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ADRIANO GARSIA
EUGÈNE RODEMICH

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MONOTONICITY OF CERTAIN FUNCTIONALS UNDER REARRANGEMENT

by A. M. GARSIA and E. RODEMICH [1].

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Introduction.

We shall be concerned here with classes of functions $f(x)$ measurable on $[0, 1]$ and satisfying a condition of the type

$$(\text{I.1}) \quad I_{\Psi, p}(f) = \int_0^1 \int_0^1 \Psi\left(\frac{f(x) - f(y)}{p(x - y)}\right) dx\, dy < \infty$$

where Ψ and p are restricted as follows

(I.2) *a)* $\Psi(u)$ is defined and continuous in $(-\infty, +\infty)$,
b) $\Psi(u) = \Psi(-u) \uparrow \infty$ (strictly) as $|u| \uparrow \infty$, and
(I.3) *a)* $p(u)$ is defined and continuous on $[-1, 1]$,
b) $p(u) = p(-u) \downarrow 0$ (strictly) as $|u| \downarrow 0$.

Let us recall that to each measurable $f(x)$ on $[0, 1]$ we can associate a non-increasing right continuous function $f^*(x)$ such that

(I.4)
$$m\{x : f(x) \geqslant \lambda\} = m\{x : f^*(x) \geqslant \lambda\} \quad \forall \lambda \in (-\infty, +\infty),$$

$f^*(x)$ is usually referred to as the « *non-increasing rearrangement of* f » and can be defined by the formula

$$(\text{I.5}) \qquad f^*(x) = \inf\{\lambda : m\{t : f(t) > \lambda\} \leqslant x\}.$$

The main contribution of this paper is the following

THEOREM I.1. — *If* Ψ *and* p *satisfy* (I.2) *and* (I.3) *and in addition* $\Psi(e^x)$ *is convex then*

$$(\text{I.6}) \qquad I_{\Psi, p}(f^*) \leqslant I_{\Psi, p}(f).$$

This result has been announced in [4] but no complete proof of it has yet appeared in print. The reader may refer to [4] for further background information. Basically our motivation for proving (I.6) has been to obtain an extension and sharpening of the following result established in [7].

THEOREM I.2. — *Let*

$$\int_0^1 \int_0^1 \Psi\left(\frac{f(x) - f(y)}{p(x-y)}\right) dx\, dy \leqslant B < \infty$$

and suppose that p *and* Ψ *in addition to* (I.2) *and* (I.3) *satisfy the condition*

$$\text{(I.7)} \qquad \int_0^1 \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u) < \infty \quad (^2),$$

then $f(x)$ *is essentially continuous and for almost all* x, y:

$$\text{(I.8)} \qquad |f(x) - f(y)| \leqslant 8 \int_0^{|x-y|} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u).$$

Here, by means of (I.6) we shall obtain a considerable improvement of (I.8) at least in the case $\Psi(u) = |u|^\alpha$ $(\alpha > 1)$, in addition we shall also be able to draw some conclusions in case (I.7) is not satisfied.

The particular cases $\Psi(u) = e^{c|u|^2}$, $\Psi(u) = e^{c|u|}$ of (I.8) have so far proved quite useful in probability theory. For instance in [7] and [5] we show that (I.8) with $\Psi(u) = e^{c|u|^2}$ implies the continuity of paths for certain Gaussian processes. Also, Getoor and Kesten in [8] use (I.8) with $\Psi(u) = e^{c|u|}$ to derive the continuity of local times for certain Markov processes. Some further applications of (I.8) have been discovered by Strook (3).

Another interesting use of I.8 can be made to obtain a rather direct and simple proof of the following result of Nisio (see [12]). If $\{\varphi_n(t)\}$ is a complete orthonormal system on [0, 1], the series

$$X_t(\omega) = \sum_{\nu=1}^{\infty} \theta_\nu(\omega) \int_0^t \varphi_\nu(\sigma)\, d\sigma$$

(2) Ψ^{-1} denotes the inverse function of Ψ.
(3) Personal communication.

when $\{\theta_\nu\}$ is a sequence of independent standard normals, converges uniformly in t with probability 1. Of course the resulting process $X_t(\omega)$ is always Brownian motion, and in this case I.8 yields estimates for the modulus of continuity of paths that are as good as any that have been obtained by other methods ([4]).

It turns out that our inequality I.6 enables us now to obtain the following result.

THEOREM I.3. — *If*

$$\int_0^1 \int_0^1 \Psi\left(\frac{f(x) - f(y)}{p(x-y)}\right) dx\, dy \leqslant B < \infty$$

where Ψ, p *satisfy* (I.2) *and* (I.3) *and in addition* $\Psi(e^x)$ *is convex then for* $0 < x \leqslant 1/2$

$$\text{(I.9)} \quad \left.\begin{array}{l} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array}\right\} \leqslant 4\int_x^{1/2} \psi^{-1}\left(\frac{B}{u^2}\right) dp(u) + 4\psi^{-1}\left(\frac{B}{x^2}\right) p(x).$$

We shall show later that (I.9) includes (I.8) at least in the cases when $\Psi(e^x)$ is convex. This given it should now be possible, essentially using the same approach adopted in [7] and [5], to obtain some information concerning Gaussian processes without continuous paths.

We shall not pursue this point here and leave it as a subject for further research.

Roughly speaking our aim here is to use the finiteness of $I_{\Psi,p}(f)$ to derive a-priori bounds for other important functionals of f such as its *modulus of continuity* or other *high order norms*. These bounds will of course involve Ψ and p but are « a-priori » in the sense that they depend on f only through the value of $I_{\Psi,p}(f)$.

It is good to give here a sample of our results.

For instance in the case $\Psi(u) = |u|^\alpha (\alpha > 1)$ we can obtain from (I.6) the following theorems.

([4]) This suggests that perhaps (I. 8) is best possible when $\Psi(u) \uparrow \infty$ very rapidly.

