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KOHUR GOWRISANKARAN

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MULTIPLY SUPERHARMONIC FUNCTIONS

by Kohur GOWRISANKAREN [1]

Dédié à Monsieur M. Brelot à l'occasion de son 70e anniversaire.

1. Introduction.

Let Ω be a locally compact, connected and locally connected Hausdorff space. We shall say that Ω is a harmonic space of Brelot if Ω has a countable base for open sets and if there is a system of harmonic functions defined on open subsets of Ω satisfying the axioms 1, 2 and 3 of Brelot [2].

Let Ω_1 and Ω_2 be harmonic spaces of Brelot. A lower semi-continuous extended real valued function v on the product $\Omega_1 \times \Omega_2$ is said to be multiply superharmonic if (i) $v \not\equiv +\infty$ and (ii) v is hyperharmonic in each variable separately. In this article we shall be concerned with multiply superharmonic functions : their boundary behaviour at the distinguished boundary, the reduced functions and the integral representation of positive functions. There are a number of questions which arise in connection with these topics. Concerning these questions we shall present here some results though not exhaustive, and describe some of the problems of interest to us.

2. Integral Representation and a Consequence.

Let Ω_1 and Ω_2 be harmonic spaces of Brelot, each one of them with a countable base for open sets consisting of

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completely determining domains [2]. Let MS^+ denote the class of all multiply superharmonic positive functions on the product $\Omega_1 \times \Omega_2$. Let MS denote the differences of elements in MS^+ defined in the natural way. Then MS, provided with the semi-norms $\varphi_1 - \varphi_2 \longmapsto \left|\int \varphi_1 \, d\rho_z^\omega - \int \varphi_2 \, d\rho_z^\omega\right|$, where $z = (x, y) \in \omega = \omega_1 \times \omega_2$, ω_1 and ω_2 are regular domains in the respective spaces, is a locally convex Hausdorff topological vector space over the real numbers. A. Drinkwater [5] showed that with the above topology MS^+ has a compact metrizable base Δ and hence deduced that every positive multiply superharmonic function φ is represented as an integral of a positive finite Radon measure on Δ carried by the set of extreme elements of Δ. She also showed that this set of extreme elements contains elements of the form $s_1(x)s_2(y)$ where s_1 and s_2 are extreme generators of the cone of positive superharmonic functions on the respective spaces Ω_1 and Ω_2. An explicit determination of all the extreme elements of Δ has not yet been done. Also, it is not known whether the measures carried by the extreme elements representing the elements of MS^+ are unique. We feel that the representing measure is not unique in general. The results of Ahern and Rudin [1] may throw some light on this problem.

Let C be the subclass of MS^+ consisting of the elements φ which verify the Cairoli condition; viz., $x \longmapsto \nu_x$ behaves hyperharmonically on Ω_1 where for every $x \in \Omega_1$, ν_x is the canonical measure, representing $\varphi(x, .)$, on a convenient compact base of the positive superharmonic functions on Ω_2, [4, 10, 11]. It was shown in [10] that C is a cone with a compact base and the extreme elements of this base are precisely of the form $s_1 s_2$ considered above. Further C is a lattice in its own order. Hence, the elements of C are represented uniquely by finite Radon measures carried by the extreme elements of a convenient compact base of C. The following is an application of the above integral representation.

Let U be the open unit disc in the plane. For any positive integer n, $N(U^n)$ denotes the Nevanlinna class of the polydisc U^n, i.e., the class of holomorphic functions f on U^n such that $\log|f|$ has a non-negative multiply harmonic majorant. The following result characterises the functions f

in $N(U^n)$ such that $-\log|f| +$ multiply harmonic positive function belongs to $C(U^n)$.

THEOREM 2.1. — *Let* $f \in N(U^n)$. *Then, the following are equivalent.*

a) $\exists$ *a positive multiply harmonic function* w *on* U^n *such* $-\log f + w$ *belongs to* $C(U^n)$.

b) $f = hg$ *where* h *is free of zeros in* $N(U^n)$ *and* $g = B_1, \ldots, B_n$ *where* B_k *is a Blaschke product for each* k.

