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A free boundary value problem in potential theory


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A FREE BOUNDARY VALUE PROBLEM IN POTENTIAL THEORY
by David KINDERLEHRER (*) and Guido STAMPACCHIA

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1. Introduction.

In this paper we shall describe the formulation and solution of a free boundary value problem in the framework of variational inequalities. For simplicity, we confine our attention to a problem in the plane which consists in finding a domain $\Omega$ and a function $u$ defined in $\Omega$ satisfying there a given differential equation together with both assigned Dirichlet and Neumann data on the boundary $\Gamma$ of $\Omega$. Under appropriate hypotheses about the given data we prove that there is a unique solution pair $\Omega, u$ which resolves this problem and that $\Gamma$ is a smooth curve.

Let $z = x_1 + ix_2 = re^{i\theta}$, $0 \leq \theta < 2\pi$, denote a point in the $z$-plane. Let us suppose, for the moment, that $F(z)$ is a function in $C^2(\mathbb{R}^2)$ which satisfies the conditions

$$\rho^{-2} F(z) \in C^2(\mathbb{R}^2)$$

$$\inf_{\rho} \rho^{-2} F(z) > 0$$

$$(1.1) \quad F_\rho(z) \geq 0 \quad z \in \mathbb{R}^2$$

$F(0) = F_\rho(0) = 0. $$

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These conditions will be weakened. Our object is to solve, in some manner, this

**Problem 1.** — To find a bounded $\Omega$ and a function $u$ such that

\begin{align*}
-\Delta u &= \varphi^{-1} F_z \quad \text{in} \quad \Omega \\
\begin{cases}
u = 0 \\
\frac{\partial u}{\partial \nu} = - F \frac{d\theta}{ds}
\end{cases} &\quad \text{on} \quad \Gamma \\
u(0) &= \gamma
\end{align*}

where $\Gamma = \partial \Omega$, $\nu$ is the outward directed normal vector and $s$ the arc length on $\Gamma$, $F$ satisfies (1.1), and $\gamma$ is given.

Supposing $\Omega$, $u$ to be a solution to Problem 1, the maximum principle for superharmonics implies that $u > 0$ in $\Omega$ since $-\Delta u \geq 0$ in $\Omega$. We assume, consequently, that $\gamma > 0$ and that $u \in C(R^2)$ with $\Omega = \{z : u(z) > 0\}$. Further, if $\Omega$ is a domain with smooth boundary $\Gamma$ and $u$ satisfies (1.2) in $\Omega$ and (1.3) on $\Gamma$ then

$$\frac{\partial u}{\partial \nu}(z) < 0 \quad \text{for} \quad z \in \Gamma$$

in view of Hopf's well known maximum principle. Therefore

$$\frac{d\theta}{ds}(z) = - \frac{1}{F(z)} \frac{\partial u}{\partial \nu}(z) > 0 \quad \text{for} \quad z \in \Gamma,$$

or the central angle $\theta$ is a strictly increasing function of the arc length parameter on $\Gamma$. Interpreting this situation geometrically, we conclude if $\Gamma$ is smooth and $u$ satisfies (1.2) in $\Omega$ and (1.3) on $\Gamma$, then $\Omega$ is starshaped with respect to $z = 0$.

We shall solve Problem 1 by means of a variational inequality suggested by the properties of a function $g(z)$ which satisfies

\begin{equation}
(1.5) \quad g_{\varphi} = - \varphi^{-1} u
\end{equation}

The idea of introducing a new unknown related to the original one through differentiation is due to C. Baiocchi [1] who
studied a filtration problem. It has subsequently been employed by H. Brézis and G. Stampacchia [5], V. Benci [2], Duvaut [6], and also in [12].

A characteristic of the present work is the logarithmic nature of a function \( g \) defined by (1.5) at \( z = 0 \). This difficulty will be overcome by considering an unbounded obstacle.

In the following section we transform our problem to one concerning a variational inequality. In § 3 we solve the variational inequality. With the aid of [4] we are able to show in § 5 that \( \Gamma \) is a Jordan curve represented by a continuous function of the central angle \( \theta \). In § 6 we use a result of [8] to conclude the smoothness of \( \Gamma \) and the existence of a classical solution to Problem 1.

2.

In this section we introduce a variational inequality and determine its relationship to Problem 1. We begin with some notations. Set \( B_r = \{ z : |z| < r \}, r > 0, \) and (1)

\[
K_r = \{ \nu \in H^1(B_r) : \nu \geq \log \rho \text{ in } B_r \text{ and } \nu = \log r \text{ on } \partial B_r \}.
\]

Define the bilinear form

\[
a(\nu, \zeta) = \int_{B_r} \nu \zeta \, dx = \int_{B_r} \left\{ \nu_\rho \zeta_\rho + \frac{1}{\rho^2} \nu_\theta \zeta_\theta \right\} \rho \, d\rho \, d\theta,
\]

\( \nu, \zeta \in H^1(B_r) \).

We always depress the dependence of \( a(\nu, \zeta) \) on \( r > 0 \). Let

\[
f \in L^p_{\text{loc}}(\mathbb{R}^2) \text{ for some } p > 2.
\]

Problem (\(*\)) — To find a pair \( r > 1 \) and \( \omega \in K_r \) such that

\[
(2.1) \quad \omega \in K_r : a(\omega, \nu - \omega) \geq \int_{B_r} f(\nu - \omega) \, dx \quad \nu \in K_r
\]

(1) Usual notation is employed for function spaces.
and the function $\tilde{w}(z)$ defined by

$$\tilde{w}(z) = \begin{cases} w(z) & z \in B_r \\ \log |z| & z \notin B_r \end{cases}$$ is in $C^1(R^2)$

The existence and other properties of a solution to Problem (*) will be investigated in the next paragraph. We note here that the restriction of $\tilde{w}$ to $B_R$ for $R > r$ will be a solution of (2.1) in $B_R$. Since this means that (2.2) will be automatically satisfied, so that $R, \tilde{w}|_{B_R} \in K_R$ is also a solution to Problem (*), we shall not distinguish between $w$ and $\tilde{w}$ in the sequel.

**Theorem 1.** — Let $\Omega, u$ be a solution of Problem 1 where $F$ satisfies (1.1) and $\gamma > 0$. Suppose that $\Gamma$ is a smooth curve. Then there exists a solution $\tilde{r}, \tilde{w} \in K_{\tilde{r}}$ of Problem (*) for $w = \tilde{w}$ such that

$$Q, = \{z : w(z) > \log \rho\} \text{ and } u(z) = \gamma(1 - \rho w_{\tilde{r}}(z)).$$

The theorem is based on the lemma below which also explains the role of the normal derivative condition in (1.3).

