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ON DEFINITIONS OF SUPERHARMONIC FUNCTIONS

by Seizô ITÔ

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1. Introduction.

The classical definition of superharmonic functions by F. Riesz [3] (see also M. Brelot [1]) can be generalized in natural way to the case of the elliptic differential operator $A$ of second order with variable coefficients ($§$ 2 of the present paper). On the other hand, L. Schwartz [4] has defined the superharmonicity with respect to the general elliptic differential operator in view-point of the theory of distribution and given an elegant proof to Riesz decomposition theorem. One may easily prove that the superharmonicity with respect to $A$ (abbreviated to $A$-superharmonicity) of the Riesz-Brelot sense implies that of Schwartz sense in case $A$ is the ordinary Laplacian.

However, in the case of the elliptic differential operator $A$ with variable coefficients, it seems not to be evident that the theory of distribution is applicable to $A$-superharmonic functions in the classical sense; in fact, even the local summability of an $A$-superharmonic function in the classical sense seems not to be trivial.

The purpose of the present paper is to prove that any $A$-superharmonic function in the Riesz-Brelot sense is locally
summable and satisfies the $A$-superharmonicity in the sense of Schwartz distribution. The $A$-superharmonicity in Schwartz sense implies the Riesz decomposition formula as shown in [4], while one may easily see that any function represented by the Riesz decomposition formula is $A$-superharmonic in the Riesz-Brelot sense. Thus we may conclude the equivalence of the $A$-superharmonicity in the Riesz-Brelot sense, that of Schwartz sense and the Riesz decomposition formula for arbitrary elliptic differential operator $A$ of second order with variable coefficients.

2. Main results.

Let $\Omega$ be a subdomain of an orientable $m$-dimensional $C^\infty$-manifold $(m \geq 2)$, and $A$ be an elliptic differential operator of the form:

$$Au(x) = \text{div} \left[ \nabla u(x) + (b(x), \nabla u(x)) + c(x)u(x) \right] = \sum_{i,j} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^j} \left[ \sqrt{a(x)} a^{ij}(x) \frac{\partial u(x)}{\partial x^j} \right] + \sum_i b^i(x) \frac{\partial u(x)}{\partial x^i} + c(x)u(x),$$

where $\|a^{ij}(x)\|$ is a contravariant tensor of class $C^2$ in $\Omega$ and is symmetric and strictly positive-definite for any $x \in \Omega$, $a(x) = \det \|a_{ij}(x)\| = \det \|a^{ij}(x)\|^{-1}$, $b(x) \equiv \|b^i(x)\|$ is a contravariant vector of class $C^2$ in $\Omega$, and $c(x)$ is a Hölder-continuous function satisfying $c(x) \leq 0$ in $\Omega$. We shall denote by $dx$ and $dS(x)$ respectively the volume element and the $m-1$ dimensional hypersurface element with respect to the Riemannian metric defined by the tensor $\|a_{ij}(x)\|$. The formally adjoint operator $A^*$ of $A$ is defined by

$$A^*u(x) = \text{div} \left[ \nabla u(x) - b(x)u(x) \right] + c(x)u(x).$$

By definition, a function $u(x)$ is said to be $A$-harmonic in $\Omega$ if it satisfies $Au = 0$ in $\Omega$, and is said to be $A$-superharmonic in $\Omega$ if it satisfies the following three conditions:

i) $-\infty < u(x) \leq \infty$ and $u(x) \not= \infty$ in $\Omega$,

ii) $u(x)$ is lower semi-continuous in $\Omega$, 

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iii) if $D$ is a domain with its compact closure $\overline{D} \subset \Omega$, and if $w(x)$ is continuous on $\overline{D}$, A-harmonic in $D$ and satisfies $w(x) \leq u(x)$ on $\partial D$, then $w(x) \leq u(x)$ holds in $D$.

We shall prove the following two theorems in § 4.

**Theorem 1.** — Any A-superharmonic function in $\Omega$ is locally summable in $\Omega$.

**Theorem 2.** — Any A-superharmonic function $u(x)$ in $\Omega$ satisfies $Au \leq 0$ in $\Omega$ in the sense of distribution.

3. Preliminary lemmas.

We shall use some properties of fundamental solutions of parabolic equations. The following facts are implied by the results of one of the author’s previous papers [3].

