ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 25, nº 3-4 (1975), p. 499-507

<http://www.numdam.org/item?id=AIF_1975__25_3-4_499_0>

Ann. Inst. Fourier, Grenoble
25, 3 et 4 (1975), 499-507.

A GENERAL DEFINITION OF CAPACITY

by Makoto OHTSUKA

Dédié à Monsieur M. Brelot à l'occasion de son 70e anniversaire.

Introduction.

During the past 20 years the notion of extremal length proved its usefulness in many branches of analysis. Given a family Γ of locally rectifiable curves in the (x, y)-plane, the extremal length of Γ is defined to be the reciprocal of the infimum of $\iint \rho^2\, dx\, dy$ for the family of Borel measurable functions $\rho \geqslant 0$ satisfying $\int_\gamma \rho\, ds \geqslant 1$ for every $\gamma \in \Gamma$.

There are also many definitions of capacity. One way to define the Newtonian capacity in R^n is to consider the class M of non-negative measures μ with finite energy. It is known that grad U^μ exists a.e., where U^μ denotes the Newtonian potential of μ. The Newtonian capacity of a compact set K is defined to be the infimum of $\int_{R^n} |\mathrm{grad}\, U^\mu|^2\, dx$ taken with respect to $\mu \in M$ satisfying $U^\mu \geqslant 1$ on K.

Recently, Meyers [2] defined $C_{k;\mu_0;p}(A)$ for $A \subset R^n$ by $\inf \int \rho^p\, d\mu_0$ taken with respect to $\rho \geqslant 0$ satisfying

$$\int k(x, y)\rho(y)\, d\mu_0(y) \geqslant 1$$

on A, where μ_0 is a non-negative measure and $k(x, y)$ is a positive lower semicontinuous function on $R^n \times R^n$.

In the present note we shall give a general definition of capacity which includes the above three quantities as special cases, and prove that this general capacity is continuous from the left.

1. General definition.

Let Ω be a space, and F be a family of non-negative functions defined on Ω such that $cf_1 \in F$ and $f_1 + f_2 \in F$ whenever $0 \leqslant c < \infty$ and $f_1, f_2 \in F$; we set $0 \cdot \infty = 0$ if $0 \cdot \infty$ happens for cf_1. It follows that $f \equiv 0$ belongs to F. Let $\Phi \not\equiv \infty$ be a non-negative functional defined on F. Assume that there exist $p, q > 0$ such that $\Phi(cf) \leqslant c^p \Phi(f)$ for any constant $c \geqslant 0$ and $f \in F$, and

$$(\Phi(f_1 + f_2))^q \leqslant (\Phi(f_1))^q + (\Phi(f_2))^q$$

if $f_1, f_2 \in F$. In addition, we assume that, if $f_1, f_2, \ldots \in F$ and $\Phi(f_{m+1} + \cdots + f_n) \to 0$ as $n, m \to \infty$, then

$$f = \sum_{k=1}^{\infty} f_k \in F$$

and $\Phi\left(\sum_{k=1}^{n} f_k\right) \to \Phi(f)$ as $n \to \infty$. It follows that

$$(\Phi(\Sigma f_n))^q \leqslant \Sigma(\Phi(f_n))^q$$

for such $\{f_n\}$.

Let Γ_0 be another space, and G be a class of subsets of Γ_0 such that $\Gamma_1, \Gamma_2, \ldots \in G$ implies $\bigcup_n \Gamma_n \in G$ and that $\Gamma \in G$ and $\Gamma' \subset \Gamma$ imply $\Gamma' \in G$. We shall say that a property holds G-a.e. on $\Gamma \subset \Gamma_0$ if the exceptional set belongs to G. For each $f \in F$ suppose a non-negative function $T_f(\gamma)$ is defined G-a.e. on Γ_0, and assume that, for any f_1, $f_2 \in F$ and $c \geqslant 0$, $T_{cf_1} = cT_{f_1}$ and $T_{f_1+f_2} = T_{f_1} + T_{f_2}$ hold and $f_1 \leqslant f_2$ implies $T_{f_1} \leqslant T_{f_2}$, where all relations as to T_{f_1} and T_{f_2} are supposed to hold wherever they are defined.

