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SOME REMARKS ON THE EXISTENCE OF A RESOLVENT

by Masanori KISHI

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1. Introduction.

In potential theory, one of the important classes of convolution kernels on a locally compact abelian group G is the class of those kernels with which resolvents are associated. Such a kernel satisfies the domination principle, and as shown in [6], it induces a Hunt convolution kernel on a quotient space G/H modulo a compact subgroup H. It is a Hunt convolution kernel if and only if it satisfies the unicity principle.

In this note, we shall deal with convolution kernels satisfying the domination principle and give some remarks on the existence of resolvents.

A convolution kernel $\varkappa$ on G is a positive (Radon) measure on G. Given a positive measure μ, the convolution $\varkappa * \mu$ is defined by

$$\int \varphi d(\varkappa * \mu) = \iint \varphi(x+y)\, d\varkappa(x)\, d\mu(y),$$

when the integral converges for every $\varphi \in C_K(G)$, the linear space of continuous functions with compact support. The convolution $\varkappa * \mu$ is a positive measure. We denote by $\mathcal{D}^+(\varkappa)$ the totality of positive measures μ such that the convolution $\varkappa * \mu$ is defined.

A family of positive measures $\{\varkappa_p\}$ $(p > 0)$ is called a resolvent, if each $\varkappa_p$ belongs to $\mathscr{D}^+(\varkappa)$ and

$$\varkappa - \varkappa_p = p\varkappa * \varkappa_p. \tag{1}$$

When $\{\varkappa_p\}$ verifies moreover the condition

$$\lim_{p \downarrow 0} \varkappa_p = \varkappa \quad \text{(vaguely)}, \tag{2}$$

we call it a resolvent of $\varkappa$ or we say that $\{\varkappa_p\}$ is associated with $\varkappa$. It is known that a resolvent $\{\varkappa_p\}$ of $\varkappa$ exists at most one and

$$\varkappa + \frac{1}{p}\varepsilon = \frac{1}{p}\sum_{n=0}^{\infty} (p\varkappa_p)^n \tag{3}$$

where ε denotes the Dirac measure at the origin and

$$(p\varkappa_p)^0 = \varepsilon, \quad (p\varkappa_p)^n = (p\varkappa_p)^{n-1} * (p\varkappa_p).$$

We denote by (R) the family of all convolution kernels on G with which resolvents are associated.

2. Relatively balayaged measure.

We say that a convolution kernel $\varkappa'$ satisfies the *domination principle relative to a convolution kernel* $\varkappa''$, if for any φ, $\psi \in C_K^+(G)$ (= the totality of non-negative functions of $C_K(G)$), $\varkappa' * \varphi \leqslant \varkappa'' * \psi$ in G, whenever $\varkappa' * \varphi \leqslant \varkappa'' * \psi$ on supp (φ), the support of φ. In this case under suitable assumptions on $\varkappa'$, we can balayage $\mu \in \mathscr{D}^+(\varkappa')$ to a given relatively compact open set ω relative to $\varkappa'$. Namely, there exists $\mu' \in \mathscr{D}^+(\varkappa')$, supported by $\bar{\omega}$, such that

$$\begin{aligned} \varkappa' * \mu' &= \varkappa'' * \mu \quad \text{in} \quad \omega \\ \varkappa' * \mu' &\leqslant \varkappa'' * \mu \quad \text{in} \quad G. \end{aligned}$$

As to the existence of μ', we refer to [3] and [4]. In case that

$$\varkappa' = \varkappa + c\varepsilon (c > 0) \qquad \text{and} \qquad \varkappa'' = \varkappa,$$

the existence of μ' is shown also by modifying Choquet-

Deny's method in [1]. In case that $\varkappa' = \varkappa'' = \varkappa$, we say simply that $\varkappa$ satisfies the domination principle and call the above μ' the balayaged measure.