THEOREM I.4. — *Let*

$$\text{(I.10)} \qquad B = \left[\int_0^1 \int_0^1 \left|\frac{f(x) - f(y)}{|x-y|^\theta}\right|^\alpha \frac{dx\,dy}{(x-y)^2}\right]^{1/\alpha} < \infty$$

where $\alpha > 1$, $\theta > 0$. *Let* $\beta = \alpha/(\alpha - 1)$. *Then if* $0 < \theta < 1/\beta$ (5), $f(x)$ *is essentially continuous and for almost all* x, y

$$\text{(I.11)} \qquad |f(x) - f(y)| \leqslant c_{\alpha,\theta} B |x-y|^\theta$$

where

$$c_{\alpha,\theta} = \frac{2}{2^\theta(\theta\beta)^{1/\beta}\log 2}.$$

THEOREM I.5. — *Let*

$$B = \left[\int_0^1 \int_0^1 ||f(x) - f(y)|\ |x-y|^{1/p}|^\alpha \frac{dx\,dy}{|x-y|^2}\right]^{1/\alpha} < \infty$$

where $\alpha, p \geqslant 1$. *Then* f *is in weak* L_p, *indeed it is in the Lorentz class* $L_{p,\alpha}$ *and we have a constant* $c_{p,\alpha}$ *such that*

(I.12)

$$\|f\|_{p,\alpha} = \left[\frac{\alpha}{p}\int_0^{1/2} [(f^*(x) - f^*(1-x))x^{1/p}]^\alpha \frac{dx}{x}\right]^{1/\alpha} \leqslant c_{p,\alpha} B.$$

To be sure both these theorems can be derived without making use of (I.6). In fact, (I.11) is qualitatively the same as (I.8), while a result similar to (I.12) has been obtained by Herz in [10] (6).

However, there are other consequences of the inequality in (I.6) that seem to be inaccessible by any other methods. For instance, without any additional difficulty I.6 enables us to treat also the limiting cases $\theta = 0$ and $p = \infty$ of Theorems (I.4) and (I.5).

These can be stated as follows:

THEOREM I.6. — *Let*

$$B = \left[\int_0^1 \int_0^1 \left[|f(x) - f(y)| \left(\log\frac{2}{|x-y|}\right)^{-\theta+1/\beta}\right]^\alpha \frac{dx\,dy}{|x-y|^2}\right]^{1/\alpha} < \infty.$$

(5) For $\theta \geqslant 1/\beta$ it can be shown that $f(x)$ is essentially constant.

(6) We are grateful here to D. Adams who made us aware that our methods can also be used to obtain estimates such as (I. 12).

where $\alpha > 1$, $\theta > 0$ *and* $\beta = \alpha/(\alpha - 1)$. *Then*

$$\int_0^1 \exp c_{\alpha,\theta} \left|\frac{f(x) - f^*(1/2)}{B}\right|^{1/\theta} dx \leqslant 2\sqrt{2}$$

where

$$c_{\alpha,\theta} = \frac{(\log 2)^{1/\theta}(\beta\theta)^{1/\beta\theta}}{2}$$

For $\theta = 0$ we have the following curious result:

THEOREM I.7. — *Let*

$$B = \left[\int_0^1 \int_0^1 \left\{|f(x) - f(y)| \left(\log \frac{1}{|x-y|}\right)^{1/\beta}\right\}^{\alpha} \frac{dx}{|x-y|^2}\right]^{1/\alpha} < \infty$$

then

$$\int_0^1 \exp \frac{1}{2} \left\{\exp\left(\frac{|f(x) - f^*(1/2)|}{B} \log 2\right)^{1/\beta}\right\} dx \leqslant 8.$$

If we change the sign of θ we get a remarkable modulus of continuity result.

THEOREM I.8. — *Let*

$$B = \left[\int_0^1 \int_0^1 \left\{|f(x) - f(y)| \left(\log \frac{2}{|x-y|}\right)^{\theta+1/\beta}\right\}^{\alpha} \frac{dx\,dy}{|x-y|^2}\right]^{1/\alpha} < \infty$$

with $\theta > 0$, $\alpha > 1$, $\beta = \alpha/(\alpha - 1)$, *then* $f(x)$ *is essentially continuous in* $[0, 1]$ *and for almost all* x, y

$$\text{(I.13)} \qquad |f(x) - f(y)| \leqslant c_{\alpha,\theta} \frac{B}{\left[\log \frac{2}{|x-y|}\right]^{\theta}}.$$

where

$$c_{\alpha,\theta} = \frac{2}{\log 2(\beta\theta)^{1/\beta}}$$

The conclusions of these last three theorems are probably best possible. Indeed, Greenhall in [9] using potential theory methods was able to show that this is the case when $\alpha = 2$.

We shall see here that an estimate analogous to (I.13) can also be derived for $\alpha > 1$ and general p, provided

$$\int_0^{1/2} \left[\frac{p(u)}{u^{2/\alpha}}\right]^{\beta} \frac{du}{u} < \infty \ (^7)$$

Using this latter estimate when $\alpha = 2$, we obtain a new path continuity result for L_2 processes. The same ideas should, of course, be fruitful in the study of general L_p processes, but we shall leave this for further research.

Also, we should point out that estimates such as those given by Theorems I.2 and I.5 have been obtained in [5] and [10] without dimension restrictions.

Now, it would be very unreasonable if an inequality similar to I.6 were not behind these higher dimensional results as it is in the one dimensional case.

It should thus make a worthwhile research project to try and extend the methods of the present paper and obtain such highly refined estimates as those given by Theorems I.6 and I.7 for every dimension.

The main difficulty in this endeavour seems to be formulating the appropriate generalization of I.6. This difficulty consists in making the right choice for f^*.

As we shall see, I.6 follows from a purely combinatorial result concerning certain lattices of intervals. Now, the latter result appears generalizable to higher dimensions, and a study of this generalization might very well lead to the proper definition of f^*.

1. The combinatorial result. Preview.

We shall now give a brief outline of the path we will pursue in presenting our results.

First of all, we show that I.6 is an immediate consequence of the following interesting inequality.

THEOREM 1.1. — *Let* $\Phi(u)$ *be a non-decreasing function of* $|u|$ *in* $(-\infty, +\infty)$. *Let* $0 < \delta < 1$ *and set*

$$(1.1) \qquad J_{\Phi,\delta}(f) = \iint_{|x-y| \leq \delta} \Phi(f(x) - f(y))\, dx\, dy.$$

(7) See Theorem 5.2 below.