**Lemma 2.1.** — Let $\Omega$ be a simply connected domain containing the origin and $\Gamma' \subset \partial \Omega$ a smooth arc. Let $F \in C^2(R^2)$ satisfy (1.1). Suppose that $u$ satisfies

$$\begin{cases} -\Delta u = \rho^{-1}F_{\rho} & \text{in } \Omega \\ u = 0 & \text{on } \Gamma' \\ \frac{\partial u}{\partial \nu} = -F \frac{d\theta}{ds} & \text{on } \Gamma'. \end{cases}$$

Let $g \in C^1(\overline{\Omega} - \{0\})$ denote any function with the property

$$g_{\rho} = -\rho^{-1}u \text{ in } \overline{\Omega} - \{0\} \text{ and } \Delta g \in C(\overline{\Omega} - \{0\}).$$

Let $\zeta \in C^\infty_c(R^2)$ vanish in a neighborhood of $\partial \Omega - \Gamma'$ and $z = 0$. Then

$$\int_{\Gamma'} \zeta_{\rho}^2 \Delta g \, d\theta = \int_{\Gamma'} \zeta F \, d\theta - \int_{\Gamma'} g_\theta(\zeta_{\rho} \, d\rho + \zeta_{\theta} \, d\theta)$$
Proof. — First we compute $\Delta g$ in $\Omega$. For this, observe that

$$- F_\varphi = (\rho u_\varphi)_\varphi + \rho^{-1} u_{\theta\theta}$$

$$= - (\rho (\rho g_\varphi)_\varphi - g_{\rho\theta})$$

$$= - \frac{\partial}{\partial \varphi} \{ \rho (\rho g_\varphi)_\varphi + g_{\theta\theta} \}$$

$$= - \frac{\partial}{\partial \varphi} (\rho^2 \Delta g).$$

Hence

$$\frac{\partial}{\partial \varphi} (\rho^2 \Delta g) = F_\varphi \text{ in } \Omega.$$  \hfill (2.4)

Let $\zeta \in C_0^\infty(B_r)$, where $\overline{\Omega} \subset B_r$, satisfy $\zeta = 0$ in a neighborhood of $\partial \Omega - \Gamma'$ and $z = 0$. Then observing that

$$- F \ d\theta = \frac{\partial u}{\partial \nu} \ ds = \rho u_\varphi \ d\theta - \frac{1}{\rho} \ u_\theta \ d\varphi,$$

$$- \int_{\Gamma'} F_\varphi \ z \ d\theta = \int_{\Gamma'} \left( \rho u_\varphi \ d\theta - \frac{1}{\rho} \ u_\theta \ d\varphi \right)$$

$$= \int_{\Omega} \left( (\rho u_\varphi) + \frac{1}{\rho} \ u_{\theta\theta} \right) \ d\varphi \ d\theta + \int_{\Omega} \left( \rho u_\varphi \ z_\varphi + \frac{1}{\rho} \ u_\theta \ z_\theta \right) \ d\varphi \ d\theta$$

$$= - \int_{\Omega} \zeta F_\varphi \ d\varphi \ d\theta - \int_{\Omega} \{ \rho (\rho g_\varphi)_\varphi \z_\varphi + g_{\rho\theta} z_\theta \} \ d\varphi \ d\theta$$

$$= - \int_{\Omega} \{ \rho (\rho^2 \Delta g - g_{\theta\theta}) \z_\varphi + g_{\rho\theta} z_\theta \} \ d\varphi \ d\theta$$

$$= - \int_{\Omega} \{ \rho \z_\varphi + \rho^2 \Delta g \z_\varphi \} \ d\varphi \ d\theta + \int_{\Omega} \{ g_{\theta\theta} \z_\varphi - g_{\rho\theta} z_\theta \} \ d\varphi \ d\theta.$$

We evaluate the first integral by (2.4). Hence

$$\int_{\Omega} \{ F_\varphi \z_\varphi + \rho^2 \Delta g \z_\varphi \} \ d\varphi \ d\theta = \int_{\Omega} \left( \z_\varphi \rho^2 \Delta g \right) \ d\varphi \ d\theta$$

$$= \int_{\Gamma'} \z_\varphi^2 \Delta g \ d\theta.$$

Turning to the second integral, we compute that

$$\int_{\Omega} \{ g_{\theta\theta} \z_\varphi - g_{\rho\theta} z_\theta \} \ d\varphi \ d\theta = \int_{\Omega} \left\{ (g_{\theta} \z_\varphi)_\varphi - (g_{\theta} \z_\varphi)_\theta \right\} \ d\varphi \ d\theta$$

$$= \int_{\Gamma'} \left\{ g_{\theta} \z_\varphi \ d\varphi + g_{\theta} \z_\theta \ d\theta \right\}.$$  \hfill (2.5)

Finally, we obtain that

$$\int_{\Gamma'} \ z_\varphi \ d\theta = \int_{\Gamma'} \rho^2 \Delta g \z \ d\theta + \int_{\Gamma'} g_{\theta} (\z_\varphi \ d\varphi + \z_\theta \ d\theta). \quad \text{Q.E.D.}$$
Lemma 2.2. — Let $\Omega$, $u$ be a solution to Problem 1 and suppose that $\Gamma = \partial \Omega$ is smooth. Set $u = 0$ in $\mathbb{R}^2 - \Omega$. Let $r$ be large enough that $\overline{\Omega} \subset B_r$ and choose

$$(2.5) \quad g(z) = \int_0^r t^{-1} u(t, 0) \, dt, \quad |z| = \rho, \quad 0 \neq z \in B_r$$

Then

$$g \in C^1(\overline{\Omega} - \{0\}), \quad \Delta g \in C(\overline{\Omega} - \{0\}),$$

and moreover

$$\Omega = \{z : g(z) > 0\} \quad \text{and} \quad \Omega = g^{-1}(0) \cap B_r - \Omega.$$