For any subdomain $D$ of $\Omega$ with compact closure $\overline{D} \subset \Omega$ and with boundary $\partial D$ of class $C^3$, there exists one and only one fundamental solution $U_D(t, x, y)$ of the initial-boundary value problem:

\[
(3.1) \quad \frac{\partial u}{\partial t} = Au \quad \text{in} \quad (0, \infty) \times D, \quad u|_{t=0} = u_0, \quad u|_{x \in \partial D} = \varphi.
\]

The function $U_D(t, x, y)$ satisfies that

\[
(3.2) \quad \begin{cases} 
U_D(t, x, y) \geq 0 \quad \text{for any} \\
\langle t, x, y \rangle \in (0, \infty) \times \overline{D} \times \overline{D}; \\
\text{the equality holds if and only if at least one} \\
of \ x \ \text{and} \ y \ \text{belongs to} \ \partial D
\end{cases}
\]

and that

\[
(3.3) \quad \frac{\partial U_D(t, x, y)}{\partial n(y)} \leq 0 \quad \text{for any} \ t > 0, \ y \in \partial D
\]

and $x \in D - \{y\}$ where $\frac{\partial}{\partial n(y)}$ denotes the exterior normal derivative at $y \in \partial D$. For any continuous functions $u_0(x)$ on $\overline{D}$ and $\varphi(t, x)$ on $[0, \infty) \times \partial D$, there exists one and only
one solution $u(t, x)$ of the initial boundary value problem (3.1) and it is given by

\begin{equation}
(3.4) \quad u(t, x) = \int_D u_D(t, x, y) u_0(y) \, dy
- \int_0^t \int_{\partial D} \frac{\partial U_D(t - \tau, x, y)}{\partial n(y)} \varphi(\tau, y) \, dS(y).
\end{equation}

In particular, if $\varphi(x)$ satisfies $A\varphi = 0$ in $D$ and $\varphi|_{\partial D} = \psi$ where $\psi$ is continuous on $\partial D$, then

\begin{equation}
(3.5) \quad \varphi(x) = \int_D u_D(t, x, y) \varphi(y) \, dy
- \int_0^t \int_{\partial D} \frac{\partial U_D(\tau, x, y)}{\partial n(y)} \psi(y) \, dS(y).
\end{equation}

**Lemma 1.** — Let $\Omega_0$ be a subdomain of $\Omega$ with its compact closure $\bar{\Omega}_0 \subset \Omega$ and with boundary $\partial \Omega_0$ of class $C^2$, $u(x)$ be an $A$-superharmonic function on $\Omega$ such that $u(x) > 0$ on $\bar{\Omega}_0$ and $\varphi(x)$ be a continuous function on $\bar{\Omega}_0$ such that $0 \leq \varphi(x) < u(x)$ on $\bar{\Omega}_0$. Then $\int_{\Omega_0} u_{\Omega_0}(t, x, y) \varphi(y) \, dy < u(x)$ on $(0, \infty) \times \bar{\Omega}_0$.

**Proof.** — The function $\varphi(t, x) = \int_{\Omega_0} u_{\Omega_0}(t, x, y) \varphi(y) \, dy$ is the solution of the initial-boundary value problem (3.1) with $D = \Omega_0$, $u_0 = \varphi$ and $\varphi = 0$. Suppose that

$$\varphi(t, x) \geq u(x)$$

at some point $(t, x) \in (0, \infty) \times \bar{\Omega}_0$, and put

$$t_1 = \inf \{t; \varphi(t, x) \geq u(x) \text{ for some } x \in \bar{\Omega}_0\}.$$ Then

\begin{equation}
(3.6) \quad 0 \leq \varphi(\tau, x) < u(x) \quad \text{whenever} \quad 0 < \tau < t_1
\end{equation}

and $x \in \bar{\Omega}_0$. By means of the continuity of $\varphi(t, x)$, lower semi-continuity of $u(x)$ and by the fact: $\varphi(t, x) = 0$ for any $x \in \partial \Omega_0$, we may find a point $x_1 \in \Omega_0$ such that

\begin{equation}
(3.7) \quad \varphi(t_1, x_1) = u(x_1) < \infty.
\end{equation}

Since $u(x) - \varphi(x)$ is positive and lower semi-continuous on
there exists a positive number $\delta$ such that

$$0 < \varphi(x) + 3\delta < u(x) \quad \text{on } \overline{\Omega}_0.$$  

Further we may find a domain $D$ with boundary $\partial D$ of class $C^3$ such that $x_1 \in D \subset \Omega_0$ and that

$$\varphi(x) < \varphi(x_1) + \delta \quad \text{and} \quad u(x) > u(x_1) - \delta \quad \text{on } \overline{D}.$$  

Combining these inequalities with (3.8), we get

$$\varphi(x) + \delta < \inf_{x \in \overline{D}} u(x) \quad \text{on } \overline{D}.$$  

