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IBRAHIM DIBAG

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DECOMPOSITION IN THE LARGE OF TWO-FORMS OF CONSTANT RANK

by Ibrahim DIBAG

0. Introduction.

"Whether a vector-bundle admits a 2-form of constant rank" has been an important question in algebraic topology ; and a good deal of research (4, 5, 10) has been done on the subject. In this thesis we shall take, apriori, a vector-bundle that does admit such a 2-form, w, of constant rank $2s$. We shall then show that, w, *locally* decomposes into a sum : $w = y_1 \wedge y_{s+1} + y_2 \wedge y_{s+2} + \cdots + y_s \wedge y_{2s}$ of products of linearly-independent 1-forms (y_i) on E. The main task of the thesis is to find necessary and sufficient conditions for, w, to have a *global* such decomposition.

We shall define a $2s$-dimensional sub-bundle S_w of E on which, w, can be regarded as a 2-form of maximal rank ; and a necessary condition for, w, to decompose globally is that S_w is a trivial (product) bundle.

Using the triviality of S_w we shall represent w, as a map $w_1 : B \to I_s$; where B is the base-space, and $I_s = SO(2s)/U(s)$ is the homogenous space ; and, w, decomposes globally if and only if w_1 lifts to $SO(2s)$.

We shall then investigate the integercohomology, $H^*(I_s ; Z)$, of I_s ; and the cohomology-mapping

$$p^* : H^*(I_s ; Z) \to H^*(SO(2s) ; Z)$$

induced by the projection $p : SO(2s) \to I_s$. We shall deduce that :

1) $H^*(I_s ; Z)$ is, additively, generated by the duals of normal cells $[2i_1 ; 2i_2 ; \cdots ; 2i_k]$ for $s > i_1 > i_2 > \cdots > i_k \geqslant 1$ and the zero-cell $[0]$.

2) $p^*[2i_1 \; ; \; 2i_2 \; ; \cdots ; \; 2i_k]^*$ is of order 2 in $H^*(SO(2s) \; ; \; Z)$. From these two statements will follow the theorem that : "A necessary condition for the liftability of w_1 is that Image of $w_1^* \subset$ Subgroup of elements of $H^*(B : Z)$ of order 2" and the corollary that :

"If $H^*(B \; ; \; Z)$ does not have any 2-torsion ; then a necessary condition for the liftability of w_1 is $w_1^* = 0$.

These results will then be applied to some special cases, and a full discussion will be given of the existence and decomposability of 2-forms of constant rank on i) spheres, ii) real, and iii) complex-projective spaces.

1. Fiber-bundle structures over two-forms of rank $2s$.

1.1. *Définitions and notation :*

Let E be a real n-dimensional inner-product space ; and as usual, identify E with its dual E^* through the metric.

Then it is well known (e.g. refer to [9]) that :

i) Any 2-form, w, on E decomposes into

$$w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$$

a sum of products of linearly-independent vectors (y_i) of E.

ii) The number of terms in any such decomposition is unique ; and is called the "rank" of w.

Thus if $\widetilde{V}_{2s}(E)$ = manifold of ordered $2s$-tuplets of linearly-independent vectors in E.

$\widetilde{A}_s(E)$ = Set of 2-forms on E of rank $2s$.

We can define $\widetilde{f}_s : \widetilde{V}_{2s}(E) \to \widetilde{A}_s(E)$ by

$$(y_1 , y_2 , \ldots , y_{2s}) \mapsto y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$$

and by the above, $\widetilde{f}_s$ is "onto". Also, the real-symplectic group, $Sp(s \; ; \; R)$ acts freely and transitively on the fibers of $\widetilde{f}_s$; and thus $\widetilde{f}_s$ factors through the orbit-space, $\widetilde{V}_{2s}(E)/Sp(s \; ; \; R)$, in a bijective fashion.

1.2. *The Principal* $Sp(s\,;R)$-bundle : $\widetilde{V}_{2s}(E)\ (\widetilde{A}_s(E)\ ;\ Sp(s\,;R))$

1.2.1. LEMMA. – *The map* $\widetilde{f}_s : \widetilde{V}_{2s}(E) \to \widetilde{A}_s(E)$ *admits a local cross-section.*

Note : In the following proof, we shall, for convenience of notation, take the definition of $\widetilde{f}_s$ to be :

$$\widetilde{f}_s(y_1\,,\ldots,y_{2s}) = y_1 \wedge y_2 + \cdots + y_{2s-1} \wedge y_{2s}\,.$$

Proof. – Choose a basis $(e_1\,,e_2\,,\ldots,e_n)$ of E. Then any $w \in \Lambda^2 E$ can be written as $w = \sum_{i<j} a_{ij}(w)\, e_i \wedge e_j$ where $a_{ij} : \Lambda^2 E \to R^1$ are continuous functions on $\Lambda^2 E$.

$Q_r = (w \in A_r(E)/a_{12}(w) \neq 0)$ is an open subset of $\widetilde{A}_r(E)$ for $1 \leqslant r \leqslant s$.

$$S_{2r} = \widetilde{f}_r^{-1}(Q_r) \subset V_{2r}(E) \quad ; \quad \widetilde{f}_r : S_{2r} \to Q_r$$

well-defined.

