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# CLOSURES OF FACES OF COMPACT CONVEX SETS

by **A.K. ROY**

## 1. Introduction.

It is well-known that one of the disconcerting facts in the theory of infinite-dimensional compact convex sets is that the closure of a face need not be a face. The main purpose of this paper is to determine necessary and sufficient conditions which ensure that this pathology does not occur for a given face. It should be emphasised that our results are purely individual in character. We do *not* characterise the class of compact convex sets which have the property that the closures of all their faces are again faces. (As a matter of fact, this appears to be a very difficult problem.) By way of applications, it is shown that several results scattered in the literature can be proved in a rather economical and uniform manner by our method.

We conclude by giving several characterizations of cases when face (C) is closed in a compact convex set  $K$ , for any closed convex subset  $C$  of  $K$  without core points. This generalises a recent result in [11]. Our method of proof is quite different.

It is a pleasure to thank Dr. A.J. Ellis for showing some interest in this investigation and for providing me with the example at the end of § 3.

## 2. Definitions & Notations.

We will work with a fixed compact convex set  $K$  in a locally convex Hausdorff topological vector space  $E$  defined over the reals

R. We assume throughout that  $K$  is "regularly embedded" in  $E$  in the sense defined in [1].

Following [1], we let  $\partial_e K$  be the set of extreme points of  $K$  and let  $C(K)$ ,  $P(K)$  and  $A(K)$  denote, respectively, the space of continuous functions, the cone of continuous convex functions and the space of continuous affine functions, on  $K$ . Let  $M_1^+(K)$  denote the convex set of probability measures on  $K$  equipped with the weak\* topology induced on it by  $M(K)$ , the dual of  $C(K)$ .

For each  $x \in K$ , we write

$$M_x = \{\mu \in M_1^+(K) : \mu(a) = a(x), \forall a \in A(K)\}$$

which is a non-empty weak\* compact convex subset of  $M_1^+(K)$ . Let  $Z_x$  denote the set of *maximal or boundary measures* [1] in  $M_x$ .

If  $f \in C(K)$ , we define

$$\hat{f}(x) = \inf \{h(x) : h \in A(K), h \geq f\};$$

which is the least upper semicontinuous (u.s.c) concave majorant of  $f$  and, dually, we define  $\check{f}$  as the greatest convex minorant of  $f$ .

If  $C$  is a proper compact convex subset of  $K$ , we define for each  $\alpha \geq 1$ ,

$$D_\alpha(C) = (\alpha C - (\alpha - 1)K) \cap K$$

and by face (C) we mean the  $\sigma$ -compact set  $\bigcup_{n=1}^{\infty} D_n(C)$ . We recall [2 : page 99] that face (C) is the smallest, not necessarily closed, face of  $K$  containing  $C$ .

If  $f$  is a function defined on  $K$  and  $S$  is a subset of  $K$ , we consistently employ the notation  $f(S) \leq \alpha$  to mean  $f(x) \leq \alpha$  for all  $x \in S$ . A similar meaning should be given to  $f(S) = 0$ .

### 3. Conditions for the closure of a face to be a face.

Let  $F \subset K$  be a face and let  $a \in A(K)$  be such that  $a \leq 0$  on  $F$ , and hence on  $\bar{F}$ . The theorem in this section is motivated by the following simple observation :

$$\widehat{0 \vee a}(x) = 0 \text{ for all } x \in F.$$

This follows from the fact (see [1]) that

$$\widehat{0 \vee a}(x) = \sup \{ \mu(0 \vee a) : \mu \text{ discrete, } \mu \in M_x \}.$$

However, since  $\widehat{0 \vee a}$  is u.s.c. we cannot, in general, assert that  $\widehat{0 \vee a}(x) = 0$  for all  $x \in \bar{F}$ . But this is the case if and only if  $\bar{F}$  is also a face.

Let

$$F^* = \{ a \in A(K) : a(F) \leq 0 \}$$

and

$$(F^*)_* = \{ x \in K : a(x) \leq 0 \forall a \in F^* \}.$$

Then we have the following.

