Preprint.
- [4] Luis A CORDERO, Special connections on almost-multifoliate Riemannian manifolds, Preprint.
- [5] Luis A. CORDERO, Sobre la geometria de las variedades multifoliadas, *III Jornadas Mat. Hispano-Lusitanas*, Sevilla 1974.
- [6] C. GODBILLON, Conferencias sobre cohomologia de Gelfand-Fuks y foliaciones, *Public. Dep. Geometria y Topologia*, Univ. Santiago de Compostela, n° 22 (1973), notas de P.M. Gadea.
- [7] S. KOBAYASHI and K. NOMIZU, Foundations of Differential Geometry, I, II. Interscience Publ. (1963-69).
- [8] K. KODAIRA and D.C. SPENCER, Multifoliate structure, *Ann. of Math.*, 74 (1961), 52-99.
- [9] D. LEHMANN, J-homotopie dans les espaces de connexions et classes exotiques de Chern-Simons, *C.R. Acad. Sc. Paris*, 275 A (1972), 835-838.
- [10] D. LEHMANN, Classes caractéristiques exotiques et J-connexité des espaces de connexions, Preprint.
- [11] I. VAISMAN, Almost-multifoliate Riemannian manifolds, *Ann. Sc. Univ. "Al. I. Cuza"*, sect. I, vol. XVI (1970), 97-104.

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