Let F be the subspace of E generated by $(e_3\,,e_4\,,\ldots,e_n)$

$$((y_1\,,y_2) \quad ; \quad (y_3\,,y_4\,,\ldots,y_{2s})) \mapsto (y_1\,,y_2\,,\ldots,y_{2s})$$

defines a continuous map

$$i : S_2 \times \widetilde{V}_{2s-2}(F) \to S_{2s}\ ; \qquad \text{and} \qquad (q\ ;\ w_0) \mapsto q + w_0$$

defines a continuous map $B : Q_1 \times A_{s-1}(F) \to Q_s$ and that

$$\widetilde{f}_s \circ i = B \circ (\widetilde{f}_1 \times \widetilde{f}_{s-1}).$$

Now, given $w \in Q_s$, we have :

$$w = \left(e_1 - \frac{a_{23}}{a_{12}}\, e_3 - \cdots - \frac{a_{2n}}{a_{12}}\, e_n\right) \wedge (a_{12}\, e_2 + \cdots + a_{1n}\, e_n) + w_0$$

where $w_0 \in \widetilde{A}_{s-1}(F)$. Let

$$y_1(w) = e_1 - \frac{a_{23}}{a_{12}}\, e_3 - \cdots - \frac{a_{2n}}{a_{12}}\, e$$

$$y_2(w) = a_{12}\, e_2 + a_{13}\, e_3 + \cdots + a_{1n}\, e_n\,.$$

Then define continuous maps :

$$k_s : Q_s \to S_2 \quad \text{by} \quad k_s(w) = (y_1(w) \ ; \ y_2(w))$$

$$p_s : Q \to \widetilde{A}_{s-1}(F) \quad \text{by} \quad p_s(w) = w_0 .$$

By définition : $B((\widetilde{f}_1 \circ k_s) \times p_s) = l_d$. We shall, now, prove by induction on s that $\widetilde{f}_s$ admits a local cross-section. For $s = 1$. Assume W.L.G. that $w \in Q_1$. Since $\widetilde{A}_{s-1}(F) = 0$; $p_1(w) = 0$.

Hence, $k_1 : Q_1 \to S_2$ yields the desired lifting of $\widetilde{f}_1$.

For $s > 1$; again assume W.L.G. that $w \in Q_s$, and that the inductive hypothesis holds for $s - 1$; i.e. there exists a neighbourhood U of $p_s(w)$ in $\widetilde{A}_{s-1}(F)$ and a lifting L_{s-1} of $\widetilde{f}_{s-1}$ over U. Then $N = p_s^{-1}(U) \subset Q_s$ is a neighbourhood for w in Q_s and hence in $\widetilde{A}_s(E)$; and

$$N \xrightarrow{k_1 \times (L_{s-1} \circ p_s)} S_2 \times V_{2s-2}(F) \xrightarrow{\quad i \quad} S_{2s} \subset \widetilde{V}_{2s}(E)$$

yields the desired lifting $L_s = i \circ (k_1 \times (L_{s-1} \circ p_s))$ of $\widetilde{f}_s$ over the neighbourhood N of w.

Q.E.D.

1.2.2. Proposition. – *$\widetilde{f}_s$ induces a principal* $Sp(s \ ; R)$*-bundle* : $\widetilde{V}_{2s}(E) \ (\widetilde{A}_s(E) \ ; \ Sp(s \ ; R))$.

Proof. – The existence of a local cross-section to $\widetilde{f}_s$ implies that $\widetilde{A}_s(E)$ and the orbit-space $\widetilde{V}_{2s}(E)/Sp(s \ ; R)$ are homeomorphic ; and that $\widetilde{f}_s$ and the projection $p : \widetilde{V}_{2s}(E) \to \widetilde{V}_{2s}(E)/Sp(s \ ; R)$ can be identified. The fact that the projection, p, induces *a* principal $Sp(s \ ; R)$-bundle follows from the fact that $Sp(s \ ; R)$ is a closed subgroup of $GL(2s \ ; R)$; and that the full-projection :

$$\widetilde{V}_{2s}(E) \to \widetilde{V}_{2s}(E)/GL(2s \ ; R) = G_{2s}(E)$$

= Grassmann-Manifold of $2s$-planes on E, induces a principal $GL(2s \ ; R)$-bundle.

1.3. *The Principal Unitary-bundle* : $V_{2s}(E) \ (A_s(E) \ ; \ U(s))$.

1.3.1. Let $V_{2s}(E)$ = Stiefel Manifold of orthonormal $2s$-frames on E. $A_s(E) = \widetilde{f}_s(V_{2s}(E))$ = Manifold of "normalized" 2-forms on E of rank $2s$. $f_s : V_{2s}(E) \to A_s(E)$ the "restriction" of $\widetilde{f}_s$ to $V_{2s}(E)$.

Then, $U(s) = Sp(s ; R) \cap O(2s)$ acts freely and transitively on the fibers of f_s ; and thus f_s factors through the orbit-space $V_{2s}(E)/U(s)$ in a bijective-fashion.

LEMMA. – *There exists a retraction* $r : \widetilde{V}_{2s}(E) \to V_{2s}(E)$ *such such that* $\widetilde{f}_s = f_s \circ r$ *when restricted to* $\widetilde{f}_s^{-1}(A_s(E))$.