We shall say that f is G-almost admissible (or simply G-alm. ad.) for $\Gamma \subset \Gamma_0$ when $f \in F$ and $T_f \geqslant 1$ G-a.e. on Γ. We set

$$G_G(\Gamma) = \inf_{\text{G-alm.ad.} f} \Phi(f)$$

if there is at least one G-alm. ad. f, and otherwise

$$C_G(\Gamma) = \infty.$$

Evidently $C_G(\Gamma) \leqslant C_G(\Gamma')$ if $\Gamma \subset \Gamma'$. We observe that $C_G(\Gamma) = 0$ for every $\Gamma \in G$ because $f \equiv 0$ is G-alm. ad. and $\Phi(0) = 0$.

We shall denote by L the family of functions $f \in F$ with finite $\Phi(f)$.

Theorem 1. — *$C_G(\Gamma) = 0$ if and only if there exists $f \in L$ such that $T_f = \infty$ G-a.e. on Γ.*

Proof. — The if part follows from the definition of C_G and the properties of Φ and T_f. To prove the only-if part take $f_n \in F$ and $\Gamma_n \in G$ for each n so that $T_{f_n} \geqslant 1$ on $\Gamma - \Gamma_n$ and $(\Phi(f_n))^q \leqslant 2^{-n}$, and set $f = \sum_n f_n$. Then $f \in F$ and

$$(\Phi(f))^q \leqslant \Sigma(\Phi(f_n))^q \leqslant 1.$$

We have

$$T_f(\gamma) \geqslant \sum_{k=1}^{m} T_{f_k}(\gamma) \geqslant m$$

for every $\gamma \in \Gamma - \bigcup_n \Gamma_n$ and m so that $T_f(\gamma) = \infty$ for every $\gamma \in \Gamma - \bigcup_n \Gamma_n$. Since $\bigcup_n \Gamma_n \in G$, our theorem is proved.

Lemma 1. — $\left(C_G\left(\bigcup_n \Gamma_n\right)\right)^q \leqslant \Sigma(C_G(\Gamma_n))^q$.

Proof. — We may assume that $\Sigma(C_G(\Gamma_n))^q < \infty$. Given $\varepsilon > 0$, let f_n be G-alm. ad. for Γ_n such that

$$(\Phi(f_n))^q \leqslant (C_G(\Gamma_n))^q + \varepsilon 2^{-n}.$$

By our assumption on Φ, $f = \Sigma f_n \in F$ and

$$(\Phi(f))^q \leqslant \Sigma(\Phi(f_n))^q.$$

Evidently f is G-alm. ad. for $\bigcup_n \Gamma_n$ so that

$$(C_G(\cup \Gamma_n))^q \leqslant (\Phi(f))^q \leqslant \Sigma(\Phi(f_n))^q \leqslant \Sigma(C_G(\Gamma_n))^q + \varepsilon.$$

This gives the required inequality.

LEMMA 2. — *Suppose* $f \in F$ *satisfies* $T_f \geqslant 1$ *on* $\Gamma - \Gamma'$, *where* $C_G(\Gamma') = 0$. *Then* $C_G(\Gamma) \leqslant \Phi(f)$.

Proof. — By Theorem 1 there exist $f' \in L$ and $\Gamma'' \in G$ such that $T_{f'} = \infty$ on $\Gamma' - \Gamma''$. For any $\varepsilon > 0$ we have $T_{f+\varepsilon f'} \geqslant 1$ on $\Gamma - \Gamma''$, and hence

$$C_G(\Gamma) \leqslant \Phi(f + \varepsilon f') \leqslant \{(\Phi(f))^q + (\varepsilon^p \Phi(f'))^q\}^{\frac{1}{q}} \to \Phi(f)$$

as $\varepsilon \to 0$. Thus $C_G(\Gamma) \leqslant \Phi(f)$.