3. Dominated convergence theorem.

For a convolution kernel $\varkappa$ in (R), we have the following dominated convergence theorem [6], [7] : Let $\{\mu_\alpha\}$ be a net of positive measures, converging vaguely to μ, and suppose that there exists $\lambda \in \mathcal{D}^+(\varkappa)$ such that $\varkappa * \lambda$ dominates every $\varkappa * \mu_\alpha$. Then $\{\varkappa * \mu_\alpha\}$ converges vaguely to $\varkappa * \mu$. This is proved by the equality (3) and Deny's convergence lemma ([2], Lemme 5.2).

We shall show the converse.

Theorem 1. — *If a convolution kernel $\varkappa$ satisfies the domination principle and has the following dominated convergence property ($*$), then a resolvent is associated with $\varkappa$:*

($*$) *A net $\{\varkappa * \mu_\alpha\}$ converges vaguely to $\varkappa * \mu$, if $\{\mu_\alpha\}$ converges vaguely to μ and every $\varkappa * \mu_\alpha$ is dominated by $\varkappa$.*

Proof. — We shall use the method of M. Itô [3]. Let p be a fixed positive number and denote by $\mathcal{W}$ the directed set of relatively compact open neighborhoods of the origin such that $\bigcup \{\omega_\alpha ; \omega_\alpha \in \mathcal{W}\} = G$. Since $p\varkappa + \varepsilon$ satisfies the domination principle relative to $\varkappa$, we have the following relatively balayaged measure μ_α, supported by $\overline{\omega}_\alpha$ for every $\omega_\alpha \in \mathcal{W}$:

$$(p\varkappa + \varepsilon) * \mu_\alpha = \varkappa \quad \text{in} \quad \omega_\alpha,$$
$$(p\varkappa + \varepsilon) * \mu_\alpha \leqslant \varkappa \quad \text{in} \quad G.$$

The net $\{\mu_\alpha\}$ being vaguely bounded, we may suppose that as $\omega_\alpha \uparrow G$, $\{\mu_\alpha\}$ converges vaguely to a positive measure $\varkappa_p$. Then by the dominated convergence property ($*$), $\{p\varkappa * \mu_\alpha\}$ converges vaguely to $p\varkappa * \varkappa_p$, and hence $(p\varkappa + \varepsilon) * \varkappa_p = \varkappa$. Thus we have constructed a resolvent $\{\varkappa_p\}$ $(p > 0)$.

To show the vague convergence $\lim\limits_{p \downarrow 0} \varkappa_p = \varkappa$, it is sufficient to remark that $\lim\limits_{p \downarrow 0} p\varkappa_p = 0$ (vaguely) and

$$\varkappa * (p\varkappa_p) \leqq x,$$

because $\varkappa$ has the dominated convergence property $(*)$. This completes the proof.

From this theorem it follows immediately that if $\varkappa$ satisfies the domination principle and if $\varkappa \in \mathscr{D}^+(\varkappa)$, then $\varkappa$ belongs to (R). It is also seen that if $\varkappa$ satifies the domination principle and if $\varkappa * \varphi(x)$, for any $\varphi \in C_K(G)$, tends to 0 as x tends to the point at infinity, then $\varkappa$ belongs to (R).

Necessary and sufficient conditions for the existence of a resolvent are also given in [3].

Lemma — *Suppose that $\varkappa$ satisfies the domination principle and there exists $\varkappa'(\neq 0) \in (R)$ such that $\varkappa' \leqslant \varkappa, \varkappa - \varkappa' \in \mathscr{D}^+(\varkappa')$ and $\varkappa'$ satisfies the domination principle relative to $\varkappa$. Then there exists $\varkappa'' \in \mathscr{D}^+(\varkappa')$ such that $\varkappa = \varkappa' * \varkappa''$.*

Proof. — Without loss of generality we may assume that $\varkappa \neq \varkappa'$. For every ω_β of the net $\mathscr{W}$ used in the proof of Theorem 1, there exists a relatively balayaged measure ν_β, supported by $\overline{\omega}_\beta$, such that

$$\begin{aligned} \varkappa' * \nu_\beta &= \varkappa \quad \text{in} \quad \omega_\beta, \\ \varkappa' * \nu_\beta &\leqslant \varkappa \quad \text{in} \quad G. \end{aligned}$$