Proof. — As we remarked in the introduction, smoothness of $\Gamma$ implies that $\Omega$ is starshaped with respect to $z = 0$. Hence if $g(z) = 0$ for $z = \rho e^{i\theta}$, then the non-negative continuous integrand in (2.5) vanishes for $te^{i\theta}, t > \rho$, so that $g(te^{i\theta}) = 0, t > \rho$. Therefore, since $u > 0$ in $\Omega$, we see that $g(z) > 0$ in $\Omega - \{0\}$ and $g(z) = 0$ in $B_r - \Omega$. Because $u$ is smooth in $\Omega$ it is easy to derive that $g \in C^1(B_r - \{0\})$. On the other hand $g$ attains its minimum on $B_r - \Omega$ whence

$$(2.6) \quad g_\theta = 0 = g_0 \quad \text{on} \quad B_r - \Omega.$$

Since $g_\rho = -\rho^{-1} u$ in $\Omega$, by (2.4),

$$(2.7) \quad \frac{\partial}{\partial \rho} (\rho^2 \Delta g) = F_\rho \quad \text{in} \quad \Omega.$$

We may integrate (2.7) in $\Omega$ since $\Omega$ is starshaped to obtain

$$\rho^2 \Delta g(z) = F(z) + \psi(0), \quad z = \rho e^{i\theta} \in \Omega,$$

where $\psi$ is a function of the central angle $\theta$ only. Now by Lemma 2.1

$$\int_{\Gamma} \zeta F(z) \, d\theta + \int_{\Gamma} \psi(\theta) \zeta \, d\theta = \int_{\Gamma} F\zeta \, d\theta - \int_{\Gamma} g_\phi(\zeta_\phi \, d\rho + \zeta_\theta \, d\theta)$$

for $\zeta \in C^\infty_0(B_r - \{0\})$. Since $g_0 = 0$ on $\Gamma \subset B_r - \Omega$ (cf. 2.6),

$$\int_{\Gamma} \psi(\theta) \zeta \, d\theta = 0 \quad \zeta \in C^\infty_0(B_r - \{0\})$$

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or
\[ \psi(\theta) = 0, \quad 0 \leq \theta < 2\pi. \]

Q.E.D.

**Proof of Theorem 1.** — As we have observed, \( \Omega \) is star-shaped with respect to \( z = 0 \) so the function \( g(z) \) defined by (2.5) satisfies the conclusions of Lemma 2.2. Let \( r \) be so large that \( \overline{\Omega} \subset B_r \) and define
\[
\omega^*(z) = \frac{1}{\gamma} g(z) + \log \rho \quad 0 \neq z \in B_r,
\]
\[
= \frac{1}{\gamma} \int_0^r t^{-1} u(t, \theta) - \gamma dt + \log r
\]
where \( \gamma = u(0) > 0 \). We shall show that \( r, \omega^* \in K_r \) is a solution to. **Problem (\ast).** Clearly \( \omega^* \) is bounded in \( B_r \) and satisfies
\[
(2.8) \quad - \Delta \omega^* = \begin{cases} f & \text{in } \Omega - \{0\} \\ 0 & \text{in } B_r - \Omega \end{cases} \quad \text{a.e.}
\]
by **Lemma 2.2** where \( f(z) = - \frac{1}{\gamma \rho^2} F(z) \). Since \( f \in C^2(\mathbb{R}^2) \), cf. (1.1), it follows from Riemann's Theorem on removable singularities that \( \omega^* \) is smooth in \( \Omega \). We observe that
\[
\omega^*(z) \geq \log \rho \quad \text{since} \quad g(z) \geq 0
\]
and \( \Omega = \{z: \omega^*(z) > \log \rho\} \). Further, \( \overline{\Omega} \subset B_r \) implies that, for \( |z| = r \),
\[
\omega^*_0(z) = \log r \\
\omega^*_1(z) = 1/r \quad \text{and} \quad \omega^*_0(z) = 0
\]
Therefore, \( \omega^* \in K_r \) and the function
\[
\tilde{\omega}^*(z) = \begin{cases} \omega^*(z) & z \in B_r \\ \log \rho & z \notin B_r \end{cases}
\]
is a \( C^1(\mathbb{R}^2) \) function. Hence (2.2) holds.

It is easy to verify (2.1). Let \( \nu \in K_r \). Then
\[
a(\omega^*, \nu - \omega^*) = \int_\Omega f(\nu - \omega^*) \, dx
\]
by (2.8) and an integration by parts, valid since \( \omega^* \in C^1(\overline{\Omega}) \).
Indeed, \( \omega^* \in C^1(\mathbb{R}^2) \), as noted above. Hence
\[
a(\omega^*, \nu - \omega^*) - \int_{B_r} f(\nu - \omega^*) \, dx = - \int_{B_r - \Omega} f(\nu - \omega^*) \, dx.
\]
Since \( f \leq 0 \) in \( B_r \) and \( \nu \in K_r \) implies
\[
0 \leq \nu - \log \rho = \nu - \omega^* \quad \text{in} \quad B_r - \Omega,
\]
the last integral is non-negative so that
\[
a(\omega^*, \nu - \omega^*) \geq \int_{B_r} f(\nu - \omega^*) \, dx \quad \nu \in K_r \quad \text{Q.E.D.}
\]

3.

This paragraph is devoted to the solution of the variational inequality Problem (*). According to a well known theorem [11], there is a solution to (2.1) for each \( r > 0 \). To establish its smoothness in \( B_r \), we shall prove that it is bounded. For once this is known, the obstacle \( \log \rho \) may be replaced by a smooth obstacle \( \psi \) which equals \( \log \rho \) when
\[
\log \rho > - \| \omega \|_{L^\infty(B_r)}
\]
and (2.1) may be solved in the convex \( K_\psi \) of \( H^1(B_r) \) functions which exceed \( \psi \) in \( B_r \) and satisfy the boundary condition \( \nu(z) = \log r, |z| = r \). The solution to this latter problem is known to be suitable smooth (cf. [10]) and is easily shown to be the solution of (2.1).

**Lemma 3.1.** — Let \( f \in L^p(B_r) \) for some \( p > 2 \) and satisfy
\[
f \leq 0 \quad \text{in} \quad B_r.
\]
Then the solution \( \omega \) of (2.1) for \( f \) satisfies
\[
\log r - c\| f \|_{L^p(B_r)} \leq \omega(z) \leq \log r \quad \text{in} \quad B_r,
\]
where \( c = c(r, p) > 0 \).