Sketch of Proof. – Let $y \in V_{2s}(E)$; and pick any orthonormal frame e in the plane of y. Then $y = u \circ e$ for some $u \in GL(2s ; R)$. Let $u = tv$ be the polar decomposition of u into an orthogonal matrix t and a positive-definite symmetric matrix v. Put $r(y) = t \circ e$. Then, independence of the definition of $r(y)$ on the frame used, and other properties of r can easily be verified.

COROLLARY. – *Let* B *be a topological-space and* $w : B \to A_s(E)$ *a continuous map ; and* $\phi : B \to \widetilde{V}_{2s}(E)$ *a lifting of* w. *Then,* $r \circ \phi$ *lifts* w *to* $V_{2s}(E)$.

1.3.2. PROPOSITION. – f_s *induces a principal* $U(s)$*-bundle* :

$$V_{2s}(E)\ (A_s(E)\ ;\ U(s))\,.$$

Proof. – Let ϕ be a cross-section to $\widetilde{f}_s$ over some compact neighbourhood $\widetilde{N}$ of $\widetilde{A}_s(E)$. Put $N = \widetilde{N} \cap A_s(E)$ and $\phi_1 = \phi/N$. Then, by the preceeding Corollary, $r\phi_1$ is a cross-section to f_s over N. Define $t : N \times U(s) \to f_s^{-1}(N)$ by $t(n, u) = u((r\phi_1)\, n)$. Then, t, is a homeomorphism (by compactness). Hence f_s is locally-trivial ; and thus induces a principal $U(s)$-bundle.

1.4. *Retraction of* $\widetilde{A}_s(E)$ *onto* $A_s(E)$.

Let $\widetilde{W}_s$ = Set of non-singular and skew-symmetric $2s \times 2s$ matrices. W_s = Set of orthogonal and skew-symmetric $2s \times 2s$ matrices.

Then, $GL(2s ; R)$ acts on W_s by $u \circ k = uku^t$ for

$$u \in GL(2s\,;R)\text{i} \quad \text{and} \quad k \in \widetilde{W}_s$$

and the subgroup, $O(2s)$, leaves W_s invariant under this action. If

$k = gv$ is the polar-decomposition of $k \in \widetilde{W}_s$; then $g \in W_s$; and thus $k \to g$ defines a projection $p : \widetilde{W}_s \to W_s$.

LEMMA. – *There exists a continuous deformation retraction of $\widetilde{W}_s$ onto W_s that commutes with the action of* $O(2s)$.

Proof. – Define a homotopy $h_r : \widetilde{W}_s \to W_s$ by

$$h_r(gv) = g((1 - r)v + rI_d)$$

Then, $h_o = I_d$; $h_I = p$; and h_r commutes with the action of $O(2s)$.

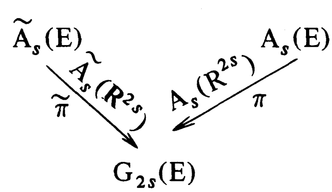
From this Lemma we recover the following :

PROPOSITION. – *There exists a retraction* $\theta : \widetilde{A}_s(E) \to A_s(E)$.

Proof. – Let's first assume that $n = 2s$. Then, an orthonormal frame e on E defines homeomorphisms ; $t_e : \widetilde{W}_s \to \widetilde{A}_s(E)$ and $t_e : W_s \to A_s(E)$ by $t_e(k) = \sum\limits_{i<j} k_{ij}\, e_i \wedge e_j$ and $t_e = t_e / W_s$.

A homotopy $f_r : \widetilde{A}_s(E) \to A_s(E)$ can be defined by $f_r = t_e \circ h_r \circ t_e^{-1}$ and it is, immediately, verified that this definition is independent of the orthonormal frame used. Thus, $\theta = f_1$ yields the desired retraction.

For $n \geqslant 2s$; we have the diagram :



a retraction θ_p ; and a homotopy $(f_r)_p : \widetilde{\pi}^{-1}(p) \to \pi^{-1}(p)$ over each $2s$-plane, $p \in G_{2s}(E)$. Then, the collections , $\theta = (\theta_p)_{p \in G_{2s}(E)}$ and $f_r = (f_r)_p$ yield the desired retraction and the homotopy respectively.

Q.E.D.

2. Decomposability of two-forms of constant rank.

2.1. *Notations and definitions :*

Let E be an R^n-bundle (with a Riemannian-metric) over a connected base-space B. Let $\widetilde{V}_{2s}(E)$, $V_{2s}(E)$, $\widetilde{A}_s(E)$, $A_s(E)$ be the associated-bundles to E with fibers $\widetilde{V}_{2s}(R^n)$, $V_{2s}(R^n)$, $\widetilde{A}_s(R^n)$, $A_s(R^n)$ respectively. A 2-form, w, on E of constant rank $2s$ is, by definition, *a* cross-section to $A_s(E)$. The maps $\widetilde{f}_s(E) : \widetilde{V}_{2s}(E) \to \widetilde{A}_s(E)$ and $f_s(E) : V_{2s}(E) \to A_s(E)$ are defined and we have the following "global" versions of Propositions 1.2.2. and 1.3.2. :

PROPOSITION 1.2.2.* – $\widetilde{f}_s(E)$ *induces a principal* $Sp(s ; R)$*-bundle.*

PROPOSITION 1.3.2.* – $f_s(E)$ *induces a principal* $U(s)$*-bundle.*

2.2. *Local-Decomposability and the Sub-bundle* S_w :