LEMMA 3.1.  $\bar{F} = (F^*)_*$

We omit the proof which is a simple application of the Hahn-Banach (separation) theorem. We will also need the following simple result.

LEMMA 3.2. — Let  $f \in P(K)$  and let  $\{f_n\}$  be a sequence of functions in  $P(K)$  converging uniformly to  $f$ . Then  $\{\hat{f}_n\}$  converges uniformly to  $f$ .

This is an obvious consequence of the fact that  $f - \epsilon \leq g \leq f + \epsilon$  implies  $\hat{f} - \epsilon \leq \hat{g} \leq \hat{f} + \epsilon$  for any  $\epsilon > 0$ .

Adopting the terminology of [6], we say that  $F^*$  is *perfect* if for any  $a \in F^*$  and  $\epsilon > 0$ , there exists  $a_\epsilon \in F^*$  such that  $0, a \leq a_\epsilon + \epsilon$ .

We can now state the main result of this section as follows :

THEOREM 3.3. — Let  $F \subseteq K$  be a proper face. Then the following are equivalent :

- (1)  $\bar{F}$  is a face.
- (2)  $F^*$  is perfect.
- (3)  $\widehat{0 \vee a}(\bar{F}) = 0 \forall a \in F^*$ .

(4)  $\widehat{0 \vee f}(\bar{F}) = 0 \ \forall f \in P(K)$  such that  $f(F) \leq 0$ .

(5) If  $f - g, f \in P(K)$  with  $f(F) \leq g(F)$ , then  $\widehat{f \vee g} = g$  on  $\bar{F}$ .

*Comments 1.* – If  $F$  is assumed to be closed, then the equivalence of (1) and a result similar to (2) has been proved in [4] by means of the “polar calculus”. However, our proof, which is an adaptation to this context of an argument in [8], and formulation are somewhat different.

2. – We should note that the statements (2) – (5) have obvious “duals” : for example, the dual of (2) is  $0 \wedge a(\bar{F}) = 0 \ \forall a \in F^0$  where

$$F^0 = \{a \in A(K) : a(F) \geq 0\}.$$

*Proof of Theorem 3.3*

(1)  $\Rightarrow$  (2). Suppose  $F^*$  is not perfect. Then  $\exists a_0 \in F^*$  and  $\epsilon_0 > 0$  such that  $\forall b \in F^*$ ,

either  $a_0 \not\leq b + \epsilon_0$   
 or  $0 \not\leq b + \epsilon_0$  }  $\cdot$  ( $\alpha$ )

If  $A(K)^+$  denotes the positive cone in  $A(K)$ , define

$$U = \{a \in A(K) : \|a\| \leq \epsilon_0\}$$

and

$$H = \{(b - p, b - q) : b \in F^*, p, q \in A(K)^+\}.$$

Then ( $\alpha$ ) can be restated as

$$(a_0, 0) + (u_1, u_2) \notin H, \forall u_1, u_2 \in U.$$

This implies that  $(a_0, 0) \notin \bar{H}$  and hence by the Hahn-Banach theorem,  $\exists \varphi \in (A(K) \times A(K))^*$  such that

$$\sup \varphi(\bar{H}) < \varphi(a_0, 0). \quad (\beta)$$

$H$  being a cone, ( $\beta$ ) says that  $\varphi \leq 0$  on  $H$ . Now, we can write

$$\varphi = \varphi_1 + \varphi_2 \quad \text{where } \varphi_i \in A(K)^* \ (i = 1, 2)$$

and

$$\varphi_1(a) = \varphi(a, 0), \varphi_2(b) = \varphi(0, b) \quad \text{for } a, b \in A(K).$$

If  $c \in A(K)^+$  then  $(-c, 0) \in H$  and hence  $\varphi_1(-c) = \varphi(-c, 0) \leq 0$ , showing that  $\varphi_1 \geq 0$ . Similarly,  $\varphi_2 \geq 0$  and thus (by [13]),

$\varphi_i(a) = \lambda_i a(x_i) \forall a \in A(K)$ , for some  $\lambda_i \in R^+$  and  $x_i \in K$  ( $i = 1, 2$ ).