THEOREM 2. — *Denote* $\{\Gamma^* \subset \Gamma_0;\ C_G(\Gamma^*) = 0\}$ *by* G_0. *Then*

$$C_{G_0}(\Gamma) = C_G(\Gamma)$$

for any $\Gamma \subset \Gamma_0$.

Proof. — We observe that $\Gamma_1, \Gamma_2, \ldots \in G_0$ implies $\bigcup_n \Gamma_n \in G_0$ in virtue of Lemma 1 and that $\Gamma \in G_0$ and $\Gamma' \subset \Gamma$ imply $\Gamma' \in G_0$. Since $G \subset G_0$, $C_{G_0}(\Gamma) \leqslant C_G(\Gamma)$. Assume that $C_{G_0}(\Gamma) < \infty$, and take $f \in F$ such that $T_f \geqslant 1$ on $\Gamma - \Gamma'$ where $\Gamma' \in G_0$. By Lemma 2 $C_G(\Gamma) \leqslant \Phi(f)$. Because of the arbitrariness of f we derive

$$C_G(\Gamma) \leqslant C_{G_0}(\Gamma).$$

The equality now follows.

THEOREM 3 (cf. [2], Theorem 4). — *Each of the following statements implies the succeeding one.*

(i) $T_{f_n} \to T_f$ *in* C_G, *namely, for any* $a > 0$,

$$C_G(\{\gamma \in \Gamma_0 - \Gamma;\ |T_{f_n}(\gamma) - T_f(\gamma)| \geqslant a\}) \to 0 \quad as \quad n \to \infty,$$

where $\Gamma \in G$ *and all* T_{f_n} *and* T_f *are defined on* $\Gamma_0 - \Gamma$; $\infty - \infty$ *is set to be* 0 *if it happens for* $T_{f_n} - T_f$.

(ii) *We can find* $\{f_{n_k}\}$ *with the property that, given* $\varepsilon > 0$, *there exists* $\Gamma' \subset \Gamma_0$ *with* $C_G(\Gamma') < \varepsilon$ *such that* $T_{f_{n_k}} - T_f \to 0$ *uniformly on* $\Gamma_0 - \Gamma'$.

(iii) *For the sequence* $\{f_{n_k}\}$ *in* (ii), $T_{f_{n_k}} \to T_f$ *on* $\Gamma_0 - \Gamma''$, *where* $C_G(\Gamma'') = 0$.

Proof. — (i) → (ii). There exist $\{f_{n_k}\}$ and $\{\Gamma_k\}$ in Γ_0 such that, for each k, $\Gamma_k \supset \Gamma$, $(C_G(\Gamma_k))^q \leqslant 2^{-k}$ and

$$|T_{f_{n_k}} - T_f| \leqslant \frac{1}{k} \quad \text{on} \quad \Gamma_0 - \Gamma_k.$$

Given $\varepsilon > 0$, choose k_0 so that $2^{-k_0+1} < \varepsilon^q$. We see that $T_{f_{n_k}} - T_f \to 0$ uniformly on $\Gamma_0 - \bigcup_{k=k_0}^{\infty} \Gamma_k$, and

$$\left(C_G\left(\bigcup_{k=k_0}^{\infty} \Gamma_k\right)\right)^q \leqslant \sum_{k=k_0}^{\infty} (C_G(\Gamma_k))^q \leqslant \varepsilon^q$$

by Lemma 1. This establishes (i) → (ii).

(ii) → (iii) is evident.