Since $\{\nu_\beta\}$ is vaguely bounded, we have a vaguely adherent measure $\varkappa''$ of $\{\nu_\beta\}$, which is a required one. We show it assuming that $\{\nu_\beta\}$ converges vaguely to $\varkappa''$. What we have to show is the vague convergence of $\{\varkappa' * \nu_\beta\}$ to $\varkappa' * \varkappa''$. Let $\{\varkappa'_p\}$ be the resolvent of $\varkappa$, and let p be fixed. By virtue of $(p\varkappa'_p)^n \leqslant p\varkappa'$ $(n \geqslant 1)$ and $\varkappa' \in \mathscr{D}^+(\varkappa - \varkappa')$, $(p\varkappa'_p)^n$ belongs to $\mathscr{D}^+(\varkappa - \varkappa')$ and hence to $\mathscr{D}^+(\varkappa)$. Therefore by Deny's convergence lemma,

$$\lim\nolimits_\beta (p\varkappa'_p)^n * \nu_\beta = (p\varkappa'_p)^n * \varkappa'' \quad \text{(vaguely)}.$$

On the other hand, we have

$$\begin{aligned} \sum_{m=n+1}^{\infty} (p\varkappa'_p)^m * \nu_\beta &= (p\varkappa'_p)^n * p\varkappa' * \nu_\beta \\ &\leqslant (p\varkappa'_p)^n * p\varkappa = \sum_{m=n+1}^{\infty} (p\varkappa'_p)^m + (p\varkappa'_p)^n * p(\varkappa - \varkappa'). \end{aligned}$$

Hence for a given positive number η and $\varphi \in C^+_K(G)$, we have, taking f in $C_K^+(G)$ such that $\varphi \leqslant \check{\varkappa}' * f$,

$$\frac{1}{p} \sum_{m=n+1}^{\infty} (p\varkappa_p')^m(\varphi) < \eta,$$
$$(p\varkappa_p')^n * (\varkappa - \varkappa')(\varphi) \leqslant (p\varkappa_p')^n * (\varkappa - \varkappa') * \varkappa'(f) < \eta$$

for every sufficiently large n. The last inequality is due to the fact that $\lim_n (p\varkappa_p')^n * \varkappa' = 0$ (vaguely) and

$$\varkappa' \in \mathcal{D}'(\varkappa - \varkappa')$$

and hence

$$\lim_n (p\varkappa_p')^n * (\varkappa - \varkappa') * \varkappa'(f) = 0.$$

Consequently we have

$$\limsup_\beta \varkappa' * \nu_\beta(\varphi) \leqslant \limsup_\beta \left(\varkappa' + \frac{1}{p}\varepsilon\right) * \nu_\beta(\varphi)$$
$$\leqslant \lim \frac{1}{p} \sum_{m=0}^{n} (p\varkappa_p')^m * \nu_\beta(\varphi) + \limsup_\beta \frac{1}{p} \sum_{m=n+1}^{\infty} (p\varkappa_p')^m * \nu_\beta(\varphi)$$
$$< \frac{1}{p} \sum_{m=0}^{\infty} (p\varkappa_p')^m * \varkappa''(\varphi) + 2\eta = \left(\varkappa' + \frac{1}{p}\varepsilon\right) * \varkappa''(\varphi) + 2\eta,$$
$$\limsup_\beta \varkappa' * \nu_\beta(\varphi) \leqslant \left(\varkappa' + \frac{1}{p}\varepsilon\right) * \varkappa''(\varphi).$$

Thus we obtain, letting p tend to infinity,

$$\limsup_\beta \varkappa' * \nu_\beta(\varphi) \leqslant \varkappa' * \varkappa''(\varphi)$$

and hence $\lim_\beta \varkappa' * \nu_\beta(\varphi) = \varkappa' * \varkappa''(\varphi)$.