If  $a \in F^*$ ,  $(a, a) \in H$  and therefore

$$\begin{aligned} 0 &\geq \varphi(a, a) = \varphi_1(a) + \varphi_2(a) \\ &= \lambda_1 a(x_1) + \lambda_2 a(x_2) \\ &= a(\lambda_1 x_1 + \lambda_2 x_2), \end{aligned}$$

showing, by lemma 3.1, that  $\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} \in \bar{F}$ . But by  $(\beta)$ ,

$$0 < \varphi(a_0, 0) = \varphi_1(a_0, 0) = \lambda_1 a_0(x_1)$$

and this shows, again by lemma 3.1, that  $x_1 \notin \bar{F}$ , so  $\bar{F}$  is not a face.

(2)  $\Rightarrow$  (3). If  $a \in F^*$  and  $\epsilon > 0$  then by assumption there exists  $a_\epsilon \in F^*$  such that

$$0, a \leq a_\epsilon + \epsilon$$

which implies that

$$\widehat{0 \vee a}(x) \leq a_\epsilon(x) + \epsilon \leq \epsilon, \forall x \in \bar{F}$$

and since  $\epsilon$  is arbitrary, we can conclude that  $\widehat{0 \vee a}(\bar{F}) = 0$ .

(3)  $\Rightarrow$  (4). If  $a_1, a_2 \in F^*$  then for each  $x \in K$ ,

$$0 \vee a_1(x) \vee a_2(x) \leq 0 \vee a_1(x) + 0 \vee a_2(x)$$

and hence

$$\widehat{0 \vee a_1 \vee a_2} \leq \widehat{0 \vee a_1 + 0 \vee a_2} \leq \widehat{0 \vee a_1} + \widehat{0 \vee a_2}$$

by the subadditivity of the  $\wedge$  function and it follows that  $0 \vee a_1 \vee a_2(\bar{F}) \leq 0$ . By induction, this is true for any finite number of  $a_i \in F^*$ . If  $f \in P(K)$  with  $f(\bar{F}) \leq 0$ , we know from [1] that  $f$  can be approximated uniformly by an increasing sequence of functions of the form  $a_1^{(n)} \vee \dots \vee a_k^{(n)}$  where  $a_i^{(n)} \in F^*$  for  $i = 1, \dots, k$ . Therefore, by lemma 3.2,  $(0 \vee a_1^{(n)} \vee \dots \vee a_k^{(n)})^\wedge$  increases monotonically to  $\widehat{0 \vee f}$  and it follows that  $\widehat{0 \vee f}(\bar{F}) = 0$ .

(4)  $\Rightarrow$  (5). If  $f(F) \leq g(F)$  then  $(f - g)(F) \leq 0$  and since  $f - g \in P(K)$ , by (4)  $\widehat{0 \vee (f - g)} = 0$  on  $\bar{F}$ . But

$$f \vee g = 0 \vee (f - g) + g$$

and therefore

$$\widehat{f \vee g} = \widehat{0 \vee (f - g) + g} \leq \widehat{0 \vee (f - g)} + \hat{g} = g \text{ on } \bar{F}$$

and (5) follows.

(5)  $\Rightarrow$  (3). Obvious.

(3)  $\Rightarrow$  (1). Let  $x \in \bar{F}$  and consider  $\mu \in M_x$ . Suppose, if possible, that  $\text{supp. } (\mu) \setminus \bar{F} \neq \emptyset$ , i.e.  $\exists y \in \text{supp. } (\mu)$ ,  $y \notin \bar{F}$ . By the Hahn-Banach theorem,  $\exists a \in A(K)$  such that

$$a(\bar{F}) \leq 0 < a(y).$$

So  $a \in F^*$ . By continuity, there exists a neighbourhood  $U$  of  $y$  with  $U \cap \bar{F} = \emptyset$  and  $a(U) \geq \alpha > 0$  for some  $\alpha$ .