Now, let $\Psi(f, g)$ be a functional on $F \times F$ such that, for any $f_1, f_2 \in F$, there exists $\tilde{f} \in F$ satisfying $\Phi(\tilde{f}) \leqslant \Psi(f_1, f_2)$ and $|T_{f_1} - T_{f_2}| \leqslant T_{\tilde{f}}$ G-a.e. on Γ_0. Then we have

THEOREM 4. — *For all* $f_1, f_2 \in F$ *and* $0 < a < \infty$ *we have*

$$C_G(\{\gamma \in \Gamma_0 - \Gamma;\, |T_{f_1}(\gamma) - T_{f_2}(\gamma)| \geqslant a\}) \leqslant a^{-p}\Psi(f_1, f_2),$$

where $\Gamma \in G$ *is chosen so that both* T_{f_1} *and* T_{f_2} *are defined on* $\Gamma_0 - \Gamma$.

Proof. — Denote $\{\gamma \in \Gamma_0 - \Gamma;\ |T_{f_1}(\gamma) - T_{f_2}(\gamma)| \geqslant a\}$ by Γ'. By our assumption there exists $\tilde{f} \in F$ such that $\Phi(\tilde{f}) \leqslant \Psi(f_1, f_2)$ and $|T_{f_1} - T_{f_2}| \leqslant T_{\tilde{f}}$ G-a.e. on Γ_0. Evidently $\tilde{f}/a$ is G-alm. ad. for Γ' so that

$$C_G(\Gamma') \leqslant \Phi\left(\frac{\tilde{f}}{a}\right) \leqslant a^{-p}\Psi(f_1, f_2).$$

Hereafter we assume that the relation

$$\limsup_{n, m \to \infty} \left\{\frac{\Phi(f_n) + \Phi(f_m)}{2} - \Phi\left(\frac{f_n + f_m}{2}\right)\right\} \leqslant 0$$

for $\{f_n\} \subset F$ implies the existence of $f \in F$ such that $\Psi(f_n, f) \to 0$ and $\Phi(f_n) \to \Phi(f)$.

THEOREM 5. — *If* $C_G(\Gamma) < \infty$, *then there exist* $f \in F$ *and* Γ' *with* $C_G(\Gamma') = 0$ *such that* $T_f \geqslant 1$ *on* $\Gamma - \Gamma'$ *and* $\Phi(f) = C_G(\Gamma)$.

Proof. — Choose $f_1, f_2, \ldots$ G-alm. ad. for Γ so that $\Phi(f_n) \to C_G(\Gamma)$. Then $(f_n + f_m)/2$ is G-alm. ad. for Γ, and hence $\Phi((f_n + f_m)/2) \geqslant C_G(\Gamma)$. Therefore

$$\limsup_{n, m \to \infty} \left\{ \frac{\Phi(f_n) + \Phi(f_m)}{2} - \Phi\left(\frac{f_n + f_m}{2}\right) \right\} \leqslant 0.$$

By our assumption there exists $f \in F$ such that $\Phi(f_n, f) \to 0$ and $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$. Choose $\Gamma' \in G$ so that $T_{f_n} \geqslant 1$ on $\Gamma - \Gamma'$ for every n. In view of (i) $\to$ (iii) of Theorem 3 and Theorem 4 we find $\{f_{n_k}\}$ and $\Gamma'' \subset \Gamma$ with $C_G(\Gamma'') = 0$ such that

$$T_f = \lim_{k \to \infty} T_{f_{n_k}} \geqslant 1 \quad \text{on} \quad \Gamma - \Gamma''.$$

We have $\Phi(f) = \lim_{n \to \infty} \Phi(f_n) = C_G(\Gamma)$.

THEOREM 6. — *If* $\Gamma_n \uparrow \Gamma$, *then* $C_G(\Gamma_n) \uparrow C_G(\Gamma)$.

