ANNALES DE L'INSTITUT FOURIER

DAVID STEGENGA

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Annales de l'institut Fourier, tome 25, nº 2 (1975), p. 213-217
<http://www.numdam.org/item?id=AIF_1975__25_2_213_0>

Ann. Inst. Fourier, Grenoble
25, 2 (1975), 213-217

A NOTE ON SPACES OF TYPE $H^\infty + C$

by David A. STEGENGA (1)

This note is concerned with a theorem of Sarason [1], [4], which states that $H^\infty + C$ is a closed subalgebra of L^∞ on the unit circle T. Here C is the space of all continuous functions on T and H^∞ is the subspace of L^∞ consisting of those bounded functions whose negative Fourier coefficients are zero. L^∞ is given the essential supremum norm and pointwise multiplication.

In [2], Rudin generalizes a proof of Sarason's theorem given by Zalcman [5], to the context of Banach spaces. With this generalization Rudin shows that various natural analogues of $H^\infty + C$ are closed subspaces, although not always a subalgebra. Among these natural analogues are those obtained by replacing the unit disc by polydiscs or by balls in $\mathbf{C}^n$, the space of n complex variables.

The purpose of this note is to show that Rudin's theorem is general enough to have a converse. The proof of the converse theorem is relatively easy but we give an example which shows that not too much more can be said about the nature of the solution in general.

I would like to thank Walter Rudin for calling my attention to to this problem.

We now proceed to prove the converse of the theorem referred to above. This is Theorem 1.2 in [2], and its statement is the following :

THEOREM 1. – *Suppose* Y *and* Z *are closed subspaces of a Banach space* X, *and suppose that there is a collection* Φ *of bounded operators taking* X *into* Y *with the following properties :*

a) *Every* $\Lambda \in \Phi$ *maps* Z *into* $Z \cap Y$.

(1) The author was supported by NSF Grant GP-36418X1.

b) $\sup \{\|\Lambda\| : \Lambda \in \Phi\} < \infty$.

c) *To every* $y \in Y$ *and to every* $\epsilon > 0$ *corresponds a* $\Lambda \in \Phi$ *such that* $\|y - \Lambda y\| < \epsilon$.

Then $Y + Z$ *is closed.*

The converse we have for this theorem is the following :

THEOREM 2. – *Suppose* Y *and* Z *are closed subspace of a Banach space* X, *and suppose that* $X = Y + Z$. *Then there exist a collection* Φ *of bounded operators taking* X *into* Y *with properties* (a), (b), *and* (c) *of Theorem 1. Furthermore,* Φ *can be chosen so that for every* $y \in Y$ *there corresponds a* $\Lambda \in \Phi$ *with* $\Lambda y = y$.

Proof. – Fix $y_0 \in Y$, by the Hahn Banach extension theorem there exist a bounded linear functional φ_1 on X of norm one, such that $\varphi_1(y_0) = \|y_0\|$. Also, there exists a bounded linear functional φ_2 on the quotient space $Y/Y \cap Z$ of norm one,

$$\text{with} \quad \varphi_2(y_0 + Y \cap Z) = \|y_0 + Y \cap Z\|.$$

Finally, there exists a $z_0 \in Y \cap Z$, with $\|y_0 - z_0\| \leqslant 2\,\|y_0 + Y \cap Z\|$ since the quotient norm is the infimum in z_0 of $\|y_0 - z_0\|$. If $y_0 \in Y \cap Z$, define $\Lambda(x) = \varphi_1(x)\, y_0\, \|y_0\|^{-1}$ and if $y_0 \notin Y \cap Z$ define

$$\Lambda(x) = \varphi_1(x)\, z_0\, \|y_0\|^{-1} + \varphi_2(y + Y \cap Z)\,(y_0 - z_0)\,\|y_0 + Y \cap Z\|^{-1}$$

for $x = y + z$, with $y \in Y$ and $z \in Z$. Note that if x is also equal to $y' + z'$, with $y' \in Y$ and $z' \in Z$, then $y + Z \cap Y = y' + Z \cap Y$ so the second definition of Λ makes sense. In either case we clearly have Λ a linear operator mapping X into Y, satisfying (a), and $\Lambda(y_0) = y_0$. We must show that $\|\Lambda\|$ is bounded independent of y_0. For $y_0 \in Y \cap Z$, we have $\|\Lambda\| = 1$ so we consider the second case $y_0 \notin Y \cap Z$. We have in this case

$$\begin{aligned}\|\Lambda(x)\| &\leqslant \|z_0\|\,\|x\|\,\|y_0\|^{-1} + \|y_0 - z_0\|\,\|y_0 + Y \cap Z\|^{-1}\,\|y + Y \cap Z\| \\ &\leqslant (\|z_0 - y_0\| + \|y_0\|)\,\|x\|\,\|y_0\|^{-1} + 2\,\|y + Y \cap Z\| \\ &\leqslant 3\,\|x\| + 2\|y + Y \cap Z\|.\end{aligned}$$

Since $X = Y + Z$, the quotient space $X/Y \cap Z$ is the direct sum of its closed subspaces $Y/Y \cap Z$ and $Z/Y \cap Z$, consequently the projection operator mapping $X/Y \cap Z$ into $Y/Y \cap Z$ is bounded. This is a well-known fact that follows easily from the closed graph theorem. This means that there is a constant $M > 0$ with

$$\|y + Y \cap Z\| \leqslant M \|y + z + Y \cap Z\| \leqslant M \|y + z\|$$

for all $y \in Y$ and $z \in Z$. Thus $\|\Lambda\|$ is bounded by $2M + 3$. The family of operators $\{\Lambda\}$ therefore satisfy property (b) and the proof is complete.