Let $\{u_n\}$ be a monotone increasing sequence of continuous functions on $\partial D$ such that $\lim_{n \to \infty} u_n(y) = u(y)$ on $\partial D$. Then we may easily show that

$$\lim_{n \to \infty} \left[ \inf_{y \in \partial D} u_n(y) \right] = \inf_{y \in \partial D} u(y).$$  

Let $\omega_n$ be the solution of elliptic boundary value problem: $Au_n = 0$ in $D$, $\omega_n|_{\partial D} = u_n$. Then $\omega_n(x) \leq u(x)$ in $D$ by means of the A-superharmonicity of $u$, and the sequence $\{\omega_n\}$ is monotone increasing. Hence

$$\omega(x) = \lim_{n \to \infty} \omega_n(x) \leq u(x)$$

exists and $\omega(x) \leq u(x)$ in $D$. Since $\omega_n(x) \geq \inf_{y \in \partial D} u_n(y)$ in $D$, we obtain from (3.10) and (3.9) that

$$\omega(x) \geq \inf_{y \in \partial D} u(y) \geq \varphi(x) + \delta \quad \text{in } D.$$  

On the other hand (cf. (3.5))

$$\omega_n(x) = \int_D u_D(t, x, y) \omega_n(y) \, dy$$

$$+ \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial u_D(\tau, x, y)}{\partial n(y)} \right\} u_n(y) \, dS(y).$$  

Let $n \to \infty$, and we obtain

$$u(x) \geq \omega(x) = \int_D u_D(t, x, y) \omega(y) \, dy$$

$$+ \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial u_D(\tau, x, y)}{\partial n(y)} \right\} u(y) \, dS(y).$$
Applying (3.4) to \( \nu(t, x) \) restricted on \((0, \infty) \times \overline{D} \), we get

\[
\nu(t, x) = \int_D U_D(t, x, y) \nu(y) \, dy \\
+ \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(t - \tau, x, y)}{\partial n(y)} \right\} \nu(\tau, y) \, dS(y) \\
\leq \int_D U_D(t, x, y) [\omega(y) - \delta] \, dy \\
+ \int_0^t d\tau \int_{\partial D} \left\{ - \frac{\partial U_D(\tau, x, y)}{\partial n(y)} \right\} u(y) \, dS(y) \\
\quad \text{(from (3.11) and (3.6))}
\]

\[
\leq u(x) - \delta \int_D U_D(t, x, y) \, dy \\
\quad \text{(from (3.12))}.
\]

In particular \( \nu(t_1, x_1) \leq u(x_1) - \delta \int_D U_D(t_1, x_1, y) \, dy \); this contradicts (3.7) since \( \int_D U_D(t_1, x_1, y) \, dy > 0 \) by (3.2).

**Remark.** Even the fact that \( \{x | u(x) = \infty\} \) has no interior point is not guaranteed before Theorem 1 is proved. So, for instance, each term in (3.12) might be \( \infty \) (where we use the usual convention rule: \( \infty \geq \infty, \infty > \text{any real number} \)). However we do not have to care for such situations in the above proof since \( U_D(t, x, y) \geq 0 \) and \( - \frac{\partial U_D(\tau, x, y)}{\partial n(y)} \geq 0 \).

**Lemma 2.** Let \( \Omega_0 \) and \( u(x) \) be as in Lemma 1. Then

\[
\int_{\Omega_0} U_{\Omega_0}(t, x, y) u(y) \, dy \leq u(x) \text{ on } (0, \infty) \times \overline{\Omega_0}.
\]

**Proof.** Let \( \gamma \) be a positive number less than \( \min_{x \in \overline{\Omega_0}} u(x) \), and \( \{\nu_n\} \) be a monotone increasing sequence of continuous functions on \( \overline{\Omega_0} \) such that \( \nu_n(x) \geq \gamma \) \((n = 1, 2, \ldots)\) and \( \lim_{n \to \infty} \nu_n(x) = u(x) \) on \( \overline{\Omega_0} \). For every \( n > \gamma^{-1} \), we apply Lemma 1 to the function \( \nu(x) = \nu_n(x) - n^{-1} \) and obtain

\[
\int_{\Omega_0} U_{\Omega_0}(t, x, y) [\nu_n(x) - n^{-1}] \, dy \leq u(x). \]

Let \( n \to \infty \), and we get the conclusion of Lemma 2.
4. Proof of Theorems.