Proof. — Denote $\lim_{n \to \infty} C_G(\Gamma_n)$ by C. Clearly $C \leqslant C_G(\Gamma)$. Hence it suffices to establish $C_G(\Gamma) \leqslant C$. We may assume that $C < \infty$. Choose f_n G-alm. ad. for Γ_n so that $\Phi(f_n) \to C$. If $m > n$, then f_m and hence $(f_n + f_m)/2$ is G-alm. ad. for Γ_n. Therefore $\Phi((f_n + f_m)/2) \geqslant C_G(\Gamma_n)$. As in the proof of Theorem 5 we find $f \in F$ and Γ' with $C_G(\Gamma') = 0$ so that $T_f(\gamma) \geqslant 1$ on $\Gamma - \Gamma'$ and $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$. By Lemma 2 $C_G(\Gamma) \leqslant \Phi(f)$. Hence

$$C_G(\Gamma) \leqslant \Phi(f) = \lim_{n \to \infty} \Phi(f_n) = C.$$

From this theorem we derive immediately

THEOREM 7. — *If* $C_G(\Gamma' \cup \Gamma'') \leqslant C_G(\Gamma') + C_G(\Gamma'')$ *for any* Γ', Γ'', *then*

$$(1) \qquad C_G\left(\bigcup_n \Gamma_n\right) \leqslant \sum_n C_G(\Gamma_n).$$

Remark. — If $\max(f_1, f_2)$ belongs to F and

$$\Phi(\max(f_1, f_2)) \leqslant \Phi(f_1) + \Phi(f_2)$$

whenever $f_1, f_2 \in F$, then $C_G(\Gamma_1 \cup \Gamma_2) \leqslant C_G(\Gamma_1) + C_G(\Gamma_2)$. It suffices to show $C_G(\Gamma_1 \cup \Gamma_2) \leqslant \Phi(f_1) + \Phi(f_2)$ when each $C_G(\Gamma_i)$ is finite and f_i is G-alm. ad. for Γ_i, $i = 1, 2$. This is actually true because $f = \max(f_1, f_2)$ is G-alm. ad. for $\Gamma_1 \cup \Gamma_2$ so that

$$C_G(\Gamma_1 \cup \Gamma_2) \leqslant \Phi(f) \leqslant \Phi(f_1) + \Phi(f_2).$$

2. Examples.

Let Ω be a general space. Hereafter take as Γ_0 the class M_0 of all non-negative measures defined on a σ-field E of sets in Ω, and let F be a family of non-negative E-measurable functions on Ω such that $cf_1 \in F$ and $f_1 + f_2 \in F$ whenever $c \geqslant 0$ and $f_1, f_2 \in F$. For every $f \in F$ we define $T_f(\mu)$ to be $\int f\, d\mu$. We take $G = \emptyset$ and denote $C_G(\Gamma)$ by $C(M)$ for $\Gamma = M \subset M_0$.

Example 1. — Let F consist of all non-negative E-measurable functions. With a fixed $m \in M_0$ set

$$\Phi(f) = \int f^p\, dm \quad \text{for} \quad f \in F$$

and $\Psi(f_1, f_2) = \int |f_1 - f_2|^p\, dm$ for $f_1, f_2 \in F$ if

$$\int f_1^p\, dm + \int f_2^p\, dm < \infty.$$

If

$$\int f_1^p\, dm + \int f_2^p\, dm = \infty,$$

then define $\Psi(f_1, f_2)$ to be ∞. Then Ψ satisfies the conditions required in § 1. We call $C(M)$ the module of M of order p. Its reciprocal is called the extremal length of M of order p. In this case we have $\sup_n f_n \in F$ for $f_1, f_2, \ldots \in F$ and $\Phi\left(\sup_n f_n\right) \leqslant \Sigma\Phi(f_n)$. We obtain the subadditivity from this immediately without appealing to Theorem 7.