In order to prove that $H^\infty + C$ is a closed subspace of L^∞, set $\Phi = \{\Lambda_n\}_{n=1}^\infty$ where Λ_n is the operator that assigns to each $F \in L^\infty$ the arithmetic mean of the first n partial sums of its Fourier series. Then with $X = L^\infty$, $Y = C$, and $Z = H^\infty$, Φ satisfies the conditions of theorem 1 and the result follows. Condition (c) is Fejer's theorem. In this example a property stronger than (c) is satisfied, namely if K is any finite subset of Y and $\epsilon > 0$ then there exists a $\Lambda \in \Phi$ with

$$\sup_{y \in Z} \|y - \Lambda y\| < \epsilon .$$

In the proof of theorem 2, each operator Λ had a small range, dim (range Λ) $\leqslant 2$. A natural question is to ask if it is possible to satisfy condition (c) modified as above to involve finitely many elements of Y. This question has a negative answer. A counter-example is provided by switching the roles of H^∞ and C in the above. Let $X = H^\infty + C$, $Y = H^\infty$, $Z = C$, and Φ a collection of bounded operators mapping $H^\infty + C$ into H^∞ satisfying (a), (b), and (c) of theorem 1. That such a collection Φ exists is the content of theorem 2. Since $H^\infty \cap C$ is the disk algebra A and each $\Lambda \in \Phi$ maps C into A we restrict our attention to C.

THEOREM 3. – *Suppose Φ is a uniformly bounded family of operators mapping* C *into* A *and satisfying condition* (c) *for* $Y = A$. *Then there exists an $\epsilon > 0$ and a finite subset* K *of* A *with*

$$\sup_{F \in K} \|F - \Lambda F\|_\infty \geqslant \epsilon$$

for all $\Lambda \in \Phi$.

Proof. – Suppose, to get a contradiction, that for each $\epsilon > 0$ and each finite subset K of A there exist a $\Lambda \in \Phi$ with

$$\sup_{F \in K} \|F - \Lambda F\|_\infty < \epsilon.$$

Since Φ is a uniformly bounded family of operators we get the same conclusion if K is a compact subset of A. Let R_θ be the translation operator on C given by

$$R_\theta F(t) = F(\theta + t)\ (-\pi \leqslant t \leqslant \pi).$$

We identify T with the interval $[-\pi, \pi]$ and consider C as the space of continuous periodic functions of period 2π. By uniform continuity, the mapping θ into R_θ is continuous in the operator norm topology. As a result we can define new bounded operators on C by

$$\tilde{\Lambda} F = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_{-\theta}\, \Lambda R_\theta F\, d\theta$$

for $F \in C$. Let $e_n(t) = e^{int}$ for each integer n, then a simple computation shows that $\tilde{\Lambda} e_n = (\Lambda e_n)^\wedge(n)\, e_n$. Since Λ maps C into A we have $\tilde{\Lambda} e_n = 0$ for $n < 0$, and so $\tilde{\Lambda}$ maps C into A. Clearly, the family of operation $\{\tilde{\Lambda}\}$ are uniformly bounded in norm. We would like to show that they satisfy condition (c). Since A is translation invariant, and $R_\theta F$ is a continuous function of θ with values in A, for each $F \in A$, we have that $K = \{R_\theta F : -\pi \leqslant \theta \leqslant \pi\}$ is a compact subset of A. Let $\epsilon > 0$, then by our assumptions there exists a $\Lambda \in \Phi$ with

$$\sup_\theta \|R_\theta F - \Lambda R_\theta F\|_\infty < \epsilon.$$

Thus we have

$$\|F - \tilde{\Lambda} F\|_\infty \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \|R_{-\theta} R_\theta F - R_{-\theta} \Lambda R_\theta F\|_\infty\, d\theta$$

$$\leqslant \sup_\theta \|R_\theta F - \Lambda R_\theta F\|_\infty < \epsilon$$

and so condition (c) is satisfied. Now let $F = \sum_n a_n e_n$ be a trigonometric polynomial and $F_1 = \sum_{n \geqslant 0} a_n e_n$. The critical property of $\tilde{\Lambda}$ is that $\tilde{\Lambda} F = \tilde{\Lambda} F_1$, consequently, for all $\tilde{\Lambda}$ we have

$$\begin{aligned}\|F_1\|_\infty &\leqslant \|F_1 - \widetilde{\Lambda}F_1\|_\infty + \|\widetilde{\Lambda}F_1\|_\infty \\ &\leqslant \|F_1 - \widetilde{\Lambda}F_1\|_\infty + \sup_{\widetilde{\Lambda}} \|\widetilde{\Lambda}\| \, \|F\|_\infty .\end{aligned}$$

By property (c), $\|F_1\|_\infty \leqslant K \|F\|_\infty$ where $K = \sup \|\Lambda\| < \infty$. This is a contradiction, since it implies that

$$\sup_{|z| \leqslant 1} |F(z)| \leqslant K \sup_{|z| \leqslant 1} |\mathrm{Re}F(z)|$$

for any F analytic in $|z| < 1$ (ReF is the real part of F), which is clearly false. The conformal map of the unit disc onto a vertical strip is an explicit counterexample. This completes the proof of the theorem.

One final remark, is that the averaging process described in theorem 3 is a method employed by Rudin [3] to show that there are no bounded projections of C onto A.

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Manuscrit reçu le 30 juillet 1974
accepté par B. Malgrange.

David A. Stegenga,
Institute for Advanced Study
School of Mathematics
Princeton, N.J. 08540 (USA).