**Proof.** — Let \( \omega_0 \) denote the solution to the Dirichlet problem
\[
- \Delta \omega_0 = f \quad \text{in} \quad B_r
\]
\[
\omega_0 = 0 \quad \text{on} \quad \partial B_r.
\]
We know that \( \omega_0 \in H^{2,p}(B_r) \) and
\[
\| \omega_0 \|_{L^\infty(B_r)} \leq c\| f \|_{L^p(B_r)}, \quad c = c(r, p) > 0.
\]
Consequently, for any $\zeta \in H^1_0(B_r)$,
\[ a(w - w_0, \zeta) = a(w, \zeta) - \int_{B_r} f \zeta \, dx. \]
We define $\nu = \max (w, w_0 + \log r) \in K_r$ so by (2.1)
\[ a(w - w_0, \nu - w) \geq 0 \]
Further, computing explicitly, we find
\[ a(w - w_0, \nu - w) = \int_{B_r} (w - w_0)_{x_i} (\nu - w)_{x_i} \, dx \]
\[ = -\int_{\nu > w} (w - w_0)_{x_i} \, dx \leq 0. \]
Hence $\text{meas} \{\nu > w\} = 0$ or $\log r + w_0 \leq w$ a.e. This proves the lower bound in view of (3.1). The same argument may be employed to prove the upper bound, with
\[ \nu = \min (w, \log r), \]
using that $f \leq 0$ in $B_r$. Q.E.D.

For general $f$, we observe that an upper bound for the solution of (2.1) is
\[ \log r + c(r, p) \| \max (0, f) \|_{L^p(B_r)}. \]

**Corollary 3.2.** Let $f \in L^p(B_r)$ for some $p > 2$, $f \leq 0$ in $B_r$, and let $w$ denote the solution to (2.1) for $f$. Then $w \in H^{2, p}(B_r)$. If $f \in C^1(\overline{B_r})$, then $w \in H^{2, \infty}_{\text{loc}}(B_r)$.

**Proof.** — This is clear from the remarks preceding the proof of the lemma. In particular, that $w \in H^{2, \infty}_{\text{loc}}(B_r)$ follows by a result of Frehse [7] (cf. also [4]).

**Lemma 3.3.** Let $g \in H^1(B_r)$ satisfy
\[ g \geq \log p \quad \text{in } B_r \]
and
\[ a(g, \zeta) - \int_{B_r} f \zeta \, dx \geq 0 \quad \text{for } 0 \leq \zeta \in H^1_0(B_r). \]
Let $w$ denote the solution of Problem (*) for $f \in L^p(B_r)$, for some $p > 2$. Then $w \leq g$ in $B_r$.

**Proof.** — This is a familiar property of supersolutions. cf. [10], [11].
THEOREM 2. — Let \( f \in L^p_{\text{loc}}(\mathbb{R}^2) \) for a \( p > 2 \) satisfy
\[
\sup_{\mathbb{R}^2} f < 0.
\]
Then there exists a solution \( r, \omega \in K_r \) to Problem \((*)\). In addition, \( \omega \in H^{2,p}(B_r) \).

Proof. — We shall construct a supersolution \( g(z) = h(\rho) \) to the form
\[
a(\omega, \zeta) - \int_B f(x) \zeta dx,
\]
for some \( r > 1 \), which satisfies
\[
(3.2) \quad h \in K_r
\]
\[
(3.3) \quad h(\rho) = 1 \quad \frac{1}{r}.
\]
Indeed, suppose that
\[
0 < \beta \leq -\sup_{\mathbb{R}^2} f \quad \text{and} \quad \beta < 2e^{-1},
\]
and define
\[
h(\rho) = \alpha + \frac{1}{4} \beta \rho^2.
\]
Then
\[
- \Delta h = - \frac{1}{\rho} (\rho h')' = - \beta \geq \sup f
\]
Assume for the moment that \((3.2)\) and \((3.3)\) are fulfilled. Then
\[
\omega \leq h \quad \text{in } B_r
\]
by the previous lemma. Moreover, since \( \log \rho \leq \omega \leq h \) we conclude from \((3.3)\) that
\[
\omega(\rho) = \frac{1}{r} \quad \text{for } |z| = r
\]
and, since \( \omega = \log r \) on \( |z| = r \),
\[
\omega_\rho(\zeta) = 0 \quad \text{for } |z| = r.
\]
Therefore \( \omega \) defined by \((2.2)\) is in \( C^1(\mathbb{R}^2) \).

It remains to find \( \alpha \) and \( r \) from the conditions \((3.2)\),
One discovers that
\[ r = \left( \frac{2}{\beta} \right)^{1/2} \geq 1 \]
and
\[ x = \log r - \frac{1}{2} = \frac{1}{2} \left( \log \frac{2}{\beta} - 1 \right) > 0. \]

To verify that \( h \in K_r \), i.e., to verify that \( h(\rho) \geq \log \rho \) knowing that \( h(r) = \log r \), note that \( h(\rho) - \log \rho \) is strictly convex and attains its (unique) minimum at the \( \rho \) where \( h_\rho = \frac{1}{\rho} = 0 \). This \( \rho = r \). Q.E.D.

We wish to point out here that ideas similar to those in the proof of Theorem 2 were also studies by H. Brezis [3].

**Corollary 3.4.** — Let \( f \in L^p_{\text{loc}}(\mathbb{R}^2) \) for \( p > 2 \) satisfy \( \sup f < 0 \). Let \( r, \omega \in K_r \) denote the solution to Problem (*) for \( f \). Then for \( R > r \), the pair \( R, \tilde{\omega} \in K_R \), where \( \tilde{\omega} \) is defined by (2.2) is a solution to Problem (*).

In view of this Corollary, we shall not distinguish between \( \omega \) and \( \tilde{\omega} \) in the sequel. Furthermore, we recall that \( \omega \in H^1_{\text{loc}}(\mathbb{R}^2) \) whenever \( f \in C^1(\mathbb{R}^2) \).