Example 2. — Let F, Φ and ψ be as above. Let $k(x, e) \geqslant 0$ be an E-measurable function of x on Ω for every fixed

$e \in E$, and a measure for every fixed $x \in \Omega$. Set

$$\nu_\mu(e) = \int k(x, e)\, d\mu(x) \quad \text{for} \quad \mu \in M_0,$$

and $N_M = \{\nu_\mu;\ \mu \in M\}$. We may consider $C(N_M)$. This gives a generalization of $C_{k;\,\mu_0;\,p}(A)$ referred to in the introduction when $M = \{\varepsilon_x;\ x \in A\}$ and

$$k(x, e) = \int_e k(x, y)\, d\mu_0(y),$$

where A is a subset of R_n, ε_x is the unit point measure at x, $k(x, y) \geqslant 0$ is E-measurable for every fixed x and μ_0 is a fixed measure in M_0. As in Example 1, (1) follows immediately.

Example 3. — Let Ω be an open set in R^n, E be the Borel class of sets in Ω, m be the Lebesgue measure, F consist of (some) non-negative p-precise functions f in Ω, take $\Phi(f) = \int |\text{grad}\, f|^p\, dm$, and define $\Psi(f_1, f_2)$ by

$$\int |\text{grad}\,(f_1 - f_2)|^p\, dm.$$

See [3] for p-precise functions and properties of these functions. In order to assure $\sum_n f_n \in F$ for every $\{f_n\} \subset F$ satisfying $\int |\text{grad}\,(f_{n+1} + \cdots + f_m)|^p\, dm \to 0$, we assume that there is a family, with positive module of order p, of curves in Ω such that every $f \in F$ tends to 0 along p-a.e. curve of the family. Then all the conditions required in the beginning of § 1 are satisfied. If $\max(f_1, f_2) \in F$ for any $f_1, f_2 \in F$, then (1) holds because $|\text{grad}\,(\max(f_1, f_2))| = |\text{grad}\, f_1|$ or $|\text{grad}\, f_2|$ m-a.e. so that $\Phi(\max(f_1, f_2)) \leqslant \Phi(f_1) + \Phi(f_2)$. Consider the case when the module of order p of the family Λ of curves terminating at $\partial\Omega$ is positive, $M = \{\varepsilon_x;\ x \in A \subset \Omega\}$ and F consists of all non-negative p-precise functions in Ω tending to 0 along p-a.e. curve of Λ. Then the capacity is called the p-capacity of A (relative to Ω).

Example 4. — Let Ω be a topological space, and E be the Borel class of sets in Ω. Let F be as in the beginning of this section, and Φ be as in § 1. Denote by S the family of lower semicontinuous functions in Ω, and define $c(K)$

for every compact set K by $\inf \Phi(f)$ for $f \in F \cap S$ satisfying $f \geqslant 1$ on K. Let us see that $c(K_n) \downarrow c(K)$ if a sequence $\{K_n\}$ of compact sets decreases to K. Suppose $f \in F \cap S$ satisfies $f(x) \geqslant 1$ on K. Since f is lower semicontinuous, $f/(1-1/n) > 1$ for each $n \geqslant 2$ on an open set ω containing K. There exists m_0 such that $K_{m_0} \subset \omega$, and hence

$$c(K) \leqslant \lim_{m\to\infty} c(K_m) \leqslant \Phi(f/(1-1/n)) \leqslant (1-1/n)^{-p}\Phi(f) \to \Phi(f)$$

as $n \to \infty$. The arbitrariness of f yields $c(K) = \lim_{m\to\infty} c(K_m)$.

This together with Theorem 6 shows that $C(M_A)$ with $M_A = \{\varepsilon_x;\ x \in A\}$ is a true capacity if it is shown that

$$c(K) = C(M_K)$$

for every compact set K; cf. [1; Part II, Chap. I]. This is the case, for example, when $\Omega = R^n (n \geqslant 3)$, F consists of all Newtonian potentials of non-negative measures with finite energy and $\Phi(f) = \int |\text{grad}\, f|^2\, dm$; the 2-capacity is equal to the Newtonian outer capacity. Evidently all $f \in F$ are superharmonic.

Another example of such a case is found in [4]. See [3] too.

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Manuscrit reçu le 25 janvier 1975.

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