Proof. — We shall prove the result in the case $n = 2$. The general case follows easily by induction. Let us first show that $b) \Longrightarrow a)$. Let $f \in N(U^2)$ such that $f = B_1B_2h$ where h is in $N(U^2)$ and free of zeros, B_1, B_2 are Blaschke products on U. Then $\log|h|$ is multiply harmonic and is the difference of two such non-negative functions. Further $-\log|B_1|$ and $-\log|B_2|$ are both non-negative multiply superharmonic and are constants in at least one of the variables. Hence we deduce that $-\log|B_1|$ and $-\log|B_2|$ are both elements of C [th. II.7, 10]. Now it is easily seen that

$$b) \Longrightarrow a).$$

Conversely, suppose f is in $N(U^2)$ and is such that there exists a positive 2-harmonic function w such that

$$v = -\log|f| + w$$

belongs to $C(U^2)$. Let ν_z, for every z in U, be the measure on U corresponding to the potential part of $v(z, .)$ on the unit disc. Since f is a holomorphic function in the Nevanlinna class, $v(z, \zeta)$ is a harmonic function in ζ except possibly at a countable set $\{\zeta_k\}$ satisfying the Blaschke condition. Hence, ν_z charges precisely the points ζ_k; i.e., ν_z is a positive measure which is a countable sum of point masses at ζ_k. But, since v is in $C(U^2)$, the sets of ν_z measure zero are independent of z in U. By symmetry, we can find a sequence, $\{z_j\}$ such that 1) this sequence satisfies the Blaschke condition and 2) for every ζ in U, $v(z, \zeta)$ is harmonic in z except at the points z_k. It follows that v is 2-harmonic on $UX(U - \{\zeta_k\}) \cup (U - \{z_j\})XU$ and further that the zero set $Z(f)$ of f is precisely $UX\{\zeta_k\} \cup \{z_j\}XU$.

Let $z_0 \in U$ and $\zeta_0 \in U$ be such that $f(z_0, \zeta_0) \neq 0$. Suppose that $f(z_0, \zeta)$ has a zero of order l at ζ_k. Let

$$g(\zeta) = [(\zeta_k - \zeta)|\bar{\zeta}_k/(1 - \bar{\zeta}_k\zeta)\zeta_k]^l.$$

Then, $\varphi - \log|g|$ is clearly a 2-superharmonic function on U^2. Since $-\log|g|$ is harmonic in z or $\equiv +\infty$ for every fixed ζ, it can be shown as in [10, Theorem II.7, Cor.] that $-\log|g|$ belongs to $C(U^2)$. Now, $\varphi(z_0, \zeta) - \log|g(\zeta)|$ is harmonic at ζ_k, and by the same argument as above, we conclude that $\varphi(z, \zeta) - \log|g(\zeta)|$ is harmonic in a neighbourhood of ζ_k, for every z in U. This shows that the order of the zero of $f(z, .)$ at ζ_k is independent of z in U. Let now B_1 (resp. B_2) be the Blaschke product corresponding to zeros at ζ_k (resp. z_j) of orders determined by $f(z_0, \zeta_k)$ (resp. $f(z_j, \zeta_0)$). We now conclude that f/B_1B_2 is a holomorphic function in $N(U^2)$ free of zeros. Hence,

$$g = B_1B_2 \quad \text{and} \quad h = f/B_1B_2$$

satisfy the requirements of the theorem. The proof is complete.

The following corollary can be proved utilising the results on integral representation of positive multiply superharmonic functions [5].

COROLLARY. — *Let f be in the Nevanlinna class of the polydisc U^n. Then f can be written in the form gh, where g is a product of Blaschke products and h is in the Nevanlinna class and further the zero set of h cannot contain a set of the form $E_1XE_2X \dots XE_n$ with $E_k = U$ for some k between 1 and n.*

3. Behaviour at the Distinguished Boundary.

Let MS^+ be the set of positive multiply superharmonic functions on the product of two Brelot spaces Ω_1 and Ω_2. Let MH^+ and P denote respectively the class of positive multiply harmonic functions and the class of those φ in MS^+ with 0 as the greatest multiply harmonic minorant. It is clear that MS^+ is the direct sum of P and MH^+ and we know that the pointwise infimum of two elements in MS^+ is also in MS^+. Hence [Theorem 5,8], corresponding to each extremal generator h_1h_2 of MH^+ [Th. 1.1, 1.2, 6], there is a ' fine filter ' $G_{h_1h_2}$ defined by

$$G_{h_1h_2} = \{E \subset \Omega_1 \times \Omega_2 : v \in P, \ v \geqslant h_1h_2 \quad \text{on} \quad \complement E\}$$

Let F_{h_1} (respectively F_{h_2}) be the fine filter on Ω_1 (resp. Ω_2) corresponding to the minimal harmonic function h_1 (resp. h_2). The following lemma shows the relation between $G_{h_1h_2}$ and $F_{h_1} \times F_{h_2}$.