Now,

$$\begin{aligned} \widehat{0 \vee a}(x) &= \sup \{ \lambda(0 \vee a) : \lambda \in M_x \} \\ &\geq \int (0 \vee a) d\mu \geq \int_U (0 \vee a) d\mu \geq \alpha \mu(U \cap \text{supp.}(\mu)) > 0, \end{aligned}$$

which contradicts (3).

This completes the proof of theorem 3.3.

**COROLLARY 3.4.** — *If  $F \subseteq K$  is a face and  $\bar{F}$  is also a face, then*

$$\bar{F} = \bigcap_{a \in F^*} (\widehat{0 \vee a})^{-1}(0)$$

*Remark.* — Suppose that  $\bar{F}$  is a proper face where  $F \subseteq K$  is a face. If  $(\bar{F})'$  denotes the *complementary set* of  $\bar{F}$ , i.e. the union of all faces disjoint from  $\bar{F}$ , then it is clear that  $(\bar{F})' \subseteq F'$ . It is natural to enquire whether this inclusion is an equality. That it is *not*, is shown by the following simple example :

$$K = M_1^+[0, 1], F = \{ \mu \in M_1^+[0, 1] : \mu[0, a] = \mu[b, 1] = 0 \}$$

where  $0 < a < b < 1$ . It is clear that  $F$  is a face as is

$$\bar{F} = \{\mu \in M_1^+[0, 1] : \text{supp. } (\mu) \subseteq [a, b]\}.$$

Now,  $\epsilon_a \in F'$  but  $\epsilon_a \notin (\bar{F})'$  as  $\epsilon_a \in \bar{F}$ , showing that  $(\bar{F})' \subsetneq F'$ .

If  $\partial_e K$  is closed then the necessary and sufficient condition for the closure of a face  $F \subseteq K$  to be a face is expressed below in a different way. This has the advantage that it gives a rather explicit description of  $\bar{F}$ .

Let  $S = \bar{F} \cap \partial_e K$  and define

$$T = \{f \in C(\partial_e K) : 0 \leq f \leq 1, f(S) = 1\}.$$

Then we have the following

**THEOREM 3.5.** — *Assume  $\partial_e K$  closed. Then  $\bar{F}$  is a face if and only if  $\bigcap_{f \in T} [f = 1]$  is closed. When this condition is satisfied,*

$$\bar{F} = \bigcap_{f \in T} [f = 1].$$

This result has been extracted from [10] and since its proof is essentially the same as in [10], modulo some trivial details, we omit it.

*An example.* — There are of course several examples in the literature showing that the closure of a face may fail to be a face (see, for instance, [2]). However, to the best of our knowledge, all these examples deal with compact convex sets, usually Choquet simplexes, with non-closed boundaries. We now present an example to show that this pathology can occur even if the boundary is closed.

Let  $\mathcal{A}$  be the disc-algebra and let  $K$  be its state space. Following [7], we let  $Z = \text{conv}(K \cup -iK)$ . Then  $K$  is just the probability measures on the unit circle  $\Gamma$ , so that  $\partial_e K$  and  $\partial_e Z$  are both closed. Every point of  $\partial_e Z$  is a split face of  $Z$  and so if  $E \subseteq \partial_e Z$ , the norm-closed convex hull of  $E$  (in  $A(Z)^*$ ) is a norm-closed split face of  $Z$  (by the  $L$ -ideal theory of [3]). However, take  $E = \gamma \cup -i\gamma$  where  $\gamma$  is a proper arc of  $\Gamma$  of length  $> 0$ . Using the fact that  $K$  is a simplex, it is easy to check that  $\text{conv}(E)$  is a face of  $Z$ . However,  $\gamma$  is not a peak set for  $\mathcal{A}$  since any function in  $\mathcal{A}$  which is constant



on  $\gamma$  is necessarily constant on  $\Gamma$ , so that  $\overline{\text{conv}}(E)$  ( $w^*$ -closure) is not a face of  $Z$  (by Theorem 2 of [7]).

#### 4. Applications.

We will now give some applications of the results established in the last section.