DEFINITION. – *A 2-form, w, on E of constant rank $2s$ is said to be locally-decomposable iff each point $x \in B$ has a neighbourhood U_x and linearly-independent 1-forms (y_i) $i = 1, \ldots, 2s$ on E over U_x* s.t. $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ *over U_x. (Or, alternatively, there exists a cross-section, y, to $\widetilde{V}_{2s}(E)$ over U_x such that $w = \widetilde{f}_s \circ y$).*

LEMMA. – *A 2-form, w, of constant rank $2s$ on E is locally-decomposable.*

Proof. – Let $x \in B$; and, c, a cross-section to $\widetilde{f}_s(E)$:

$$\widetilde{V}_{2s}(E) \to \widetilde{A}_s(E)$$

over a neighbourhood N of $w(x)$ in $\widetilde{A}_s(E)$. Then, the composite $w^{-1}(N) \xrightarrow{w} N \xrightarrow{c} \widetilde{V}_{2s}(E)$ defines a cross-section $y = cw$ to $\widetilde{f}_s(E)$ over $w^{-1}(N)$ such that $\widetilde{f}_s \circ y = w$. Q.E.D.

Given a 2-form, w, of constant rank $2s$; then at each point $x \in B$, $w(x)$ determines a $2s$-dimensional subspace $S_{w(x)}$ of E_x on which it is of maximal rank ; and local decomposability of w, immetiately yields the following :

PROPOSITION. — *The union* $S_w = \bigcup_{x \in B} S_{w(x)}$ *is a sub-bundle of* E ; *and, w, being a* 2*-form on* S_w *of maximal-rank determines a reduction of its structure group from* $GL(2s\,;R)$ *to* $Sp(s\,;R)$.

This Proposition, clearly, demonstrates that the "existence of a 2-form of constant rank on E" (which is assumed apriori in the thesis) is, already, a strong condition ; and will be useful in proving non-existence theorems about 2-forms of constant rank on spheres and projective-spaces in the last-chapter.

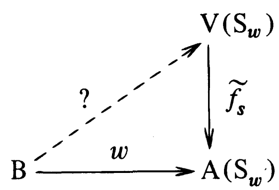
2.3. *Decomposition of* 2*-forms of constant rank :*

Let $\widetilde{V}(S_w)$, $V(S_w)$, $\widetilde{A}(S_w)$, $A(S_w)$ be the associated-bundles to S_w with fibers $\widetilde{V}(R^{2s})$, $V(R^{2s})$, $\widetilde{A}(R^{2s})$, $A(R^{2s})$ respectively.

DEFINITION. — *w is said to be decomposable iff*

$$w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$$

for linearly-independent 1*-forms* (y_i) *on E. (Or, alternatively, the diagram : admits a lifting).*



An immediate consequence of this definition is the following :

Observation. — If, *w*, is decomposable ; then S_w is a trivial (product)-bundle.

Let $r : \widetilde{V}(S_w) \to V(S_w)$ and $\theta : \widetilde{A}(S_w) \to A(S_w)$ be the retractions of Sections 1.3. and 1.4. (respectively) defined globally on S_w.

DEFINITION. — *The "normalization" of, w, is defined to be the composite* $\theta w : B \xrightarrow{w} \widetilde{A}(S_w) \xrightarrow{\theta} A(S_w)$ *and is a "normalized"* 2*-form of rank* 2*s. (i.e. a cross-section to* $A(S_w)$*).*

DEFINITION. – *A normalized 2-form, w, of rank 2s decomposes orthogonally iff* $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ *for orthonormal-frame* $y = (y_1, \ldots, y_{2s})$ *on* S_w.

PROPOSITION. – A *2-form, w, of constant rank 2s decomposes iff its normalization decomposes orthogonally.*

Proof. – Suppose, w, decomposes. i.e. there exists a continuous map $L : B \to \widetilde{V}(S_w)$ such that $\widetilde{f_s} \circ L = w$. Since θ is a retraction ; $w \simeq \theta w$, and thus $\widetilde{f_s} \circ L \simeq \theta w$. Since $\widetilde{f_s}$ is a fibration ; by the covering-homotopy-theorem ; there exists a lifting $T : B \to \widetilde{V}(S_w)$ of θw to $\widetilde{V}(S_w)$ and by the "global-version" of Corollary 1.3.1. rT is a lifting of θw to $V(S_w)$. Thus, θw decomposes orthogonally.

Conversely, suppose θw decomposes orthogonally ; i.e. that there exists a lift $k : B \to V(S_w)$ of θw to $V(S_w)$. Then,

$$f_s \circ k = \theta w \simeq w \ ;$$

and again, by the covering homotopy theorem, there exists a lifting of, w, to $\widetilde{V}(S_w)$.

Q.E.D.

By Observation 2.3., a necessary condition for w to decompose is that S_w is a trivial (product)-bundle. Let's choose a particular product representation : $S_w = B \times R^{2s}$ which gives rise to further product representations : i) $V(S_w) = B \times V(R^{2s}) = B \times O(2s)$ and ii) $A(S_w) = B \times A(R^{2s}) = B \times O(2s)/U(s)$ and a representation of θw as a map $w_1 : B \to O(2s)/U(s)$.