Then, for each measurable f *on* $[0, 1]$ *we have*

$$(1.2) \qquad J_{\Phi,\delta}(f^*) \leqslant J_{\Phi,\delta}(f).$$

Next we derive (1.2) from the following discretized version of it:

THEOREM 1.2. — *Let* $\Phi(u)$ *be a non-decreasing function of* $|u|$ *in* $(-\infty, +\infty)$ *and let* $1 \leqslant M \leqslant n$ *be integers. Then for any reals*

$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$$

and any permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ *of* $(1, 2, \ldots, n)$ *we have*:

$$(1.3) \qquad \sum_{|i-j|\leqslant M} \Phi(x_i - x_j) \leqslant \sum_{|i-j|\leqslant M} \Phi(x_{\sigma_i} - x_{\sigma_j}).$$

Our next task is to eliminate the presence of Φ in (1.3). This is done by showing that (1.3) is a simple consequence of a purely combinatorial result. The latter can be stated as follows.

For given integers $1 \leqslant M \leqslant n$ let $\mathcal{D}_M$ denote the set of intervals $a = (i, j)$ $(1 \leqslant i < j \leqslant n)$ such that $|i - j| \leqslant M$. We introduce in $\mathcal{D}_n$ a lattice structure by saying that

$$a = (i, j) < a' = (i', j')$$

if and only if

$$i' \leqslant i < j \leqslant j'.$$

I.e., if and only if a is *contained* in a' in the usual sense.

Now, given a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of

$$(1, 2, \ldots, n)$$

we set for each $a = (i, j)$

$$\sigma a = \begin{cases} (\sigma_i, \sigma_j) & \text{if} \quad \sigma_i < \sigma_j, \\ (\sigma_j, \sigma_i) & \text{if} \quad \sigma_j < \sigma_i. \end{cases}$$

Furthermore let, for each $1 \leqslant M \leqslant n$, $\sigma\mathcal{D}_M$ denote the image of $\mathcal{D}_M$ under this map σ.

This given, (1.3) is shown to be an easy consequence of the following result.

THEOREM 1.3. — *For each* $1 \leqslant M \leqslant n$ *and each permutation* $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, *there exists a one-to one map*

$\pi : \mathcal{D}_M \longleftrightarrow \sigma\mathcal{D}_M$ *such that*

$$\text{(1.4)} \qquad \pi a > a \quad \forall a \in \mathcal{D}_M.$$

Our final step in the proof of I.6 is to show that a certain very simple inductive procedure leads to a construction of this map π, for each M and each given σ.

It is worth noting at this point that in our first proof of Theorem 1.3 we only deduced the *existence* of π. This was done by an application of what is commonly known in combinatorial theory as the « marriage theorem ». This approach reduces Theorem 1.3 to proving a further combinatorial lemma (Theorem 4.1 below).

In the summer of 1969 H. Taylor found an ingenious proof of this lemma, (see [15]). Thereby a complete proof of I.6 could finally be put together.

In this paper we show that Taylor's lemma is essentially none other than the inequality (1.3) in the particular case that $\Phi(u)$ is the step function

$$\Phi(u) = \begin{cases} 1 & \text{if} \quad |u| \geq 1, \\ 0 & \text{if} \quad |u| < 1. \end{cases}$$

We thus obtain here a new and somewhat simpler proof of this remarkable result proved by Taylor.

Our next and final task is to present some of the consequences of I.6. In doing this we are led to derive inequalities concerning monotone functions f satisfying conditions such as

$$I_{\Psi,p}(f) < \infty.$$

This will deliver us theorems such as those stated in the introduction. In particular we shall derive Theorem I.3, which by the way antedated Taylor's work and was our original motivation for proving I.6.

We terminate the paper with some applications and comments that might be useful for further research.

2. The reduction to a finite problem.

We start by proving that Theorem I.1 follows from Theorem 1.1.

To this end, let Ψ and p satisfy the conditions (I.2), and (I.3) and suppose further that $\Psi(e^x)$ is convex. Since we can assume without loss that Ψ is absolutely continuous, for all $0 < u < 1$ and $\Delta > 0$ we have

$$\Psi\left(\frac{\Delta}{p(u)}\right) = \Psi\left(\frac{\Delta}{p(1)}\right) + \int_u^1 \Psi'\left(\frac{\Delta}{p(\delta)}\right)\frac{\Delta}{[p(\delta)]^2}\, dp\,(\delta).$$

Setting

$$\Phi(u) = \Psi'(u)u,$$

this relation can be rewritten in the form

$$\Psi\left(\frac{\Delta}{p(u)}\right) = \Psi\left(\frac{\Delta}{p(1)}\right) + \int_0^1 \Phi\left(\frac{\Delta}{p(\delta)}\right)\chi(u \leqslant \delta)\frac{dp(\delta)}{p(\delta)}\ (^8)$$

Replacing Δ by $|f(x) - f(y)|$, u by $|x - y|$ and integrating over the square $[0, 1] \times [0, 1]$ we get

$$\begin{aligned} I_{\Psi,p}(f) &= \int_0^1\int_0^1 \Psi\left(\frac{f(x) - f(y)}{p(1)}\right) dx\, dy \\ &+ \int_0^1\int_0^1 \left\{\int_0^1 \Phi\left(\frac{|f(x) - f(y)|}{p(\delta)}\right)\chi(|x - y| \leqslant \delta)\frac{dp(\delta)}{p(\delta)}\right\} dx\, dy. \end{aligned}$$

Thus, Fubini's theorem gives

$$\begin{aligned} I_{\Psi,p}(f) &= \int_0^1\int_0^1 \Psi\left(\frac{f(x) - f(y)}{p(1)}\right) dx\, dy \\ &+ \int_0^1 \left\{\iint_{|x-y|\leqslant\delta} \Phi\left(\frac{|f(x) - f(y)|}{p(\delta)}\right) dx\, dy\right\}\frac{dp(\delta)}{p(\delta)}. \end{aligned}$$

Since each of the functions

$$\Psi\left(\frac{u}{p(1)}\right), \qquad \Phi\left(\frac{|u|}{p(\delta)}\right)$$

satisfies the hypotheses of Theorem 1.1 the inequality (1.2) gives

$$\begin{aligned} I_{\Psi,p}(f) &\geqslant \int_0^1\int_0^1 \Psi\left(\frac{f^*(x) - f^*(y)}{p(1)}\right) dx\, dy \\ &+ \int_0^1 \left\{\iint_{|x-y|\leqslant\delta} \Phi\left(\frac{|f^*(x) - f^*(y)|}{p(\delta)}\right) dx\, dy\right\}\frac{dp(\delta)}{p(\delta)}. \end{aligned}$$

(8) For any statement $\mathcal{A}$ we set $\chi(\mathcal{A}) = 1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A}) = 0$ if $\mathcal{A}$ is false.