LEMMA 3.1. — *The filter* $G_{h_1h_2}$ *is finer than* $F_{h_1} \times F_{h_2}$.

Proof. — It is enough to show that every set of the form $E_1 \times E_2$ in $F_{h_1} \times F_{h_2}$ is also in $G_{h_1h_2}$. Given a set of this form, there exist potentials p_1 and p_2 on Ω_1 and Ω_2 respectively such that $p_i \geqslant h_i$ on $\Omega_i \setminus E_i$ for $i = 1, 2$. Let $v = p_1h_2 + h_1p_2$. Then $v \in P$ and clearly v majorises h_1h_2 on $\Omega_1 \times \Omega_2 \setminus (E_1 \times E_2)$. The proof is complete.

It is known that for positive harmonic functions u_1 on Ω_1, u_2 on Ω_2, the limits of $[w(x, y)/u_1(x)u_2(y)]$ as x and y tend to the fine boundaries of the spaces and independent of each other do not exist in general, even when $w > 0$ is a multiply harmonic function [9]. The following however shows that the limit following the product of fine filters exist for certain special classes.

THEOREM 3.2. — *Let* $u_i > 0$ *be positive harmonic function on* Ω_i, μ_i *the canonical Radon measure representing* u_i *carried by the set of extreme harmonic functions* Δ_1^i *of a conveniently chosen compact base of positive superharmonic functions on* Ω_i, *for* $i = 1, 2$. *Let* $w > 0$ *be a multiply harmonic function on* $\Omega_1 \times \Omega_2$ *such that the measure on* $\Delta_1^1 \times \Delta_1^2$ *representing* w *has a Radon-Nikodym derivative* f *with respect to* $\mu_1 \otimes \mu_2$ *such that* f *is a bounded uniformly continuous function on* $\Delta_1^1 \times \Delta_1^2$. *Then,* w/u_1u_2 *has finite limit following* $F_{h_1} \times F_{h_2}$, *equal to* f, *for* $\mu_1 \otimes \mu_2$ *almost every* (h_1, h_2) *in* $\Delta_1^1 \times \Delta_1^2$.

Proof. — The space $C_u(\Delta_1^1)$ of bounded uniformly continuous functions on Δ_1^1 provided with the topology of uniform convergence on compact subsets of Ω_1 is separable. Hence, there exists a countable dense subset $\{f_n\}_{n\in\mathbf{N}}$ for this space. Hence, it is possible to find a Borel set $E_1 \subset \Delta_1^1$ of μ_1-measure zero such that, for all $n \in \mathbf{N}$, $\frac{1}{u_1(x)}\int h(x)f_n(h)\mu_1\,(dh)$ has the fine limit f_n, except for elements in E_1. Now, given any f in $C_u(\Delta_1^1)$ there exists a sequence f_{n_k} which converges uniformly on Δ_1^1 to f. This implies that $\int f_{n_k}(h)h(x)\mu_1\,(dh)$ converges uniformly on Ω_1 to $\int f(h)h(x)\mu_1\,(dh)$. Hence, we deduce that except for elements in E_1, $\frac{1}{u_1(x)}\int f(h)h(x)\mu_1\,(dh)$ has the fine limit equal to f. Similarly we can find a set $E_2 \subset \Delta_1^2$ of μ_2-measure zero such that for all g in $C_u(\Delta_1^2)$, the fine limit of $\frac{1}{u_2(y)}\int h(y)g(h)\mu_2\,(dh)$ exists and equals g for all elements of Δ_1^2 except possibly in E_2.