θw decomposes orthogonally iff w_1 lifts to $O(2s)$. Since B is connected and w_1 continuous ; we may, without loss of generality assume that $w_1(B) \subset I_s = SO(2s)/U(s)$; and then lifting w_1 to $O(2s)$ is equivalent to lifting it to $SO(2s)$. We can summarize this in a single :

THEOREM. – A *2-form, w, of constant rank 2s decomposes iff*

i) S_w *is a trivial (product)-bundle.*

ii) *The representation of its normalization as a map* w_1 :

$$B \to I_s = SO(2s)/U(s)$$

arising from any trivialization of S_w *lifts to* $SO(2s)$.

The method used above was to assume the existence apriori, of a metric on E (and thus on S_w) ; and to show that, w, decomposes iff its normalization (with respect to this metric) decomposes orthogonally.

A more and invariant approach does not pre-suppose the existence of a metric on S_w. w, determines a reduction of the structure-group of S_w to $Sp(s\,;R)$; and since $U(s)$ is a maximal compact subgroup of $Sp(s\,;R)$; it undergoes a further reduction to $U(s)$; and thus S_w admits a unique Hermitian metric. Then, w, becomes normalized with respect to the corresponding real-metric, and thus decomposes iff it decomposes orthogonally. The rest of the theory goes as before ; and one, again, obtains the above theorem with obvious modifications.

3. Cohomology of I_s.

3.1. *Preliminaries* :

Let $x \in P^{n-1}$; and ϕ_x be the "reflection" through the hyperplane perpendicular to x ; and ϕ_0 the reflection corresponding to the initial point $(1, 0, \ldots, 0)$. Then, we imbed $P^{n-1} \subset SO(n)$ by $x \to \phi_x \phi_0$. We, now, list the following standard results ; and for proofs we refer the reader to [8] pp. 40-45.

Observation : i) $P^{n-1} \cap SO(n-1) = P^{n-2}$. ii) $P^i \circ P^j = P^j \circ P^i$ *and* iii) $P^i \circ P^i = P^i \circ P^{i-1}$ *in* $SO(n)$.

Let P^{n-1}/P^{n-2} be the space obtained by collapsing P^{n-2} to a point ; and $SO(n)/SO(n-1)$ the left coset-space.

LEMMA. – *The natural-map* $T : P^{n-1}/P^{n-2} \to SO(n)/SO(n-1)$ *is a "homeomorphism".*

PROPOSITION. – *The matrix-multiplication*

$$m : (P^n \times SO(n) \,;\, P^{n-1} \times SO(n)) \to (SO(n+1) \,;\, SO(n))$$

is a relative-homeomorphism.

THEOREM. – $SO(n)$ *is a cell-complex with normal cells*

$$[i_1 \; ; i_2 \; ; \cdots ; i_k] \quad \text{for} \quad n > i_1 > i_2 > \cdots > i_k \geqslant 1$$

given by

$$E^{i_1} \times E^{i_2} \times \cdots \times E^{i_k} \to P^{i_1} \times P^{i_2} \times \cdots \times P^{i_k} \xrightarrow{m} SO(n)$$

and the zero-cell [0] *; and matrix-multiplication* m :

$$SO(n) \times SO(n) \to SO(n)$$

is a cellular-map.

3.2. *Cellular Structure of* I_s :

Observation : $I_s = SO(2s)/U(s) = SO(2s-1)/U(s-1)$.

Proof. – Obviously, $SO(2s-1) \cap U(s) = U(s-1)$ and

$$SO(2s-1) \circ U(s) = SO(2s)$$

by a dimension argument. Thus,

$$I_s = SO(2s-1) \circ U(s)/U(s) = SO(2s-1)/U(s-1).$$

Q.E.D.

Let $\overline{P}^{2s+1}$ and $\overline{P}^{2s}$ denote the images of P^{2s+1} and P^{2s} under the projections $SO(2s+2) \to I_{s+1}$ and $SO(2s+1) \to I_{s+1}$ respectively. We, then, have the following :

LEMMA. – $\overline{P}^{2s+1} = \overline{P}^{2s}$

Proof. – It is an immediate consequence of the fact that the "composite" $P^{2s+1} \subset SO(2s+2) \to I_{s+1}$ factors through $P_s(C)$; and that $P^{2s} \subset P^{2s+1} \to P_s(C)$ is "onto".

Q.E.D.

Let $v : SO(2s) \times I_s \to I_s$ be the action of $SO(2s)$ on I_s. Then, we obtain the analogue of Proposition 3.1. for I_s :

PROPOSITION. – $v : (P^{2s} \times I_s \; ; \; P^{2s-1} \times I_s) \to (I_{s+1} \; ; \; I_s)$ *is a relative-homeomorphism* which in turn becomes the key in the proof of the following

THEOREM. – I_s *is a cell-complex consisting of even-dimensional normal-cells* $[2i_1 ; 2i_2 ; \cdots ; 2i_k]$ *for* $s > i_1 > i_2 > \cdots > i_k \geqslant 1$, *given by*

$$E^{2i_1} \times \cdots \times E^{2i_k} \to P^{2i_1} \times \cdots \times P^{2i_k} \xrightarrow{m} SO(2s) \xrightarrow{\text{proj}^n} I_s$$

and the zer-cell [O] ; *and the action-map* $v : SO(2s) \times I_s \to I_s$ *is cellular.*

Proof. – We prove the theorem by induction on s.