Reversing our steps by means of another use of Fubini's theorem we deduce

$$I_{\Psi,p}(f) \geqslant I_{\Psi,p}(f^*).$$

Thus Theorem 1.1 implies Theorem I.1 as asserted.

Our next step is to derive Theorem 1.1 from Theorem 1.2.

Let then $\Phi(u)$ be a non-decreasing function of $|u|$ in $(-\infty, +\infty)$. Suppose first that $f(x)$ is constant on the intervals

$$I_\nu^{(m)} = \left[\frac{\nu-1}{2^m}, \frac{\nu}{2^m}\right) \quad \nu = 1, 2, \ldots, 2^m.$$

In other words

$$f(x) = \sum_{\nu=1}^{2^m} y_\nu \chi_{I_\nu^{(m)}}(x).$$

Note then that for any $n \geqslant m$ we also have

$$f(x) = \sum_{\nu=1}^{2^n} y_\nu^{(n)} \chi_{I_\nu^{(n)}}(x)$$

where

$$y_\nu^{(n)} = 2^n \int_{I_\nu^{(n)}} f(x)\, dx.$$

It is easy to see that in this case

$$f^*(x) = \sum_{\nu=1}^{2^n} y_{\sigma_\nu^{(n)}} \chi_{I_\nu^{(n)}}(x)$$

where $(\sigma_1, \sigma_2, \ldots, \sigma_{2^n})$ is the permutation of $(1, 2, \ldots, 2^n)$ which arranges $y_1^{(n)}, y_2^{(n)}, \ldots, y_{2^n}^{(n)}$ in decreasing order.

This given, when $\frac{1}{2^n} < \delta$ we have

$$\iint_{|x-y|\leqslant\delta} \Phi(f(x) - f(y))\, dx\, dy = \sum_{|i-j|\leqslant 2^n\delta} \Phi(y_i^{(n)} - y_j^{(n)}) \frac{1}{2^{2n}} + e_n;$$

$$\iint_{|x-y|\leqslant\delta} \Phi(f^*(x) - f^*(y))\, dx\, dy$$
$$= \sum_{|i-j|\leqslant 2^n\delta} \Phi(y_{\sigma_i}^{(n)} - y_{\sigma_j}^{(n)}) \frac{1}{2^{2n}} + e_n^*,$$

where e_n and e_n^* are correction terms which in absolute value do not exceed

$$2^{n+1}\Phi(2 \max |f(x)|) \frac{1}{2^{2n}}.$$

Since this quantity tends to zero as $n \to \infty$, Theorem 1.2 and a passage to the limit as $n \to \infty$ yields

$$J_{\Phi,\delta}(f^*) \leqslant J_{\Phi,\delta}(f) \tag{2.1}$$

for these special choices of $f(x)$.

Let us suppose now that $f(x)$ is a bounded measurable function on $[0, 1]$, say $|f(x)| \leqslant M$, and let for each integer m

$$f_m(x) = \sum_{\nu=1}^{2^m} y_\nu^{(m)} \chi_{I_\nu^m}(x)$$

where $y_\nu^{(m)}$ is the average of f in $I_\nu^{(m)}$, i.e.:

$$y_\nu^{(m)} = 2^m \int_{I_\nu^{(m)}} f(t)\, dt.$$

From what we have shown we deduce that

$$J_{\Phi,\delta}(f_m^*) \leqslant J_{\Phi,\delta}(f_m) \quad \forall m = 1, 2, \ldots \tag{2.2}$$

It is well known that

$$f_m(x) \to f(x) \quad \text{a.e. in } [0, 1]$$

and since $|f_m(x)| \leqslant M \ \forall m$, the dominated convergence theorem gives

$$\lim_{m\to\infty} J_{\Phi,\delta}(f_m) = J_{\Phi,\delta}(f).$$

At this moment, we need only have

$$f_m^*(x) \to f^*(x) \quad \text{a.e.} \quad \text{in} \quad [0, 1], \tag{2.3}$$

for then, a passage to the limit in (2.2) and Fatou's lemma yields (2.1) for all bounded measurable f.

If f is not bounded, we can proceed as follows. For each integer $m \geqslant 1$ we set

$$g_m(x) = \begin{cases} m & \text{if } f(x) > m, \\ f(x) & \text{if } |f(x)| \leqslant m, \\ -m & \text{if } f(x) < -m. \end{cases}$$

Now, it is easy to see that

$$|g_m(x) - g_m(y)| \leqslant |f(x) - f(y)| \quad \forall x, y \in [0, 1].$$

We thus have

$$J_{\Phi,\delta}(g_m) \leqslant J_{\Phi,\delta}(f).$$

Since each g_m is bounded, (2.1) yields

$$(2.4) \qquad J_{\Phi,\delta}(g_m^*) \leqslant J_{\Phi,\delta}(f) \quad \forall m.$$

Thus, again if we have

$$(2.5) \qquad g_m^*(x) \to f^*(x) \quad \text{a.e. in } [0, 1]$$

passing to the limit as $m \to \infty$ in (2.4), Fatou's lemma yields (2.1) for all measurable f.

To complete our argument we need only verify (2.3) and (2.5).

Now, these two facts are immediate consequences of the following:

LEMMA 2.1. — *If* $\{f_n\}$ *and* f *are integrable in* $[0, 1]$ *and*

$$(2.6) \qquad \lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|\, dx = 0$$

then

$$f_n^*(x) \to f^*(x)$$

holds at all points of continuity of $f^*(x)$.

This result must be known, but for lack of a specific reference we shall briefly indicate how it can be proved. To this end observe first that from 2.6 it follows that

$$\int_0^x f_n^*(t)\, dt \to \int_0^x f^*(t)\, dt$$

uniformly for $x \in [0, 1]$.