**PROPOSITION 4.1.** — *If  $K$  has the property that  $\hat{f}$  is continuous for all  $f \in P(K)$ , then the closure of every face  $F$  in  $K$  is again a face.*

*Proof.* — Let  $a \in F^*$ . As we have already remarked,  $\widehat{0 \vee a}(F) = 0$ . But  $0 \vee a \in P(K)$  and therefore  $\widehat{0 \vee a}$  is continuous by assumption. Thus  $\widehat{0 \vee a}(\bar{F}) = 0$  and we can use (3) of Theorem 3.3 to conclude that  $\bar{F}$  is a face.

*Remark.* — This result was first proved in [10] by a more elaborate method.

**DEFINITION 4.2** (after [12]). — *A compact convex set  $K$  has the strong equal support property (s.e.s.p. for short) if (i)  $\partial_e K$  is closed and (ii) for any  $x \in K$  and  $\mu, \nu \in Z_x$  we have  $\text{supp.}(\mu) = \text{supp.}(\nu)$ .*

We now prove

**PROPOSITION 4.3.** — *If  $K$  has the strong equal support property, then the closure of every face  $F$  in  $K$  is again a face.*

*Proof.* — If  $S = \bar{F} \cap \partial_e K$  then it is rather easy to show from the defining property of a face that  $\bar{F} = \overline{\text{conv}}(S)$ . Let  $a \in F^*$ . Then  $\widehat{0 \vee a}(F) = 0$ . If  $x \in \bar{F}$ , there exists a probability measure  $\mu$  on  $S$  representing  $x$  (See [13]) :  $\mu$  is obviously maximal since  $S \subseteq \partial_e K$ , hence all  $\lambda \in Z_x$  have their supports in  $S$  by the s.e.s.p. But

$$\widehat{0 \vee a}(x) = \lambda_1(0 \vee a) \text{ for some } \lambda_1 \in Z_x$$

$$= 0$$

and hence  $\bar{F}$  is a face by Theorem 3.3.

PROPOSITION 4.4. — *Let  $K$  be a Choquet simplex. If  $F \subseteq K$  is a face then  $\bar{F}$  is a face if and only if  $\partial_e \bar{F} \subseteq \partial_e K$ .*

*Proof.* — If  $\bar{F}$  is a face then by the Krein-Milman theorem,  $\partial_e \bar{F}$  is non-empty and it is clear that  $\partial_e \bar{F} \subseteq \partial_e K$ .

On the other hand, suppose  $F$  is a face with the property that  $\partial_e \bar{F} \subseteq \partial_e K$ . Consider  $a \in F^*$ . If  $x \in \partial_e \bar{F} \subseteq \partial_e K$ , by Herve's criterion [1],

$$\widehat{0 \vee a}(x) = (0 \vee a)(x) = 0.$$

But  $K$  being a simplex,  $\widehat{0 \vee a}$  is an u.s.c. affine function (by the Choquet-Meyer theorem), hence by the Bauer maximum principle,  $\widehat{0 \vee a}(\bar{F}) = 0$  and we see that  $\bar{F}$  is a face.

COROLLARY 4.5. — *If  $K$  is a Choquet simplex and if  $S$  is a compact subset of  $\partial_e K$  then  $\text{conv}(S)$  is a face of  $K$ .*

*Proof.* — This is immediate from the preceding Proposition, once we observe that  $F = \text{conv}(S)$  is a face in  $K$  and that  $\partial_e \bar{F} = S \subseteq \partial_e K$ .

*Remarks 1.* — It would be interesting to know whether Prop. 4.4 extends to compact convex sets with the equal support property [12].

2. — In [15], the following result of Mokobodzki is proved : *If  $K$  is a Choquet simplex and if  $B$  is a compact convex subset of  $K$  with  $\partial_e B \subseteq \partial_e K$  then  $B$  is a face of  $K$ .*

This again follows immediately from the preceding discussion once we note that  $F = \text{conv}(\partial_e B)$  is a face and that  $\partial_e \bar{F} = \partial_e B \subseteq \partial_e K$ .