For $s = 1$; I_1 is just the zero-cell O ; and thus $v : SO(2) \times I_1 \to I_1$ is, obviously, cellular. By the preceding proposition, I_{s+1} is the adjunction-space : $I_{s+1} = I_s \vee_v (P^{2s} \times I_s)$. We, now, apply the following standard Lemma : "If K and L′ are cell-complexes ; L a subcomplex of K and $v : L \to L'$ a cellular-map ; then the adjunction-space, $K \vee_v L'$ is a cell-complex having L′ as a subcomplex ; and the images of the cells of (K – L) as the remaining cells" with

$$K = P^{2s} \times I_s \quad ; \quad L = P^{2s-1} \times I_s \quad ; \quad L' = I_s$$

By the inductive hypothesis, $v : SO(2s) \times I_s \to I_s$; and hence its restriction to the subcomplex, $P^{2s-1} \times I_s$, is cellular ; and thus we deduce that, I_{s+1}, is a cell-complex having I_s as a subcomplex ; and the v-images of the cells of $(P^{2s} - P^{2s-1}) \times I_s$ as the remaining cells. By the inductive-hypothesis, the cells of I_s are normal cells $[2i_1 ; \cdots ; 2i_k]$ for $s > i_1 > \cdots > i_k \geqslant 1$, and the zero-cell [O] ; and the v-images of the cells of $(P^{2s} - P^{2s-1}) \times I_s$ are normal-cells $[2s ; 2i_2 ; \cdots ; 2i_k]$ for $s > i_2 > \cdots > i_k \geqslant 1$. The proof will be complete once we prove that : $v : SO(2s+2) \times I_{s+1} \to I_{s+1}$ is cellular ; and this is done in five steps :

i) $v : P^{2s} \times I_s \to I_{s+1}$ is cellular.

ii) $v : SO(2s+1) \times I_s \to I_{s+1}$ is cellular.

iii) $v : SO(2s+1) \times I_{s+1} \to I_{s+1}$ is cellular.

iv) $v : P^{2s+1} \times I_{s+1} \to I_{s+1}$ is cellular.

v) $v : SO(2s+2) \times I_{s+1} \to I_{s+1}$ is cellular.

Only iv) has a non-trivial proof which can be outlined as follows :

Proof of iv). – By iii) the restriction of, v, to the subcomplex, $P^{2s} \times I_{s+1}$ of, $P^{2s+1} \times I_{s+1}$, is cellular ; and thus it suffices to prove that :

$$v(P^{2s+1} \; ; (I_{s+1})^{2q}) \subset (I_{s+1})^{2(s+q)}$$

Let $s+1 > i_1 > i_2 > \cdots > i_k \geqslant 1$ and $i_1 + i_2 + \cdots + i_k = q$

$$v(P^{2s+1} \; ; \overline{P^{2i_1} \times \cdots \times P^{2i_k}}) = \overline{P^{2s+1} \times P^{2i_1} \times \cdots \times P^{2i_k}}$$

$$= \overline{P^{2i_1} \times \cdots \times P^{2i_k} \times P^{2s+1}} = v(P^{2i_1} \times \cdots \times P^{2i_k} \; ; \overline{P}^{2s+1})$$

$$= v(P^{2i_1} \times \cdots \times P^{2i_k} \; ; \overline{P}^{2s}) = \overline{P^{2s} \times P^{2i_1} \times \cdots \times P^{2i_k}}$$

$$= v(P^{2s} \; ; \overline{P^{2i_1} \times P^{2i_2} \times \cdots \times P^{2i_k}}) \subset v((SO(2s+1))^{2s} \; ; (I_{s+1})^{2q})$$

$\subset (I_{s+1})^{2(s+q)}$ by Part iii).

Q.E.D.

COROLLARY. – *The projection $p : SO(2s) \to I_s$ is cellular ; and maps normal cells $[2i_1 \; ; 2i_2 \; ; \cdots \; ; 2i_k]$ of $SO(2s)$ onto normal cells $[2i_1 \; ; 2i_2 \; ; \cdots \; ; 2i_k]$ of I_s. The images of the remaining cells,* i.e. *$[j_1 \; ; j_2 \; ; \cdots \; ; j_k]$ where j_t is odd for some $1 \leqslant t \leqslant k$ are contained in a skeleton of lower dimension.*

3.3. *Integer-Cohomology of I_s and the Lifting Problem :*

Since I_s is a cell-complex consisting of even dimensional cells only ; the co-boundary operator is identically zero ; and hence the $2q^{th}$-cohomology group $H^{2q}(I_s \; ; Z)$ coincides with $2q^{th}$-cochains, $C^{2q}(I_s \; ; Z)$, which is the free abelian group generated by the duals $[2i_1 \; ; \cdots \; ; 2i_k]^*$ of normal cells $[2i_1 \; ; \cdots \; ; 2i_k]$ for $q = i_1 + \cdots + i_k$.

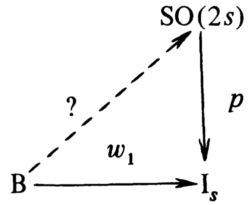
PROPOSITION. – *Image $p^* \subset$ Subgroup of elements of*

$$H^*(SO(2s) \; ; Z)$$

of order 2.

Proof. – By the above ; $p^*[2i_1 \; ; \cdots \; ; 2i_k]^* = [2i_1 \; ; \cdots \; ; 2i_k]^*$ and $2[2i_1 \; ; \cdots \; ; 2i_k]^* = \delta[2i_1 - 1 \; ; \cdots \; ; 2i_k]^*$ in $SO(2s)$.