In fact (see remarks below) it can be shown that for any two integrable functions $f_1(t)$ and $f_2(t)$ and for any $x \in [0, 1]$ we have

$$(2.7) \quad \left|\int_0^x f_1^*(t)\, dt - \int_0^x f_2^*(t)\, dt\right| \leqslant \int_0^1 |f_1(t) - f_2(t)|\, dt.$$

This given, using the monotonicity of f_n^*, for any

$$x_0 \in [0, 1]$$

and any $\Delta > 0$ we get

$$\frac{1}{\Delta}\int_{x_0}^{x_0+\Delta} f^*(t)\,dt - e_n \leqslant f^*_n(x_0) \leqslant \frac{1}{\Delta}\int_{x_0-\Delta}^{x_0} f^*(t)\,dt + e_n$$

where we may take

$$e_n = \frac{2}{\Delta}\int_0^1 |f_n(t) - f(t)|\,dt.$$

Passing to the limit as $n \to \infty$ we derive

$$\frac{1}{\Delta}\int_{x_0}^{x_0+\Delta} f^*(t)\,dt \leqslant \liminf_{n\to\infty} f^*_n(x_0)$$
$$\leqslant \limsup_{n\to\infty} f^*_n(x_0) \leqslant \frac{1}{\Delta}\int_{x_0-\Delta}^{x_0} f^*(t)\,dt.$$

From these estimates the conclusion of the lemma is easily established.

Remarks. — The inequality in (2.7) can be established in the following manner. We first note that for any integrable f we have

$$\text{(2.8)} \qquad \int_0^x f^*(t)\,dt = \inf_\lambda \left\{\lambda x + \int_0^1 (f(t) - \lambda)^+\,dt\right\}.$$

Setting for a moment

$$\theta_f(\lambda) = \lambda x + \int_0^1 (f(t) - \lambda)^+\,dt$$

we see that

$$|\theta_{f_1}(\lambda) - \theta_{f_2}(\lambda)| \leqslant \int_0^1 |f_1(t) - f_2(t)|\,dt.$$

Thus

$$|\inf \theta_{f_1}(\lambda) - \inf \theta_{f_2}(\lambda)| \leqslant \sup_\lambda |\theta_{f_1}(\lambda) - \theta_{f_2}(\lambda)|$$
$$\leqslant \int_0^1 |f_1(t) - f_2(t)|\,dt.$$

and (2.7) follows from (2.8).

The identity in (2.8) is easily verified if we observe that for any λ we have

$$\text{(2.9)} \qquad \int_0^1 (f(t) - \lambda)^+\,dt = \int_0^1 (f^*(t) - \lambda)^+\,dt,$$

and this in turn can be derived from the fact that f and f^*

have the same distribution. Actually, 2.9 can just as well be derived directly from the definition of f^*.

Indeed, since we shall need it later on, let us show here that

$$(2.10) \qquad \int_0^1 \Phi(f(t))\, dt = \int_0^1 \Phi(f^*(t))\, dt$$

holds for all $\Phi(u)$ of the form

$$\Phi(u) = \int_{-\infty}^{u} p(\lambda)\, d\lambda$$

where $p(\lambda)$ is non negative and integrable in $(-\infty, +\infty)$.

To this end, for convenience set

$$\gamma_f(\lambda) = m\{t : f(t) > \lambda\},$$

and note that

$$(2.11) \quad f^*(x) = \inf\{\lambda : \gamma_f(\lambda) \leqslant x\} = \sup\{\lambda : \gamma_f(\lambda) \geqslant x\}.$$

A multiple application of Fubini's theorem then yields

$$\begin{aligned}
\int_0^1 \Phi(f(t))\, dt
&= \int_{-\infty}^{+\infty} \int_0^1 \chi(\lambda < f(t))\, dt\, p(\lambda)\, d\lambda \\
&= \int_{-\infty}^{+\infty} p(\lambda)\gamma_f(\lambda)\, d\lambda = \int_{-\infty}^{+\infty} p(\lambda) \int_0^1 \chi(\gamma_f(\lambda) \geqslant x)\, dx\, d\lambda \\
&= \int_0^1 \int_{-\infty}^{+\infty} p(\lambda)\chi(\lambda \leqslant f^*(x))\, d\lambda\, dx \\
&= \int_0^1 \Phi(f^*(x))\, dx.
\end{aligned}$$

The fact that f and f^* have the same distribution can now be derived from (2.10). Indeed, the dominated convergence theorem yields immediately

$$\begin{aligned}
m\{x : f^*(x) > \lambda\} &= \frac{d}{d\lambda} \int_0^1 (f^*(x) - \lambda)^+\, dx \\
&= \frac{d}{d\lambda} \int_0^1 (f(x) - \lambda)^+\, dx = m\{x : f(x) > \lambda\}.
\end{aligned}$$

3. Combinatorial arguments.

We show now that Theorem 1.3 implies Theorem 1.2. Remembering the notation of Section 1 we see that if we set for each $a = (i, j)$

$$\varphi(a) = \Phi(|x_i - x_j|)$$

then the inequality in (1.3) can be rewritten in the form

$$\sum_{a \in \mathfrak{D}_M} \varphi(a) \leqslant \sum_{a \in \mathfrak{D}_M} \varphi(\sigma a),$$

or better

$$\sum_{a \in \mathfrak{D}_M} \varphi(a) \leqslant \sum_{a \in \sigma\mathfrak{D}_M} \varphi(a).$$

Note then that, since $\Phi(u)$ is monotone non-decreasing, the condition

$$x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$$

makes the function $\varphi(a)$ monotone non-decreasing with respect to the lattice ordering we introduced on $\mathscr{D}_n$. More explicitely, if $a = (i, j)$, $a' = (i', j')$ and

$$i' \leqslant i < j \leqslant j'$$

then

$$\varphi(a) = \Phi(x_i - x_j) \leqslant \Phi(x_{i'} - x_{j'}) = \varphi(a').$$

This given, from Theorem 1.3 we get (using 1.4)

$$\sum_{a \in \mathfrak{D}_M} \varphi(a) \leqslant \sum_{a \in \mathfrak{D}_M} \varphi(\pi a) = \sum_{a \in \sigma\mathfrak{D}_M} \varphi(a)$$

which is our desired inequality. We are thus reduced to proving Theorem 1.3.

To help us visualize our steps we shall now recast the whole setup in a geometric form.