**Proof.** — We need only verify (2.1) in \( B_R \). Let \( \zeta \in C_0^\infty(B_R) \). Then

\[ a(\tilde{\omega}, \zeta) = \int_{B_R} \omega \zeta_{x_i} dx + \int_{B_R} \frac{\partial}{\partial x_i} \log \rho \zeta_{x_i} dx \]
\[ = - \int_{B_R} \Delta \omega \zeta dx + \int_{|z|=r} \omega \zeta r d\theta + \int_{B_{R-\rho}} \Delta \log \rho \zeta dx \]
\[ - \int_{|z|=r} \frac{1}{r} \zeta r d\theta \]

since \( \zeta \) has support in \( B_R \). Now \( \tilde{\omega} \in C^1(B_R) \) implies, in particular, that \( \omega_{\tilde{\omega}}(z) = \frac{1}{r} \) for \( |z| = r \) and the two integrals over \( |z| = r \) cancel. Hence

\[ a(\tilde{\omega}, \zeta) = - \int_{B_R} \Delta \omega \zeta dx \]
\[ = \int_\Omega f \zeta dx, \quad \Omega = \{ z : \omega(z) > \log \rho \}. \]
Now given $\nu \in \mathcal{K}_R$,  
\[ a(\mathcal{\tilde{w}}, \nu - \mathcal{\tilde{w}}) - \int_{B_{\mathcal{\tilde{w}}}} f(\nu - \mathcal{\tilde{w}}) \, dx = - \int_{B_{\mathcal{\tilde{w}}} - \Omega} f(\nu - \mathcal{\tilde{w}}) \, dx \geq 0 \]
where the last integral is non-negative because $\mathcal{\tilde{w}} = \log \nu$ in $B_{\mathcal{\tilde{w}}} - \Omega$ and $f < 0$. This verifies (2.1). Q.E.D.

4.

Here we show that the set where the solution to Problem (*) exceeds $\log \nu$ is starshaped under an assumption about $f$. First we prove a lemma which is useful also in the succeeding sections. It is a form of converse to Lemma 2.1 with an analogous proof.

**Lemma 4.1.** — Let $f \in L^p_{\text{loc}}(\mathbb{R}^2)$ for some $p > 2$ satisfy $\sup f < 0$. Let $r, w \in \mathcal{K}_r$ denote the solution to Problem (*) for $f$ and define 
\[ u(z) = 1 - p^w, z \in B_r \]
and 
\[ \Omega = \{ z \in B_r : w(z) > \log \nu \}. \]

i) Then $u \in H^{1,p}(B_r)$.
ii) Let $\omega \subset B_r$ be open and suppose that $- \Delta w = f$ in $\omega$. Then 
\[ - \Delta u = - p^{-1}(p^2f)_{\nu} \quad \text{in} \quad \omega \]

iii) Suppose that $f \in C^1(B_r)$ and that $\Gamma'$ is a smooth (open) arc in $\partial \Omega$. Then 
\[ \frac{\partial u}{\partial \nu} = \rho^2 f \frac{d\theta}{ds} \quad \text{on} \quad \Gamma' \]
where $\nu$ denotes the outward directed normal vector on $\Gamma'$.

**Proof.** — Since $f \in L^p_{\text{loc}}(\mathbb{R}^2)$, $p > 2$, $w \in H^{3, p}(B_r)$, so $u = 1 - \sum x_i \frac{\partial u}{\partial x_i} \in H^{1, p}(B_r)$. The statement (4.1) will be understood in the sense of distributions.
Let \( \zeta \in C^\infty_0(\omega) \). Then
\[
\int_\omega u_{x_1} \zeta_{x_1} \, dx = \int_\omega \left( \rho u_\partial \zeta_\partial + \frac{1}{\rho} u_\theta \zeta_\theta \right) \, d\rho \, d\theta
\]
\[
= \int_\omega \left\{ \rho (1 - \rho \omega_\rho) \zeta_\partial + \frac{1}{\rho} (1 - \rho \omega_\rho) \zeta_\theta \right\} \, d\rho \, d\theta
\]
\[
= - \int_\omega \left\{ \rho (\rho \omega_\rho) \zeta_\partial + \omega_\theta \zeta_\theta \right\} \, d\rho \, d\theta.
\]

We integrate by parts in the last term, first with respect to \( \rho \) and then with respect to \( \theta \), to obtain
\[
\int_\omega u_{x_1} \zeta_{x_1} \, dx = \int_\omega \frac{1}{\rho} (\rho^2 f)_\rho \zeta_\rho \, d\rho \, d\theta.
\]

We turn now to the proof of iii). Suppose that \( \Gamma' \) has a Hölder continuous tangent vector as a function of the arc-length parameter. In \( \Omega \), that \( \omega(z) > \log \rho \) implies
\[- \Delta \omega = f,\]
whence
\[- \Delta u = - \frac{1}{\rho} (\rho^2 f)_\rho \quad \text{in} \quad \Omega.
\]
Moreover, \( \omega_\rho(z) = \frac{1}{\rho} \) for \( z \in \delta \Omega \) so \( u = 0 \) on \( \Gamma' \subset \delta \Omega \).

From this and the fact \( f \in C^1(\overline{\Omega}) \) we may conclude that \( u \in C^{1,\lambda}(\Omega \cup \Gamma') \) for some \( \lambda > 0 \). Let \( \zeta \in C^\infty_0(\overline{\Omega}) \) with \( \text{supp} \, \zeta \cap (\delta \Omega - \Gamma') = \emptyset \). Then
\[
(4.3) \quad \int_{\Gamma'} u_\nu \zeta \, ds = \int_{\Gamma'} \zeta \left( \rho u_\nu \, d\theta - \frac{1}{\rho} u_\theta \, d\rho \right)
\]
\[
= \int_{\Gamma'} \zeta \left( (\rho u_\rho)_\rho + \frac{1}{\rho} u_\theta \right) \, d\rho \, d\theta + \int_{\Omega} \zeta_\rho (\rho u_\rho + \frac{1}{\rho} u_\theta) \, d\rho \, d\theta
\]
\[
= \int_{\Omega} \zeta (\rho^2 f)_\rho \, d\rho \, d\theta
\]
\[
- \int_{\Omega} \left\{ \rho (\rho \omega_\rho) \zeta_\partial + \omega_\theta \zeta_\rho - \omega_\theta \zeta_\rho + \omega_\rho \zeta_\theta \right\} \, d\rho \, d\theta
\]
\[
= \int_{\Omega} \zeta (\rho^2 f)_\rho \, d\rho \, d\theta + \int_{\Omega} \left( \omega_\theta \zeta_\rho - \omega_\rho \zeta_\theta \right) \, d\rho \, d\theta.
\]
Since \( -\Delta \omega = f \) in \( \Omega \), we evaluate the first integral to yield
\[
(4.4) \quad \int_{\Gamma'} ((\rho^2 f)_{\rho} \zeta - \rho^2 \Delta \omega \zeta_{\rho}) \, d\rho \, d\theta = \int_{\Gamma} \zeta_{\rho} \rho^2 f \, d\theta.
\]
On the other hand, \( \omega_0 = 0 \) on \( \Gamma' \subset B_r - \Omega \), therefore
\[
\int_{\Omega} (\omega_{0\rho} \zeta_{\rho} - \omega_{\rho 0} \zeta_0) \, d\rho \, d\theta = \int_{\Omega} \{(\omega_0 \zeta_{\rho})_{\rho} - (\omega_0 \zeta_{\rho})_{\rho}\} \, d\rho \, d\theta
\]
\[= - \int_{\Gamma} \omega_0 (\zeta_{\rho} \rho + \zeta_0 \theta) \, d\theta = 0.
\]
Finally, from (4.3) and (4.4) we obtain that
\[
\int_{\Gamma} u \zeta \, ds = \int_{\Gamma} \rho^2 f \zeta \, ds, \quad \zeta \in C^1_0(B_r), \quad \text{supp} \, \zeta \cap (\partial \Omega - \Gamma') = \emptyset.
\]