Using this lemma we shall remark the following

THEOREM 2. — *Let* $\{\varkappa_p\}$ $(p > 0)$ *be a resolvent such that*

$$\varkappa - \varkappa_p = p\varkappa * \varkappa_p.$$

If $\varkappa_0 = \lim_{p \downarrow 0} \varkappa_p$ *(vaguely) does not coincide with* $\varkappa$, *then*

$$\varkappa - \varkappa_0 \notin \mathcal{D}^+(\varkappa).$$

Proof. — Put $\sigma = \varkappa - \varkappa_0 (\geqslant 0)$. Then for every $p > 0$, we have $\sigma * p\varkappa_p = \sigma$ and we see that $\varkappa = \varkappa_0 + \sigma$ satisfies the domination principle. Assuming that $\sigma \in \mathcal{D}^+(\varkappa)$, we shall show that $\varkappa$ belongs absurdly to (R). We may assume that $\varkappa_0 \neq 0$. Then by the above lemma, there exists

$$\varkappa_0' \in \mathcal{D}^+(\varkappa_0)$$

such that $\varkappa = \varkappa_0 * \varkappa_0'$, because $\varkappa_0$ satisfies the domination principle relative to $\varkappa$.

Now let $\{\mu_\alpha\}$ be a net of positive measures converging vaguely to μ and assume that $\varkappa * \mu_\alpha \leqslant \varkappa$. We shall show the vague convergence of $\{\varkappa_0 * \mu_\alpha\}$ to $\varkappa_0 * \mu$. Then we have

$$\lim_\alpha \varkappa * \mu_\alpha = \varkappa * \mu \quad \text{(vaguely)},$$

because $\{\sigma * \mu_\alpha\}$ converges vaguely to $\sigma * \mu$ by the facts $\sigma * \mu_\alpha \leqslant \varkappa$ and $\sigma \in \mathscr{D}^+(\varkappa)$. Let φ be a function in $C_K^+(G)$, and take f in $C_K^+(G)$ such that $\varphi \leqslant \check{\varkappa}_0' * f$. For a fixed $p > 0$, we have $(p\varkappa_p)^m * \varkappa \leqslant \varkappa$ and

$$\lim_\alpha (p\varkappa_p)^m * \mu_\alpha = (p\varkappa_p)^m * \mu \quad \text{(vaguely)}.$$

On the other hand, we have

$$\begin{aligned}\frac{1}{p} \sum_{m=n+1}^{\infty} (p\varkappa_p)^m * \mu_\alpha(\varphi) &= (p\varkappa_p)^n * \varkappa_0 * \mu_\alpha(\varphi) \\ &\leqslant (p\varkappa_p)^n * \varkappa * \mu_\alpha(f) \leqslant (p\varkappa_p)^n * \varkappa(f) = \frac{1}{p} \sum_{m=n+1}^{\infty} (p\varkappa_p)^m * \varkappa_0'(f).\end{aligned}$$

Since $\varkappa_0'$ belongs to $\mathscr{D}^+(\varkappa_0)$, we have, given a positive number η,

$$\frac{1}{p} \sum_{m=n+1}^{\infty} (p\varkappa_p)^m * \mu_\alpha(\varphi) < \eta$$

for any sufficiently large n. Therefore we have

$$\begin{aligned}\limsup_\alpha \left(\varkappa_0 + \frac{1}{p}\varepsilon\right) * \mu_\alpha(\varphi) &\leqslant \frac{1}{p} \sum_{m=0}^{n} (p\varkappa_p)^m * \mu(\varphi) + \eta \\ &\leqslant \frac{1}{p} \sum_{m=0}^{\infty} (p\varkappa_p)^m * \mu(\varphi) + \eta = \left(\varkappa_0 + \frac{1}{p}\varepsilon\right) * \mu(\varphi) + \eta.\end{aligned}$$

Consequently $\lim_\alpha \varkappa_0 * \mu_\alpha = \varkappa_0 * \mu$ (vaguely). This completes the proof.