### 5. Compactness of face (C).

Considerations of subsets of  $K$  of the form *face*  $(x)$ ,  $x \in K$ , have proved useful in several contexts : for example, they are important

in the local version of Choquet's Uniqueness Theorem [9] and in Wil's proof of the existence and uniqueness of central measures for points of  $K$  (see [1]). Their usefulness is also suggested by the following simple result :

**PROPOSITION 5.1.** — *If  $K$  is metrisable then every closed face  $F$  of  $K$  has the form  $\text{face}(x)$  for some  $x \in K$ .*

*Proof.* — The metrisability implies that  $K$ , and hence  $F$ , is separable. Let  $\{k_n\}$  be a dense subset of  $F$  and define

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} k_n$$

It is clear that this series defines an element  $x$  of  $F$ , and that  $k_n \in \text{face}(x) \subseteq F$ ; it follows that  $\bar{F} = \text{face}(x)$ .

In view of the preceding remarks, it is natural to look for conditions which ensure that  $\text{face}(x)$  is closed. This was recently done in [11] where it is proved, among other things, that  $\text{face}(x)$  is closed iff  $\text{face}(x) = D_n(x)$  for some  $n$  (see § 2 for the definition of  $D_n(x)$ ). We propose to generalize this result in the theorem below. It should be pointed out that the proof of the implication (1)  $\Rightarrow$  (2) in this theorem follows an argument used in [14] in a different situation. We denote by  $P(K)^+$  the cone of non-negative continuous convex functions on  $K$ .

**THEOREM 5.2.** — *Let  $C$  be a proper compact convex subset of  $K$  without core points. Then the following are equivalent :*

- 1) *face(C) is closed.*
- 2) *face(C) =  $D_n(C)$  for some  $n$ .*
- 3) *If  $f_m \in P(K)^+$  and  $\lim_{m \rightarrow \infty} f_m(u) = 0$  uniformly for  $u \in C$  then  $\lim_{m \rightarrow \infty} \hat{f}_m(y) = 0$  uniformly for  $y \in \text{face}(C)$ .*
- 4) *face(C) is norm-closed in the space  $A(K)^*$ .*

Moreover, all the above statements are implied by  
 (\*) :  $\text{lin } M_c^+$  is a norm-closed (or weak\*-closed) subspace of  $M(K)$ , where  $M_c^+ = \{\mu \in M_1^+(K) : \text{resultant } (\mu) \in C\}$ .

*Proof.* — (1)  $\Rightarrow$  (2). Suppose face (C) is closed and face (C)  $\neq D_n(C)$  for all  $n$ . This means that given any  $n \in \mathbb{N}^+$  (= set of positive integers),  $\exists y_n \in \text{face (C)}$  such that  $y_n \not\leq n2^n C$ . (By this is meant that

$$y_n \not\leq n2^n u, \forall u \in C).$$

Define  $y = \sum_{n=1}^{\infty} 2^{-n} y_n$ . Now  $y \in \text{face (C)}$  as this set is closed by assumption. But then  $y \not\leq mC$  for all  $m \in \mathbb{N}^+$  which is a contradiction as  $\text{face (C)} = \bigcup_{m=1}^{\infty} D_m(C)$ .

(2)  $\Rightarrow$  (3). By (2), if  $y \in \text{face (C)}$  then  $y \leq nu$  for some  $u \in C$ , hence  $\hat{f}_m(y) \leq n \hat{f}_m(u)$  as  $\hat{f}_m$  is a concave function and thus  $\lim_{m \rightarrow \infty} \hat{f}_m(y) = 0$  uniformly on face (C) if  $\lim_{m \rightarrow \infty} \hat{f}_m(u) = 0$  uniformly on C.