**Theorem 3.** — Let \( f \in L^p_{\text{loc}}(\mathbb{R}^2) \) satisfy \( \sup f < 0 \) and \( \varphi^{-1}(\varphi^2 f) \rho \leq 0 \). Let \( r, \omega \in K_r \) denote the solution of Problem (*) for \( f \) and set
\[
\Omega = \{ z : \omega(z) > \log \varphi \}
\]
Then \( \Omega \) is starshaped with respect to \( z = 0 \).

**Proof.** — Consider, as in the preceding proposition,
\[
u(z) = 1 - \rho \varphi \omega(z), \quad z \in B_r,
\]
and note that \( u \in C^{0,1-\frac{2}{p}}(B_r) \) and \( u = 0 \) on \( \Gamma \subset B_r - \Omega \), \( \Gamma = \partial \Omega \). By the hypothesis on \( f \) and (4.1),
\[
\int_{\Omega} u \varphi \zeta_{\rho} \, dx = - \int_{\Omega} \rho^{-1}(\rho^2 f)_{\rho} \zeta \, dx \geq 0 \quad \text{for} \quad 0 \leq \zeta \in C^1_0(\Omega).
\]
The maximum principle may now be applied to conclude that
\[
u(z) \geq \min_{\Gamma} u = 0 \quad \text{for} \quad z \in \Omega.
\]
Hence the function
\[
g(z) = - \log \rho + \omega(z), \quad 0 \neq z \in B_r
\]
is decreasing on each ray \( \rho e^{i\theta}, 0 < \rho < r \), because it has derivative
\[
g_{\rho}(z) = - \frac{1}{\rho} (1 - \rho \varphi \omega(z)) = - \frac{1}{\rho} u(z) \leq 0, \quad z \in B_r, \quad z \neq 0.
\]
Therefore, given \( z = \rho e^{i\theta} \) with \( \omega(z) > \log \rho \), then

\[
\omega(te^{i\theta}) > \log t \quad \text{for} \quad t \leq \rho.
\]

This proves that \( \Omega \) is starshaped.

Q.E.D.

5.

In this paragraph we initiate the study of the free boundary determined by a solution to Problem (*) . To begin, we fix an \( f \in C^1(\mathbb{R}^2) \) which satisfies

\[
\sup_{\mathbb{R}^2} f < 0 \quad \text{and} \quad (\partial^2 f)_{\rho} \leq 0 \quad \text{in} \quad \mathbb{R}^2
\]

and let \( r, \omega \in K_r \) denote the solution to Problem (*) for \( f \).

As before, set

\[
\Omega = \{z : \omega(z) > \log \rho \}
\]

and let

\[
E = \overline{B}_r - \Omega.
\]

Observe that, by Theorem 3, \( E \) is starshaped with respect to the point at \( \infty \) in the sense that

\[
z \in E, \ t \geq 1 \quad \text{and} \quad |tz| \leq r \quad \text{implies} \quad tz \in E.
\]

Define

\[
\mu(\theta) = \inf \{\rho : z = \rho e^{i\theta} \in E\}, \ 0 \leq \theta < 2\pi,
\]

Note that \( \mu(\theta) \) is lower semicontinuous since \( E \) is closed.

For given \( z_n = \rho_n e^{i\theta_n} \), \( \rho_n = \mu(\theta_n) \), and \( z_n \to z = \rho e^{i\theta} \), we conclude that \( z \in E \) and hence \( \rho \geq \mu(\theta) \). In addition

\[
E = \{z = \rho e^{i\theta} : \mu(\theta) \leq \rho \leq r\}
\]

by the starshaped quality of \( E \) and \( \Omega \). In the next lemma, we utilize that the characteristic function of \( E, \varphi_E, \) is of bounded variation in \( \mathbb{R}^2 \) which follows from [4] (Corollary 2.1).

**Lemma 5.1.** — Let \( f \) satisfy (5.1). Then \( \mu(\theta) \) defined by (5.2) is a lower semi-continuous function of bounded variation.
Proof. — The characteristic function of $E$, $\varphi_E \in BV(R^2)$ as we have noted. This means that
\[ \left| \int_{R^2} \varphi_E \zeta_{x_i} \, dx \right| \leq C \sup_{R^2} |\zeta|, \quad \zeta \in H^{1,\infty}_0(R^2) \]
for $i = 1, 2$ and some $C > 0$. Hence by Fubini's Theorem and (5.3)
\[
\int_0^{2\pi} \int_{\mu(\theta)}^r \zeta_{x_i} \rho \, d\rho d\theta = \int_0^{2\pi} \int_0^r \varphi_E \zeta_{x_i} \rho \, d\rho d\theta \\
= \int_{R^2} \varphi_E \zeta_{x_i} \rho \, d\rho d\theta \\
\leq C \|\zeta\|_{L^\infty(R^2)} \text{ for } \zeta \in H^{1,\infty}_0(R^2).
\]

In particular, we choose $\zeta = \zeta(\theta) \in C^1([0,2\pi])$, periodic of period $2\pi$, and $\eta(\rho)$ a function vanishing identically in a neighborhood of 0 in $\Omega$, identically one in a neighborhood of $E$, and vanishing outside, say, $B_{2\pi}$. Applying the above to the product $\zeta(\theta) \eta(\rho)$ we see that
\[
\int_0^{2\pi} \int_{\mu(\theta)}^r \left( \frac{1}{\rho} \zeta' \right) \rho \, d\rho d\theta = - \int_0^{2\pi} \zeta'(\theta)(\rho - \mu(\theta)) \, d\theta \\
= \int_0^{2\pi} \mu(\theta) \zeta'(\theta) \, d\theta
\]
and hence, by the foregoing,
\[
\left| \int_0^{2\pi} \mu(\theta) \zeta'(\theta) \, d\theta \right| \leq C \sup_{0 \leq \theta \leq 2\pi} |\zeta|, \quad \zeta \in C^1([0,2\pi]).
\]