We deal with finite subsets of the upper half plane

$$\mathscr{U} = \{(x, y) : y \geqslant 0\}.$$

We introduce in $\mathscr{U}$ a partial ordering by saying that

$$(x, y) < (x', y')$$

if and only if

$$x' - y' \leqslant x - y \leqslant x + y \leqslant x' + y'.$$

In other words $(x, y) < (x', y')$ if and only if the triangle with vertices

$$(x, y), \qquad (x - y, 0), \qquad (x + y, 0)$$

is contained in the triangle with vertices

$$(x', y'), \qquad (x' - y', 0), \qquad (x' + y', 0)$$

Now, given n distinct reals

$$u_1, u_2, \ldots, u_n$$

not necessarily in decreasing order, we represent in $\mathcal{U}$ the couples

$$\{u_i, u_j\} \quad i, j = 1, 2, \ldots, n \ (i \neq j)$$

by the points

$$a(u_i, u_j) = \left(\frac{u_i + u_j}{2}, \frac{|u_i - u_j|}{2}\right).$$

We also let

$$\tilde{\mathcal{D}}_{\mathrm{M}}(u_1, u_2, \ldots, u_n)$$

for a given $1 \leqslant \mathrm{M} \leqslant n$ represent the subset of u consisting

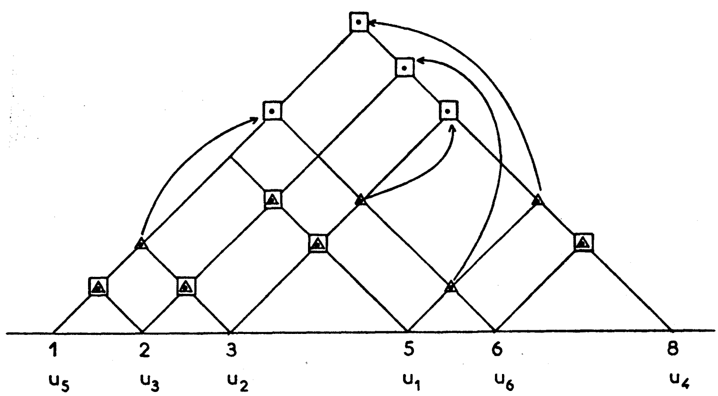

Fig. 1.

of the points $a(u_i, u_j)$ with $|i - j| \leqslant \mathrm{M}$. In other words

$$\tilde{\mathcal{D}}_{\mathrm{M}}(u_1, u_2, \ldots, u_n) = \left\{(x, y) : x = \frac{u_i + u_j}{2}, y = \frac{|u_i - u_j|}{2} \ 1 \leqslant |i - j| \leqslant \mathrm{M}\right\}.$$

We then let $\mathcal{D}_{\mathrm{M}}(u_1, u_2, \ldots, u_n)$ be none other than the set $\tilde{\mathcal{D}}_{\mathrm{M}}(u_1^*, u_2^*, \ldots, u_n^*)$ where

$$u_1^*, u_2^*, \ldots, u_n^*$$

are obtained by rearranging $u_1, u_2, \ldots, u_n$ in increasing order.

In Figure 1 we have illustrated the situation in case $n = 6$, $M = 2$ and

$$(u_1, u_2, \ldots, u_6) = (5, 3, 2, 8, 1, 6).$$

There, dots represent points of $\tilde{\mathscr{D}}_6$ and $\mathscr{D}_6$, squares represent points of $\tilde{\mathscr{D}}_2$ and triangles points of $\mathscr{D}_2$.

This given we prove the following result

THEOREM 3.1. — *Given* $(u_1, u_2, \ldots, u_n)$ *and* $1 \leqslant M \leqslant n$ *we can construct a one-to-one map* $\tau : \tilde{\mathscr{D}}_M \to \mathscr{D}_M$ *such that*

$$\tau a < a \quad \forall a \in \tilde{\mathscr{D}}_M.$$

It is easy to see that Theorem 3.1 implies Theorem 1.3 when we take $(u_1, u_2, \ldots, u_n) = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\pi = \tau^{-1}$. A moment's reflection should also reveal that the greater generality of Theorem 3.1 is only apparent. Indeed, the two theorems are perfectly equivalent.

We now get on with the proof of Theorem 3.1. Note first that since $\tilde{\mathscr{D}}_n = \mathscr{D}_n$, the result is entirely trivial when $M = n$.

We shall thus proceed by induction on n. We assume the theorem true for a given M and $n = N - 1$ and prove it for the same M and $n = N$.

Let then $u_1, u_2, \ldots, u_N$ be given and suppose that $u_N = u_k^*$. I.e. suppose that u_N is the k-th largest of the $u's$. Let us assume for simplicity that

$$M + 1 \leqslant k \leqslant n - M. \tag{3.2}$$

By the induction hypothesis we can find a one-to-one map τ_1 of $\tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_{N-1})$ onto $\mathscr{D}_M(u_1, u_2, \ldots, u_{N-1})$ which is decreasing in the lattice ordering we have introduced in u.

To construct our desired map τ of $\tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_N)$ onto $\mathscr{D}_M(u_1, u_2, \ldots, u_N)$ we shall simply modify and extend slightly this map τ_1.

To this end observe that $\tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_N)$ contains $\tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_{N-1})$. Their difference is the set $\tilde{S}$ consisting of the points

$$a(u_N, u_{N-1}), \quad a(u_N, u_{N-2}), \ldots, \quad a(u_N, u_{N-M}).$$

The sets $\mathscr{D}_M(u_1, u_2, \ldots, u_N)$ and $\mathscr{D}_M(u_1, u_2, \ldots, u_{N-1})$ differ in a more substantial way. Indeed, $\mathscr{D}_M(u_1, u_2, \ldots, u_N)$ contains the set Q consisting of the 2M points

$$a(u_k^*, u_{k-1}^*), \quad a(u_k^*, u_{k-2}^*), \quad \ldots, \quad a(u_k^*, u_{k-M}^*)$$

and

$$a(u_k^*, u_{k+1}^*), \quad a(u_k^*, u_{k+2}^*), \quad \ldots, \quad a(u_k^*, u_{k+M}^*)$$

none of which is in $\mathscr{D}_M(u_1, u_2, \ldots, u_{N-1})$.

On the other hand, since the increasing rearrangement of $u_1, u_2, \ldots, u_{N-1}$ is

$$u_1^*, u_2^*, \ldots, \quad u_{k-1}^*, u_{k+1}^*, \quad \ldots, \quad u_N^*$$

the set $\mathscr{D}_M(u_1, u_2, \ldots, u_{N-1})$ contains the set S whose points are

$$a(u_{k-M}^*, u_{k+1}^*), \quad a(u_{k-M+1}^*, u_{k+2}^*), \quad \ldots, \quad a(u_{k-1}^*, u_{k+M}^*)$$

none of which is in $\mathscr{D}_M(u_1, u_2, \ldots, u_N)$.