THEOREM. – *A necessary condition for the lifting of the diagram :*



is that :

Image $w_1^* \subset$ *Subgroup of elements of* $H^*(B ; Z)$ *of order* 2.

COROLLARY. – *If* $H^*(B ; Z)$ *contains no 2-torsion ; then a necessary condition for the liftability of* w_1 *is that* $w_1^* = 0$.

4. Applications.

4.1. *Lower-Dimensional Spaces :*

We now, combine Theorems 2.3. and 3.4. with elementary obstruction theory to obtain the following :

PROPOSITION. – *Let, w, be a 2-form of constant rank* $2s (s > 1)$ *on an* R^n*-bundle* E *over a connected base-space* B *whose cohomology vanishes in dimensions greater than or equal to four. Necessary and sufficient conditions for, w, to decompose are* i) S_w *is a trivial (product)-bundle ; and* ii) $2w_1^* = 0$ *in* $H^2(B ; z)$ *where*

$$i \in H^2(I_s ; Z) = Z$$

is the generator and w_1 *is the representation of, w, arising from any trivialization of* S_w.

When B is an orientable 3-manifold, the tangent-bundle T(B) of B is trivial ; and S_w is the pull-back of the tangent-bundle $T(S^2)$ of the 2-sphere by the Gauss-Map $P : B \to S^2$; and thus the first Chern-Class, $c_1(S_w) = 2P^*(i)$, where $i \in H^2(S^2 ; Z)$ is the generator. Also by Alexander Duality, $2P^*(i) = 0$ iff $P^*(i) = 0$. Applying Theorem 2.3. yields the observation – A nowhere-vanishing 2-form, w, on an orientable 3-manifold decomposes iff $P^*(i) = 0$.

If we further specialize by taking B to be an open connected domain in R^3 and use the Hopf-Classification Theorem that $[P] \to P^*(i)$ is an isomorphism : $[B ; S^2] \to H^2(B ; Z)$; we obtain :

COROLLARY. – *A nowhere-vanishing 2-form, w, on an open connected domain* B *of* R^3 *decomposes iff the Gauss-Map* $P : B \to S^2$ *for* S_w *is null-homotopic.*

4.2. *Methods of Constructing p-forms on Spheres :*

i) *"From constant* $(p+1)$*-forms on* R^n*".*

Let $w \in \Lambda^{p+1} R^n$; and define $t : S^{n-1} \to \Lambda^p R^n$ by $t(x) = \delta_x(w)$ for all $x \in S^{n-1}$, where δ_x is the "adjoint" of the wedge-product map, $d_x : \Lambda R^n \to \Lambda R^n$ given by $d_x(y) = x \wedge y$. Then

$$\delta_x t(x) = \delta_x \circ \delta_x (w) = 0 \ ;$$

and thus, t, is a differentiable p-form on S^{n-1}.

ii) *"From constant p-forms on* R^n*"*

Let $w \in \Lambda^p R^n$. Then $t(x) = \delta_x \circ d_x(w) = w - d_x \circ \delta_x(w)$ for $x \in S^{n-1}$ defines a differentiable p-form, t, on S^{n-1} which is called the "tangential component" of w.

PROPOSITION. – *The tangential-component of a normalized* 2*-form of maximal-rank on* R^{2n} *is a* 2-form on S^{2n-1} *of constant rank* $(2n-2)$.

Proof. – $w = x \wedge \delta_x(w) + t(x)$ for all $x \in S^{n-1}$. The transformation on R^{2n} given by $x \to \delta_x(w)$ has square equal to minus Identity; and thus $\delta_{\delta_x(w)}(t(x)) = 0$ which implies that $t(x) \in \Lambda^2 U_x$ for

$$U_x = (x \ ; \ \delta_x(w)) \ ;$$

and hence $\text{rank}(w) = \text{rank}(x \wedge \delta_x(w)) + \text{rank}\ t(x)$.

Note. – $t(-x) = t(x)$; and thus, t, also defines a 2-form on P^{2n-1} of constant rank $(2n-2)$.

4.3. *Existence and decomposability of 2-forms of constant rank on spheres :*

PROPOSITION. – S^{4n+3} *admits a 2-form of constant rank* $4n$.

Proof. – Represent $S^{4n+3} = Sp(n+1)/Sp(n)$; and let

$$w_0 = e_1 \wedge e_{2n+1} + \cdots + e_{2n} \wedge e_{4n}$$

be a "normalized" 2-form at the distinguished point e_{4n+3}. For $x \in S^{4n+3}$, take any $u \in Sp(n+1)$ such that $u(e_{4n+3}) = x$; and define $w(x) = (\Lambda^2 u)\, w_0$. Since, $Sp(n) \subset U(2n)$ leaves w_0-invariant ; w is a well defined 2-form on S^{4n+3} of constant rank $4n$. Q.E.D.

Note. – i) $w(e^{i\theta}x) = e^{2i\theta}w(x)$ and ii) $\delta_{J(x)}(w(x)) = 0$ where J is multiplication by $i = \sqrt{-1}$; and thus, w, defines a 2-form on $P_{2n+1}(C)$ (and hence on P^{4n+3}) of constant rank $4n$.