We may invoke the Riesz Representation Theorem to the functional
\[ \zeta \to \int_0^{2\pi} \zeta'(\theta) \mu(\theta) \, d\theta \]
defined and uniformly bounded on the dense subset $C^1([0,2\pi])$ of $C^0([0,2\pi])$ to infer the existence of $g(\theta) \in BV(0,2\pi)$ with the properties
\[
\int_0^{2\pi} \zeta'(\theta) \mu(\theta) \, d\theta = - \int_0^{2\pi} \zeta(\theta) \, dg(\theta) = \int_0^{2\pi} \zeta'(\theta) g(\theta) \, d\theta.
\]
In particular, $\mu(\theta) - g(\theta) = \text{const. a.e.}$, which we may take to be zero, so that
\[
(5.4) \quad \mu(\theta) = g(\theta) \text{ a.e. in } [0,2\pi].
\]
We proceed to show that \( \mu(\theta) = g(\theta) \) everywhere. We may assume that \( g \) is lower semicontinuous. Let us agree to further modify \( g \) so that

\[
(5.5) \quad g(\theta) = \liminf_{t \to \theta} g(t)
\]

It follows that \( \mu(\theta) \leq g(\theta) \). For suppose that \( g(\theta) < \mu(\theta) \) and select \( \theta_k \to \theta \) such that \( g(\theta_k) = \lim_{k \to \infty} g(\theta_k) \). Since \( \mu \) is lower semi-continuous given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[
\mu(\theta) - \varepsilon < \mu(t) \quad \text{for} \quad |t - \theta| < \delta.
\]

Hence for \( k \) so large that

\[
|g(\theta_k) - g(\theta)| < \varepsilon
\]

we may find a neighborhood \( I_k = (\theta_k - \delta_k, \theta_k + \delta_k) \),

\[
I_k \cap I_h = \emptyset \quad \text{for} \quad h \neq k,
\]

of \( \theta_k \) with the property

\[
\Var g \geq \max_{I_k} g - \min_{I_k} g \geq g(t) - (g(\theta) - \varepsilon) \quad \text{for any} \quad t \in I_k
\]

\[
\geq \mu(t) - (g(\theta) - \varepsilon) \quad \text{for almost all} \quad t \in I_k
\]

by (5.4). Hence, by our choice of \( \varepsilon \),

\[
\Var g \geq \mu(\theta) - g(\theta) - 2\varepsilon > 0
\]

Consequently, \( \Var g = +\infty \), a contradiction. Therefore once (5.5) is assumed, \( \mu(\theta) \leq g(\theta) \) in \([0, 2\pi]\). Observe that \( g \) satisfying (5.5) has no inessential discontinuities.

Consider the set

\[
F = \{z : \rho e^{i\theta} : g(\theta) \leq \rho \leq r\} \subset E \quad \text{since} \quad \mu \leq g.
\]

Since the points \( \theta \) in \([0, 2\pi]\) for which \( g \neq \mu \) have measure zero,

\[
N = E - F = \{z = \rho e^{i\theta} : \mu(\theta) \leq \rho < g(\theta)\}
\]

satisfies \( \meas N = 0 \). Furthermore \( F \) is closed by lower semi-continuity of \( g \) so \( \overline{B_r - F} \) is open, \( \Omega \subset \overline{B_r - F} \), and

\[
\overline{B_r - F} = \Omega \cup N.
\]
Recall here that \( \varphi \in H^{3, \infty}(B_r) \) since \( f \in C^1(B_r) \) by Corollary 3.2. Inasmuch as \( -\Delta \varphi = f \) in \( \Omega \), we see that \( -\Delta \varphi = f \) a.e. in \( \Omega \cup N \). Since \( \Omega \cup N \) is open, we may deduce that

\[
-\Delta \varphi = f \quad \text{in} \quad \Omega \cup N
\]

and

\[
\varphi \in C^2,\lambda(\Omega \cup N) \quad \text{for} \quad 0 < \lambda < 1.
\]

Now consider \( u(z) = 1 - \rho \varphi_{\rho}(z), \ z \in B_r \), which satisfies

\[
\int_{\Omega \cup N} u_{x_i} \zeta_{x_i} \, dx = -\int_{\Omega \cup N} \frac{1}{\rho} (\rho^2 f) \varphi \, \zeta \, dx, \ \zeta \in C^0_0(\Omega \cup N)
\]

by Lemma 4.2 (ii). Hence \( u \in C^1(\Omega \cup N) \) and

\[
\int_{\Omega \cup N} u_{x_i} \zeta_{x_i} \, dx > 0 \quad \text{when} \quad 0 \leq \zeta \in C^0_0(\Omega \cup N)
\]

so that by the strong maximum principle

\[
u(z) > \min_{\Omega \cup N} u = 0
\]

because \( \delta(\Omega \cup N) \subset B_r - \Omega \) where \( \varphi_{\rho} = \frac{1}{\rho} \) and \( \omega_0 = 0 \).

In particular, \( u(z) = 0 \) for \( z \in \delta(\Omega \cup N) \). However, if \( z \in N \)

\[
\varphi_{\rho}(z) = \frac{1}{\rho} \quad \text{and} \quad \omega_0(z) = 0
\]

so that

\[
u(z) = 1 - \rho \varphi_{\rho}(z) = 0,
\]

a contradiction. Therefore \( N = \emptyset \), and

\[
\mu(\theta) = g(\theta), \quad 0 \leq \theta \leq 2\pi. \quad \text{Q.E.D.}
\]

**Theorem 4.** — Let \( f \in C^1(R^3) \) satisfy (5.1) and let \( r, \omega \in K_r \), denote the solution to Problem (*) for \( f \). Let

\[
\Omega = \{z: \omega(z) > \log \rho\}.
\]