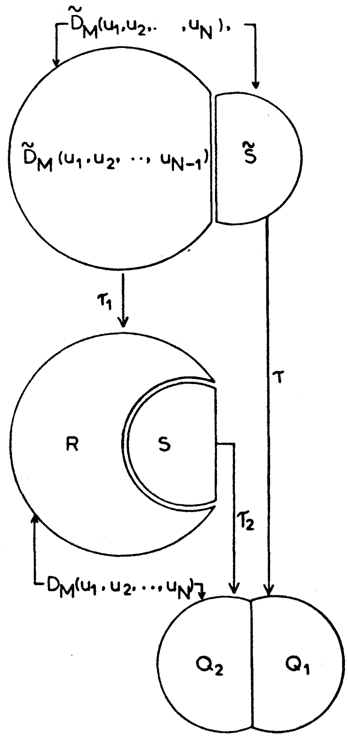

Fig. 2.

If we keep in mind the definitions of $\tilde{S}$, Q, S and set

$$R = \mathscr{D}_M(u_1, u_2, \ldots, u_{N-1}) \cap \mathscr{D}_M(u_1, u_2, \ldots, u_N),$$

we can recapitulate all our findings in the following formulae

$$\tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_N) = \tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_{N-1}) + \tilde{S}$$
$$\mathscr{D}_M(u_1, u_2, \ldots, u_{N-1}) = R + S$$
$$\mathscr{D}_M(u_1, u_2, \ldots, u_N) = R + Q$$

In figure 2 we have represented schematically these various sets and maps.

We are now ready to describe our construction of the map τ.

First of all we let τ be equal to τ_1 for those points of $\tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_{N-1})$ which are sent into R by τ_1. Next we define τ on $\tilde{S}$ by finding for each element of $\tilde{S}$ a « most economical » image of it in Q. This will use up exactly half of Q, say Q_1. The other half of it, say Q_2, will turn out to be « below » S, i.e. we shall be able to construct a map τ_2 of S into Q_2 which is one-to-one and decreasing.

We then let τ be τ_1 followed by τ_2 on those points of $\tilde{\mathscr{D}}_M(u_1, u_2, \ldots, u_{N-1})$ which are sent into S by τ_1.

Figure 3 should help understanding how this program can be carried out. There we have represented things when M=5, squares indicate the elements of S and circles the elements of Q.

Our first task is to give a recipe for defining τ on $\tilde{S}$. Note that all the points of $\tilde{S}$ lie on the 45° lines through $(u_k^*, 0)$. Say M_1 of them are on the right of $(u_k^*, 0)$ and M_2 are on the left. We match the least one on the right with $a(u_k^*, u_{k+1}^*)$, the next on the right with $a(u_k^*, u_{k+2}^*), \ldots,$

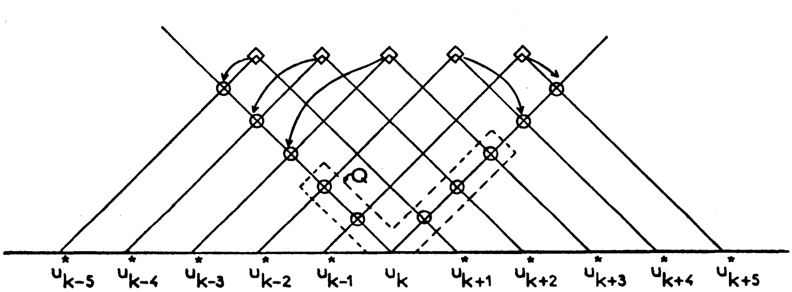

Fig. 3.

largest on the right with $a(u_k^*, u_{k+M_1}^*)$. On the left we proceed in the same way. The least element of $\tilde{S}$ on the left is matched with $a(u_k^*, u_{k-1}^*)$, the next with $a(u_k^*, u_{k-2}^*)$, ..., the largest with $a(u_k^*, u_{k-M_2}^*)$.

This defines τ on $\tilde{S}$.

In figure 3 we have illustrated the case in which $M_1 = 3$, $M_2 = 2$. With this choice of τ on $\tilde{S}$, Q_2 consists of the M points

$$(3.3) \qquad \begin{array}{lll} a(u_k^*, u_{k+M_1+1}^*), & \ldots, & a(u_k^*, u_{k+M}^*), \\ a(u_k^*, u_{k-M_2-1}^*), & \ldots, & a(u_k^*, u_{k-M}^*). \end{array}$$

We need now give a recipe for the map τ_2. However, we have no choice. We must let

$$(3.4) \qquad \begin{array}{l} \tau_2 a(u_{k-M_2-1}^*, u_{k+M_1}^*) = a(u_k^*, u_{k-M_2-1}^*) \\ \tau_2 a(u_{k-M_2-2}^*, u_{k+M_1-1}^*) = a(u_k^*, u_{k-M_2-2}^*), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_2 a(u_{k-M}^*, u_{k+1}^*) = a(u_k^*, u_{k-M}^*). \end{array}$$

and

$$\begin{array}{l} \tau_2 a(u_{k-M_2}^*, u_{k+M_1+1}^*) = a(u_k^*, u_{k+M_1+1}^*) \\ \tau_2 a(u_{k-M_2+1}^*, u_{k+M_1+2}^*) = a(u_k^*, u_{k+M_1+2}^*), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_2 a(u_{k+1}^*, u_{k+M}^*) = a(u_k^*, u_{k+M}^*). \end{array}$$

A look at Figure 3 should perhaps be more illuminating than the above formulas as to what we are doing here.

To complete our proof we need only say what must be done in case one or both the inequalities in (3.2) are not satisfied.

To this end we need only observe that in view of the definition of M_1 and M_2 we must necessarily have

$$1 \leqslant k - M_2 < k + M_1 \leqslant n.$$

This is because there must be at least M_2 points $(u_i^*, 0)$ to the left of $(u_k^*, 0)$ and M_1 to the right.

This means that the definition of τ on $\tilde{S}$ given above makes sense in any case.

In general S and Q may not have M and 2M points respectively as they do when the inequalities in 3.2 hold true.

However, to each point lost by S there corresponds, under the map τ_2 defined in 3.4, a point lost by Q_2.

So again we can use (3.4) to define τ_2 as a map between what remains of S and what remains of Q_2. Indeed, we can use the formulas in (3.4) whenever both sides make sense.