Combining Proposition 2.2 with the Standard Theorem of [7] pp. 144 ; we obtain the following :

Statement. – The existence of a 2-form of constant rank $2s$ *on* S^n *implies :*

i) *the existence of a field of* $2s$*-frames on* S^n *for* $4s \leqslant n$.

ii) *the existence of a field of* $(n-2s)$-frames on S^n for $4s > n$.

and using Adams' results on Vector Fields on Spheres ; we deduce :

COROLLARY 1. – S^{4n+1} *does not admit a 2-form of constant rank* $2s$ for $0 < s < 2n$.

COROLLARY 2. – S^{2n} *does not admit a 2-form of constant rank* $2s$ for $0 < s < n$.

It is also a consequence of Adams' results and Kirchoff's Theorem (Refer to [7] pp. 217) that S^2 and S^6 are the only even dimensional spheres which are almost-complex, i.e. admit 2-forms of maximal rank. We can, now, summarize all these results in the following :

THEOREM. – 1) *The only even dimensional spheres which admit 2-forms of constant rank are* S^2 *and* S^6 *which admit 2-forms of maximal rank. None of these forms can be decomposed.*

2) *The only non-zero 2-forms of constant rank on* S^{4n+1} *are those of rank* $4n$*, and none of these forms can be decomposed.*

3) S^{4n+3} *admits 2-forms of constant ranks* 2, $4n$, $4n+2$. Those *of constant rank* 2 *always decompose ; whereas those of constant rank* $4n$ and $4n+2$ *cannot be decomposed for* $n \geqslant 2$. *A 2-form,* w, on S^7 *of constant rank* 4 *decomposes iff* i) S_w is a trivial bundle ; *and* ii) $\partial[w_1] \in \pi_6 U(2)$ *vanishes, where* w_1 *is the representation of the normalization of* w *(with respect to the canonical Riemannian-Metric on* S^7*) arising from any trivialization of* S_w *as a map*

$$w_1 : S^7 \to I_2 \ ; \quad \text{and} \quad \partial : \pi_7 I_2 \to \pi_6 U(2)$$

is the boundary-operator of the exact homotopy sequence of the fibration $SO(4) \to I_2$.

A 2-form, w, *on* S^7 *of constant rank* 6 *decomposes iff* i)

$$\partial [P] \in \pi_6 SO(6)$$

vanishes ; where $P : S^7 \to S^6$ *is the Gauss-Map for* S_w, *and*

$$\partial : \pi_7 S^6 \to \pi_6 SO(6)$$

is the boundary-operator of $SO(7) \to S^6$. ii) $\partial[w_1] \in \pi_6 U(3)$ vanishes ; *where* $w_1 : S^7 \to I_3$ *is the representation of the normalization of* w, and $\partial : \pi_7 I_3 \to \pi_6 U(3)$ *is the boundary-operator of* $SO(6) \to I_3$.

Remark. – The above theorem solves completely the existence and decomposability problem of 2-forms of constant rank for S^{2n}, S^{4n+1}, and for S^{4n+3} up to S^{15}. The first unsolved case is the existence question of 2-forms of constant rank 10 on S^{15}. The next is the existence question of 2-forms of constant rank 16 and 18 on S^{23}.

4.4. *Existence and Decomposability of 2-forms of constant rank on Projective Spaces :*

Parts 1 and 2 and most of 3 of the preceeding Theorem go through unchanged for real-projective spaces. The only changes in Part 3 are i) 2-forms, w, on P^{4n+3} of constant rank 2 decompose iff $c_1(S_w) \in H^2(P^{4n+3}; Z) = Z_2$ vanishes. ii) The discussions for 2-forms on S^7 do not have their analogues for P^7 ; since, w, can no longer be represented as an element of $\pi_7 I_2$ or $\pi_7 I_3$. A necessary condition for the decomposability of such forms is the decomposability of the corresponding forms on S^7 (which can be determined by the previous Theorem). However, whether this is sufficient is not known.

The case of the complex projective spaces can be best summarized in the following :

PROPOSITION. – P (C), *being a complex analytic manifold, admits a* 2*-form of constant rank* $2n$.

The only non-zero 2*-forms on* $P_{2n}(C)$ *of constant rank are those of constant rank* $4n$ *which cannot be decomposed.*

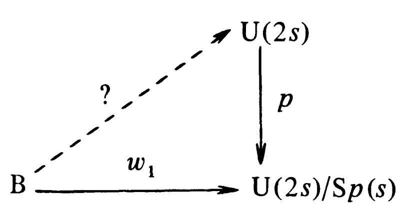
$P_{2n+1}(C)$, *admits* 2*-forms of constant ranks* $4n+2$ and $4n$ *which cannot be decomposed for* $n \geqslant 2$.

4.5. *Translation-Invariant* 2*-forms on Lie-Groups :*

PROPOSITION. – *A Lie-Group,* G, *admits translation-invariant* 2*-forms of constant rank* $2s$ *for* $2s \leqslant \dim G$; *and any translation-invariant* 2*-form on* G *decomposes.*

Appendix

The analogous problem of decomposing a 2-form of constant rank on a *complex* vector-bundle is attacked in exactly the same way ; and is reduced to the lifting-problem of the diagram :



One then investigates integer-cohomology of the homogenous-space, $U(2s)/Sp(s)$; and the Kernel of the map, p^* :

$$H^*(U(2s)/Sp(s)) \to H^*(U(2s))$$

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Ibrahim DIBAG,
Department of Mathematics
Middle-East Technical University
Ankara (Turquie).