Now consider any function on $\Delta_1^1 \times \Delta_1^2$ of the form $f(h_1)g(h_2)$ where $f \in C_u(\Delta_1^1)$ and $g \in C_u(\Delta_1^2)$. Since μ_1 and μ_2 are totally finite measures we deduce that the limit of $[1/u_1(x)u_2(y)]\iint f(h_1)g(h_2)h_1(x)h_2(y)\mu_1\,(dh_1)\mu_2\,(dh_2)$ following

$$\mathcal{F}_{h_1} \times \mathcal{F}_{h_2}$$

exists for $\mu_1 \otimes \mu_2$ almost every element of $\Delta_1^1 \times \Delta_1^2$; in fact the exceptional set is contained in $(\Delta_1^1 \times E_2 \cup E_1 \times \Delta_1^2)$. Evidently the same conclusion is true for any finite linear combination of such functions. Now a straight-forward application of Stone-Weierstrass theorem gives the required result.

The following theorem can be proved using techniques similar to that found in [3].

THEOREM 3.3. — *Let* U *be the open unit disc in the plane and* T *the unit circle. Let* w *be a bounded multiply harmonic function on* U^k. *Then,* w *has a finite limit following the product of fine filters, for Lebesgue almost every element of* T^k.

The following questions arise naturally in the above considerations; partial answers to some of them are known.

1) Let u_1, u_2, μ_1 and μ_2 be as in the Theorem 3.2. Let f be a bounded $\mu_1 \otimes \mu_2$ measurable function on $\Delta_1^1 \times \Delta_1^2$.

Let $w = \int h_1 h_2 f(h_1, h_2)(\mu_1 \otimes \mu_2)(dh_1\, dh_2)$. Does w/u_1u_2 have finite limit following the product of fine filters for $\mu_1 \otimes \mu_2$ almost every element of $\Delta_1^1 \times \Delta_1^2$?

2) Let $u > 0$ be a multiply harmonic function on $\Omega_1 \times \Omega_2$ and let ν be the canonical measure on $\Delta_1^1 \times \Delta_1^2$ representing u. Let $w > 0$ be any multiply superharmonic function on $\Omega_1 \times \Omega_2$. Do the limits of w/u following $G_{h_1h_2}$ exist ν-almost every-where?

3) In the particular case of $\Omega_1 = \Omega_2 = U$, is there any relation between the limits following $G_{h_1h_2}$ and the restricted non-tangential limits [12]?

4. Reduced function and capacity.

Let $v > 0$ be a multiply superharmonic function on the product $\Omega_1 \times \Omega_2$ of two Brelot spaces. For any

$$E \subset \Omega_1 \times \Omega_2,$$

we define the reduced function $B(E, v) = \inf.\{w : w \in MS^+$ and $w \geqslant v$ on $E\}$. The reduced functions satisfy the usual subadditive and regularisation properties [7]. We have the following property.

THEOREM 4.1. — *Let $v_i > 0$ be superharmonic functions on Ω_i, $i = 1, 2$. Then $B(\omega_1 \times \omega_2, v_1v_2) = R_{v_1}^{\omega_1} R_{v_2}^{\omega_2}$.*

Proof. — Clearly $R_{v_1}^{\omega_1} R_{v_2}^{\omega_2}$ is a multiply superharmonic function > 0 on $\Omega_1 \times \Omega_2$ and equals v_1v_2 on $\omega_1 \times \omega_2$; hence it majorises the reduced function on the left side. On the other hand, let $w > 0$ be multiply superharmonic on $\Omega_1 \times \Omega_2$ and $\geqslant v_1v_2$ on $\omega_1 \times \omega_2$. Then, for each $y \in \omega_2$, $w(x, y) \geqslant v_1(x)v_2(y)$ for all $x \in \omega_1$. Hence,

$$w(x, y) \geqslant v_2(y) R_{v_1}^{\omega_1}(x)$$

for all $x \in \Omega_1$ and each $y \in \omega_2$. Now, we get that for every $x \in \Omega_1$ and all $y \in \Omega_2$, $w(x, y) \geqslant R_{v_1}^{\omega_1}(x) R_{v_2}^{\omega_2}(y)$. This completes the proof.

The reduced function is closely related to the notion of capacity and convergence theorems, etc. The following result suggests a possible approach to the treatment of these questions.