(3)  $\Rightarrow$  (2). Suppose  $\text{face (C)} \neq D_n(C) \forall n \in \mathbb{N}^+$ . This means that given  $n \in \mathbb{N}^+$ ,  $\exists y_n \in \text{face (C)}$  such that  $y_n \not\leq nC$  but  $y_n \leq n^{m(n)} u_n$  for some  $u_n \in C$  and for some sufficiently large  $m(n) \in \mathbb{N}^+$ . Since  $(nC - y_n) \cap \tilde{K} = \emptyset$ , where  $\tilde{K}$  is the (closed) cone generated by K, a standard Hahn-Banach argument shows that  $\exists a_n \in A(K)^+$  such that

$$a_n(y_n) > n a_n(u) \forall u \in C$$

and

$$a_n(y_n) \leq n^{m(n)} a_n(u_n)$$

These inequalities imply that  $\sup \{a_n(u) : u \in C\} > 0$

and

$$a_n(y_n) > n \sup \{a_n(u) : u \in C\}.$$

Let

$$b_n = a_n/n \sup \{a_n(u) : u \in C\}$$

Then

$$b_n \in A(K)^+ \text{ and if } u \in C,$$

$$b_n(u) = a_n(u)/n \sup \{a_n(u) : u \in C\} \leq \frac{1}{n}$$

showing that  $\lim_{n \rightarrow \infty} b_n(u) = 0$  uniformly for  $u \in C$  : however,  $b_n(y_n) > 1$  and so  $b_n$  does not tend to zero uniformly on face (C), contradicting (3).

(2)  $\Rightarrow$  (1). Obvious.

(2)  $\Rightarrow$  (4). Obvious.

(4)  $\Rightarrow$  (2). By the regular embedding of  $K$  in  $A(K)^*$ , we can regard each  $x \in K$  as a member of the unit ball of  $A(K)^*$ . As a norm-closed subset of the complete metric space  $A(K)^*$ , face (C) is complete and hence by the Baire Category Theorem, some  $D_{n_0}(C)$  must have non-empty relative interior, i.e. there exists some  $y_0 \in D_{n_0}(C)$  such that for some neighbourhood of the origin,

$$U = \{u \in A(K)^* : \|u\| < \eta\}$$

$$(y_0 + U) \cap \text{face}(C) \subseteq D_{n_0}(C).$$

Let  $y \in \text{face}(C)$  and define

$$z = \frac{\eta}{2 + \eta} y + \frac{2}{2 + \eta} y_0 \in \text{face}(C)$$

Then 
$$\|z - y_0\| = \frac{\eta}{2 + \eta} \|y - y_0\| \leq \frac{2\eta}{2 + \eta} < \eta$$

and hence 
$$z \in (y_0 + U) \cap \text{face}(C) \subseteq D_{n_0}(C).$$

Therefore, for some  $c \in C$  and  $k \in K$ ,

$$\frac{\eta}{2 + \eta} y + \frac{2}{2 + \eta} y_0 = n_0 c - (n_0 - 1) k$$

or, 
$$c = \frac{1}{n_0} \cdot \frac{\eta}{2 + \eta} y + \left(1 - \frac{1}{n_0} \cdot \frac{\eta}{2 + \eta}\right) k'$$

where 
$$k' = \frac{\left(1 - \frac{1}{n_0}\right) k + \frac{1}{n_0} \cdot \frac{2}{2 + \eta} y_0}{1 - \frac{1}{n_0} \cdot \frac{\eta}{2 + \eta}} \in K.$$

Thus, 
$$y \in D_\alpha(C) \quad \text{where} \quad \alpha = \left(1 + \frac{2}{\eta}\right) n_0$$

and this implies (2).

This completes the proof of the equivalence of (1), (2), (3) and (4). As far as statement (\*) is concerned, first note that since

$M_c^+$  is a  $w^*$  compact subset of  $M_i^+(K)$ , by a known result [1 : page 112]  $\text{lin } M_c^+$  is  $w^*$  closed iff  $\text{lin } M_c^+$  is norm-closed. Now the proof follows exactly the argument used in [11] to prove (vi)  $\Rightarrow$  (i) in Theorem 1.9 of that paper.

*Remarks 1.* – The use of the Baire Category Theorem above was suggested by the proof of a similar result in [5]. The argument proving (1)  $\Rightarrow$  (2) could also be used here.

2. – We have not been able to decide whether any of the first four statements in Theorem 5.2 implies (\*).

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