4. Square root of a convolution kernel.

Let $\varkappa$ be in (R). It is shown in [5] that the convolution kernel $\varkappa^\alpha$ $(0 < \alpha < 1)$ defined by the following integral is in (R):

$$\varkappa^\alpha(\varphi) = \frac{\sin \alpha\pi}{\pi} \int_0^\infty p^{-\alpha} \varkappa_p(\varphi)\, dp \ (\varphi \in C_K(G)),$$

and that if $0 < \alpha, \beta < 1, \alpha + \beta < 1$, then

$$\varkappa^{\alpha} * \varkappa^{\beta} = \varkappa^{\alpha+\beta}.$$

As shown in [7] we have also

$$\varkappa^{\alpha} * \varkappa^{1-\alpha} = \varkappa \quad (0 < \alpha < 1).$$

Hence setting $\varkappa' = \varkappa^{1/2}$ we have $\varkappa' * \varkappa' = \varkappa$. Namely if $\varkappa$ is in (R), its square root exists in (R). (Needless to say, there exists $\varkappa \in$ (R), which has a square root not in (R).) We shall discuss the converse.

THEOREM 3. — *Let $\varkappa$ be a convolution kernel satisfying the domination principle, and suppose that there exists a square root $\varkappa'$ of $\varkappa$ satisfying the domination principle. Then $\varkappa$ belongs to* (R).

Proof. — Let $\{\mu_\alpha\}$ be a net of positive measures converging vaguely to μ and suppose that $\varkappa * \mu_\alpha \leqslant \varkappa$. We shall first show the vague convergence of $\{\varkappa' * \mu_\alpha\}$ to $\varkappa' * \mu$, assuming $\varkappa' \neq 0$. $\varkappa'$ being in $\mathscr{D}^+(\varkappa')$, it is in (R) and has the dominated convergence property. For every $V_\beta \in \mathscr{V}$, a net of compact neighborhoods of the origin such that $V_\beta \uparrow G$, we have a balayaged measure ν_β, supported by $\overline{\complement V_\beta}$, such that

$$\begin{aligned} \varkappa' * \nu_\beta &= \varkappa' \quad \text{in} \quad \complement V_\beta \\ \varkappa' * \nu_\beta &\leqslant \varkappa' \quad \text{in} \quad G. \end{aligned}$$

Evidently $\{\nu_\beta\}$ converges vaguely to 0 as $V_\beta \uparrow G$, and hence $\{\varkappa' * \nu_\beta\}$ converges vaguely to 0, by $\varkappa' * \nu_\beta \leqslant \varkappa'$ and by the dominated convergence property. Then by

$$\varkappa' * (\varkappa' * \nu_\beta) \leqslant \varkappa' * \varkappa', \quad \{\varkappa * \nu_\beta\}$$

converges vaguely to 0. Therefore, for any positive number η and any $\varphi \in C_K^+(G)$, there exists $V_\beta \in \mathscr{V}$ such that

$$\varkappa * \nu_\beta(f) < \eta,$$

where f is a function in $C_K^+(G)$ such that $\varphi \leqslant \check{\varkappa}' * f$. Then

$$\begin{aligned}\limsup_{\alpha} \varkappa' * \mu_\alpha(\varphi) &\leqslant \lim (\varkappa' - \varkappa' * \nu_\beta) * \mu_\alpha(\varphi) \\ &\quad + \limsup_{\alpha} \varkappa' * \nu_\beta * \mu_\alpha(\varphi) \\ &\leqslant (\varkappa' - \varkappa' * \nu_\beta) * \mu(\varphi) + \limsup_{\alpha} \varkappa * \nu_\beta * \mu_\alpha(f) \\ &\leqslant \varkappa' * \mu(\varphi) + \eta.\end{aligned}$$

This proves the vague convergence of $\{\varkappa' * \mu_\alpha\}$ to $\varkappa' * \mu$. We now apply again the dominated convergence property of $\varkappa'$ to conclude $\lim_{\alpha} \varkappa * \mu_\alpha = \varkappa * \mu$ (vaguely) and complete the proof.

Similarly as above we can show the following

THEOREM 3'. — *Let $\varkappa$ be a convolution kernel satisfying the domination principle and suppose that there exist convolution kernels $\varkappa', \varkappa''$ in* (R) *such that $\varkappa = \varkappa' * \varkappa''$. Then $\varkappa$ is in* (R).

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