THEOREM 4.2. — *Let* X *and* Y *be regular Hausdorff topological spaces. Let* K *(resp.* K_0*) denote the class of all compact subsets (resp. finite unions of compact rectangles) of* $X \times Y$. *Let* φ *be a non-negative valued set function on* K_0 *satisfying* 1) φ *is monotone non-decreasing,* 2) *strongly subadditive and* 3) *outer regular* i.e., *for* $C \in K_0$, $\varphi(C) < +\infty$ *and any* $\varepsilon > 0$, *there exists an open set* $V \supset C$ *such that for all* $D \in K_0$ *and* $C \subset D \subset V$, $\varphi(D) < \varphi(C) + \varepsilon$.

Then, φ can be extended to a capacity on K and this extension is unique.

Proof. — Let K be any compact set in $X \times Y$. Let us first of all show that K is the intersection of all sets in K_0 containing K. Since K is a compact set in a Hausdorff space, K is the intersection of all open sets containing K. Hence, it is sufficient to show that given an open set V containing K, there exists a set $C \in K_0$ such that $K \subset C \subset V$. Now, it is possible to choose a finite number points $(x_1, y_1), \ldots, (x_n, y_n)$ in K, and neighbourhoods $V_{x_i} \times W_{y_i}$ of (x_i, y_i) such that (i) $\overline{V}_{x_i} \times \overline{W}_{y_i} \subset V$ and (ii) $\bigcup_{i=1 \text{ to } n} (V_{x_i} \times W_{y_i}) \supset K$. Let $K_i = K \cap (\overline{V}_{x_i} \times \overline{W}_{y_i})$ and A_i and B_i be respectively the projections of K_i to X and Y. Then, $A_i \times B_i$ are compact rectangles and $C = \bigcup_{i=1 \text{ to } n} A_i \times B_i$ is in K_0 and clearly $K \subset C \subset V$.

Now let, K be any compact set of $X \times Y$. Let $\{C_i\}_{i \in I}$ be the collection of all sets in K_0 which contain K. Since K_0 is closed for finite intersections, $\{C_i\}$ is decreasing directed and we have just shown that $K = \bigcap_{i \in I} C_i$. It follows that the numbers $\varphi(C_i)$ are decreasing directed and we set

$$\tilde{\varphi}(K) = \inf . \{\varphi(C_i) : i \in I\}.$$

It is clear that $\tilde{\varphi} \geqslant 0$ and monotone non-decreasing on K. The strong subadditivity of $\tilde{\varphi}$ on K is an immediate consequence of the strong subadditivity of φ on K_0. Finally, to show the outer regularity of $\tilde{\varphi}$, suppose $K \in K$ is that $\tilde{\varphi}(K) < +\infty$ and $\varepsilon > 0$. There exists a set $C \in K_0$ such that $C \subset K$ and $\varphi(C) < \tilde{\varphi}(K) + \varepsilon/2$. Choose an open set $V \supset C$ such that for all C' in K_0 with

$$C \subset C' \subset V, \quad \varphi(C') < \varphi(C) + \varepsilon/2.$$

Let K' be any compact set such that $K \subset K' \subset V$. Then, there exists a D in K_0 such that $K' \subset D \subset V$. Now, $D \cup C$ is again a finite union of compact rectangles and hence belongs to K_0 and $D \cup C \subset V$. Hence

$$\tilde{\varphi}(K') \leqslant \varphi(D \cup C) \leqslant \varphi(C) + \varepsilon/2 \leqslant \tilde{\varphi}(K) + \varepsilon.$$

This proves that $\tilde{\varphi}$ is outer regular, showing that $\tilde{\varphi}$ is a strong capacity.

It is obvious from the definition of $\tilde{\varphi}$ that $\tilde{\varphi}$ coincides with φ on K_0. Finally, suppose ψ any strong (or even weak) capacity on K such that ψ restricted to K_0 is φ. Given any compact set of K and any $\varepsilon > 0$, there exists an open set $V \supset K$ and a set C in K_0 with

$$K \subset C \subset V$$

such that $\psi(K) > \psi(C) - \varepsilon = \varphi(C) - \varepsilon \geqslant \tilde{\varphi}(K) - \varepsilon$. Hence, $\psi(K) \geqslant \tilde{\varphi}(K)$. However,

$$\psi(K) \leqslant \inf \{\varphi(C) : C \in K_0,\, C \supset K\} = \tilde{\varphi}(K).$$

The proof is complete.

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Kohur Gowrisankaran,
McGill University
Montreal, Québec, Canada.