Let \( u(x) \) be an arbitrary A-superharmonic function on \( \Omega \). For any given subdomain \( \Omega_0 \) of \( \Omega \) with compact closure \( \overline{\Omega}_0 \subseteq \Omega \) and with boundary \( \partial \Omega_0 \) of class \( C^3 \), we may assume in proofs of Theorems 1 and 2 that \( u(x) > 0 \) on \( \overline{\Omega}_0 \) because, if \( \inf_{x \in \overline{\Omega}_0} u(x) = \alpha \leq 0 \), we may replace \( u(x) \) by
\[
  u(x) + (1 - \alpha)u_0(x)
\]
where \( u_0 \) is the solution of the elliptic boundary value problem: \( Au_0 = 0 \) in \( \Omega_0 \), \( u_0|_{\partial \Omega_0} = 1 \).

**Proof of Theorem 1.** — Let \( D \) be an arbitrary subdomain of \( \Omega \) with compact closure \( \overline{D} \). By the A-superharmonicity of \( u \), we may find a point \( x_0 \in \Omega \) where \( u(x_0) < \infty \). Let \( \Omega_0 \) be a subdomain of \( \Omega \) such that \( \overline{D} \cup \{x_0\} \subseteq \Omega_0 \), \( \overline{\Omega}_0 \) is compact and \( \partial \Omega_0 \) is of class \( C^3 \). Then, as we have noticed above, we may assume that \( u(x) > 0 \) on \( \overline{\Omega}_0 \). Hence
\[
(4.1) \quad u(x) \geq \int_{\Omega_0} U_{\Omega_0}(t, x, y)u(y) \, dy \text{ on } (0, \infty) \times \overline{\Omega}_0
\]
by Lemma 2. We fix a positive number \( t_0 \). Then, since \( U_{\Omega_0}(t_0, x, y) > 0 \) on \( \Omega_0 \times \Omega_0 \) and \( \beta = \min_{r \in \overline{D}} U_{\Omega_0}(t_0, x_0, y) > 0 \) by (3.2), it follows from (4.1) that
\[
(4.2) \quad u(x_0) \geq \int_D \beta u(y) \, dy,
\]
which implies \( \int_D u(y) \, dy \leq u(x_0)/\beta < \infty \), q.e.d.

**Proof of Theorem 2.** — Let \( \varphi(x) \) be an arbitrary non-negative valued function of class \( C^2 \) and with compact support in \( \Omega \), and \( \Omega_0 \) be a subdomain of \( \Omega \) containing the support of \( \varphi \) and such that \( \overline{\Omega}_0 \) is compact and \( \partial \Omega_0 \) is of class \( C^3 \). It suffices to prove that
\[
\int_{\Omega_0} u(x) A^*\varphi(x) \, dx \leq 0.
\]
We may assume that \( u(x) > 0 \) on \( \overline{\Omega}_0 \) as we have noticed above. Hence we have by Lemma 2
\[
(4.2) \quad \int_{\Omega_0} u(y) \left\{ \int_{\Omega_0} \varphi(x)U_{\Omega_0}(t, x, y) \, dx - \varphi(y) \right\} \, dy
\]
\[
= \int_{\Omega_0} \varphi(x) \left\{ \int_{\Omega_0} U_{\Omega_0}(t, x, y)u(y) \, dy - u(x) \right\} \, dx \leq 0.
\]
On the other hand, since
\[
\frac{\partial}{\partial t} \int_{\Omega_0} \varphi(x)U_{\Omega_0}(t, x, y) \, dx = \int_{\Omega_0} \varphi(x) \frac{\partial U_{\Omega_0}(t, x, y)}{\partial t} \, dx
\]
\[
= \int_{\Omega_0} \varphi(x) \cdot A_x U_{\Omega_0}(t, x, y) \, dx = \int_{\Omega_0} A^* \varphi(x) \cdot U_{\Omega_0}(t, x, y) \, dx
\]
(the subscript \( x \) to \( A \) indicates to operate \( A \) to \( U_{\Omega_0}(t, x, y) \) as a function of \( x \)), we get
\[
\lim_{t \to 0} \frac{\partial}{\partial t} \int_{\Omega_0} \varphi(x)U_{\Omega_0}(t, x, y) \, dx = A^* \varphi(y)
\]
boundedly in \( y \in \Omega_0 \); accordingly
\[
\lim_{t \to 0} \frac{1}{t} \left\{ \int_{\Omega_0} \varphi(x)U_{\Omega_0}(t, x, y) \, dx - \varphi(y) \right\}
\]
\[
= \lim_{t \to 0} \frac{1}{t} \int_0^t \left\{ \frac{\partial}{\partial \tau} \int_{\Omega_0} \varphi(x)U_{\Omega_0}(\tau, x, y) \, dx \right\} d\tau = A^* \varphi(y)
\]
boundedly in \( y \). Combining this result with (4.2), we obtain
\[
\int_{\Omega_0} A^* \varphi(y) \cdot u(y) \, dy \leq 0,
\]
q.e.d.

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