Then the boundary \( \Gamma \) of \( \Omega \) has the representation

\[
\Gamma: \rho = \mu(\theta), \quad 0 \leq \theta \leq 2\pi
\]

where \( \mu \) is a continuous function of bounded variation.
Proof. — Let \( \mu(\theta) \) be defined by (5.2) so that the conclusion of Lemma 5.1 holds. Suppose that \( \theta = 0 \) is a discontinuity of \( \mu \). Then \( \theta = 0 \) is a jump discontinuity so that

\[
\lim_{\theta \to 0^-} \mu(\theta) = L > \lim_{\theta \to 0^+} \mu(\theta) = \mu(0)
\]

without any loss in generality. For \( \varepsilon > 0 \) sufficiently small, there is a \( \delta > 0 \) so that the segments

\[
\{ z = \rho e^{i\theta} : 0 \leq \rho \leq L - \varepsilon \} \subset \Omega \quad \text{for} \quad \delta < \theta < 0
\]

and

\[
\{ z = \rho e^{i\theta} : \mu(0) + \varepsilon \leq \rho \leq \rho \} \subset \mathcal{E}.
\]

Hence we may find a disc \( B_\eta(z_0), z_0 = \frac{1}{2} (L + \mu(0)) \), such that

\[
B_\eta(z_0) \cap \Omega = \{ z \in B_\eta(z_0) : \text{Im} \, z < 0 \}
\]

Let \( \sigma = \{ z : \text{Im} \, z = 0, z_0 - \eta < \text{Re} \, z < z_0 + \eta \} \) and set

\[
u = 1 - \rho \omega_r.
\]

It follows that \( u \in C^1(\sigma \cup \Omega \cap B_\eta(z_0)) \) and \( u \) attains its minimum value zero at each point of \( \sigma \) by Hopf’s maximum principle and Lemma 4.1 (ii). Therefore

\[
\frac{\partial u}{\partial \nu}(z) < 0 \quad \text{for} \quad z \in \sigma.
\]

But according to Lemma 4.1 (iii) with \( \Gamma' = \sigma \)

\[
\frac{\partial u}{\partial \nu}(z) = \rho^2 f(z) \frac{d\theta}{ds}(z) = 0 \quad \text{for} \quad z \in \sigma
\]

since \( \theta = 0 \) on \( \sigma \). This is a contradiction. Q.E.D.

6.

In this paragraph we show that \( \Gamma \) has a smooth parameterization and that a solution to Problem 1 exists in the classical sense. For this, we employ the results of [8]. In the case where \( f \) is real analytic, these questions may be treated by the results of H. Lewy [9].
THEOREM 5. — Let \( f \in C^1(\mathbb{R}^2) \) satisfy \( \sup f < 0 \) and \( (\rho^2 f) \rho \leq 0 \) in \( \mathbb{R}^2 \). Let \( r, w \in K_r \) denote the solution to Problem (*) for \( f \) and \( \Gamma \) the boundary of \( \Omega = \{ z : w(z) > \log \rho \} \). Then \( \Gamma \) has a \( C^{1,\gamma} \) parameterization, \( 0 < \gamma < 1 \).

Proof. — From Theorem 4 it is known that \( \Gamma \) is a Jordan curve. We now apply [8] (Theorem 1). Let \( z_0 \in \Gamma \) and set \( \omega = B_\varepsilon(z_0) \cap \Omega, \varepsilon < |z_0| \), and consider

\[
g(z) = -\frac{1}{z} + \frac{1}{2} (w_{\varepsilon}(z) - iw_{\varepsilon}(z)) \quad z \in \overline{\Omega} - \{0\}.
\]

From the known regularity of \( w, g \in H^{1,\omega}(\omega) \). Furthermore

\[
g(z) = \frac{1}{4} \Delta w(z) = -\frac{1}{4} f(z), \quad z \in \omega
\]

\[
g(z) = 0 \quad z \in \Gamma \cap \overline{\omega}
\]

Since \( -\frac{1}{4} f(z) > 0 \) in \( B_\varepsilon(z_0) \), we may conclude that a conformal mapping \( \varphi \) of \( G = \{|t| < 1, \text{Im} \ t > 0\} \) onto \( \omega \) which maps \( -1 < t < 1 \) onto \( \Gamma \cap \overline{\omega} \) has boundary values in \( C^{1,\gamma} \) for every \( \tau, 0 < \tau < 1 \).

THEOREM 6. — Let \( F \in C^1(\mathbb{R}^2) \) satisfy \( \rho^{-2} F \in C^1(\mathbb{R}^2) \) and

\[
\inf \rho^{-2} F > 0 \quad F_\rho \geq 0 \quad F(0) = F_\rho(0) = 0.
\]

Then there exists a domain \( \Omega \) and a function \( u \in H^{1,\omega}_{loc}(\mathbb{R}^2) \) such that

(6.1) \( -\Delta u = \rho^{-1} F_\rho \quad \text{in} \quad \Omega \)

(6.2) \[
\begin{cases}
  u = 0 \\
  u_{\nu} = -F \frac{d\theta}{ds} \quad \text{a.e. on} \quad \Gamma \\
  u(0) = \gamma
\end{cases}
\]

where \( \nu \) is the outward directed normal vector and \( s \) is the arclength of \( \Gamma \) and \( \gamma > 0 \) is given.

Proof. — Given \( F \), define \( f(z) = -\frac{1}{\gamma \rho^2} F(z) \) and observe
that sup $f < 0$ and $(\rho^2 f)_\rho \leq 0$ in $\mathbb{R}^2$. Denote by $r, \omega \in K_r$ the solution to Problem (*) for $f$ and define
\[
u(z) = \gamma(1 - \rho \omega_\rho(z)) \quad z \in \mathbb{R}^2.
\]
Then, in view of Corollary 3.2, $u \in H^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$ and satisfies (6.1) (by Lemma 4.1), (6.2), and (6.4). Moreover,
\[
\Omega = \{z : u(z) > 0\}.
\]
According to Theorem 5, $\Gamma$ has a $C^{1,\gamma}$ parameterization $t \to \varphi(t)$, $t$ real, where we may assume that
\[
\varphi : \{t : \text{Im} t > 0\} \to \Omega
\]
is a conformal mapping. It is known that $\varphi'(t) \neq 0$ a.e., $-\infty < t < \infty$. In a neighborhood of any $t_0$ for which $\varphi'(t_0) \neq 0$, the tangent angle to $\Gamma$ is of class $C^{0,\gamma}$. From this one checks that $u_\nu$ is continuous in a neighborhood of $\varphi(t_0)$ in $\overline{\Omega}$, e.g., by use of conformal mapping. Now Lemma 4.1 (iii) may by applied to verify (7.3) on this neighborhood of $\varphi(t_0)$ in $\Gamma$.

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