4. Taylor's lemma, comments.

Let us take a look at Theorem 1.3 from this geometric view point. We can visualize the sets $\mathcal{D}_M$ defined in Section 1 as subsets of the upper half plane or better subsets of the set

$$\mathcal{D}_n = \left\{(x, y) : x = \frac{i+j}{2},\ y = \frac{j-i}{2} 1 \leqslant i < j \leqslant n\right\}.$$

Clearly we can construct $\mathcal{D}_n$ by drawing on the upper half plane the 45° lines through the points $(i, 0)$ $i = 1, 2, \ldots, n$ then obtain the points of $\mathcal{D}_n$ as the intersection of these lines.

For each $M \geqslant 1$ $\mathcal{D}_M$ consists of those points of $\mathcal{D}_n$ that are at a distance of no more than M/2 from the x-axis.

For a given $a = \left(\frac{i+j}{2}, \frac{|i-j|}{2}\right) \in \mathcal{D}_n$ we set

$$R_a = \{a' \in \mathcal{D}_n : a' > a\}$$

Clearly R_a is the « rectangle » with one vertex at a and the diagonally opposite vertex at $\left(\frac{n+1}{2}, \frac{n-1}{2}\right)$.

Given a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ the mapping of $\mathcal{D}_n$ into itself associated to σ in Section 1 can be written in the form

$$\sigma a = \left(\frac{\sigma_i + \sigma_j}{2}, \frac{|\sigma_i - \sigma_j|}{2}\right) \qquad \text{for} \qquad a = a(i, j).$$

To produce the map π of Theorem 1.3 we need only choose for each $a \in \mathcal{D}_M$ a point πa in the set

$$F_a = R_a \cap \sigma \mathcal{D}_M. \tag{4.1}$$

Of course we want also

$$\pi a_1 \neq \pi a_2 \qquad \text{when} \qquad a_1 \neq a_2.$$

But this is precisely a set up to which Phillip Hall's theorem on the selection of distinct representatives applies.

For the benefit of the reader who may not be acquainted with this theorem we shall state it in the specific form needed here.

Let X be a finite set of points. Let $\mathscr{F} = \{F_a : a \in I\}$ be an indexed family of subsets of X (the F_a's need not be distinct). We say that we have a « *system of distinct representatives for* $\mathscr{F}$ *in* X » if and only if we have a one-to-one map π of I into X such that

$$\pi a \in F_a \quad \forall a \in I.$$

P. Hall's theorem, sometimes referred to as the « marriage theorem », can be stated as follows:

« *Given* X *and* $\mathscr{F} = \{F_a : a \in I\}$ *it is possible to select a system of distinct representatives for* $\mathscr{F}$ *in* X *if and only if for any k-tuple* $a_1, a_2, \ldots, a_k \in I$ *we have*:

$$|F_{a_1} \cup F_{a_2} \cup \cdots \cup F_{a_k}| \geqslant k \text{ (9)}$$

Clearly this condition is necessary for the existence of the map π. Indeed, the set

$$F_{a_1} \cup F_{a_2} \cup \cdots \cup F_{a_k}$$

must at least contain the points

$$\pi a_1, \pi a_2, \ldots, \pi a_k.$$

As for the sufficiency, a very lucid proof may be found in [14].

Using P. Hall's theorem with $X = \sigma\mathscr{D}_M$, $I = \mathscr{D}_M$ and F_a given by (4.1) we immediately derive that Theorem 1.3 is equivalent to the following result:

THEOREM 4.1. — *Given any integers* $1 \leqslant M \leqslant n$, *any permutation* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *of* $(1, 2, \ldots, n)$ *and any rectangles* $R_{a_1}, R_{a_2}, \ldots, R_{a_k}$ *we have*

$$(4.2) \quad |(R_{a_1} \cup R_{a_2} \cup \ldots \cup R_{a_k}) \cap \sigma\mathscr{D}_M| \geqslant |(R_{a_1} \cup R_{a_2} \cup \ldots \cup R_{a_k}) \cap \mathscr{D}_M|.$$

(9) If S is a set by |S| we mean the number of elements of S.

Thus we see that another way of establishing Theorem 1.2 is to prove this result first then obtain Theorem 1.3 by means of the P. Hall theorem.

This is essentially the path followed in the first proof of Theorem I.1.

In fact, for a while, the inequality (4.2) remained the missing link in our work until H. Taylor found a proof of it.

Theorem 4.1 is what we referred to in Section 1 as « Taylor's lemma ».

We can make a few further remarks concerning the inequality (4.2). At first sight it may seem rather fortuitus that we are able to prove Theorem 1.2 by proving Theorem 1.3. For, there could be other ways for the inequality

$$(4.3) \qquad \sum_{|i-j|\leqslant M} \Phi(x_i - x_j) \leqslant \sum_{|i-j|\leqslant M} \Phi(x_{\sigma_i} - x_{\sigma_j})$$

to be true without necessarily having to pair off each term in the left sum with a bigger term in the right sum.

We shall show here that actually this is the only way Theorem 1.2 can be proved.

To this end note that given any reals

$$x_1 < x_2 \cdots < x_n$$

the function

$$\Phi(x_i - x_j) = \chi(|x_i - x_j| \geqslant 1)$$

takes the value one precisely on a subset

$$R_{a_1} \cup R_{a_2} \cup \cdots \cup R_{a_k}.$$

Thus for such a choice of Φ the inequality (4.3) reduces to (4.2). This given, to show that Theorem 1.2 implies Theorem 1.3 we need only establish the following:

Theorem 4.2. — *Given any $a_1, a_2, \ldots, a_k$ in $\mathcal{D}_n$ we can find reals $x_1, x_2, \ldots, x_n$ such that*

$$(4.6) \quad R_{a_1} \cup R_{a_2} \cup \cdots \cup R_{a_k} = \{a(i,j) : |x_i - x_j| \geqslant 1\}.$$

Proof. — Let

$$a_\nu = a(\alpha_\nu, \beta_\nu) = \left(\frac{\alpha_\nu + \beta_\nu}{2}, \frac{|\alpha_\nu - \beta_\nu|}{2}\right).$$

By throwing away a few unnecessary $a'_\nu s$ we can assume that

$$\alpha_1 < \alpha_2 < \cdots < \beta_k$$
$$\beta_1 < \beta_2 < \cdots < \beta_k$$

and, of course, we have

$$\alpha_\nu < \beta_\nu.$$

Let us set

$$\alpha_0 = 0, \ \beta_{k+1} = n + 1,$$

and

$$b_i = a(\alpha_i + 1, \beta_{i+1} - 1) = \left(\frac{\alpha_i + \beta_{i+1}}{2}, \frac{\beta_{i+1} - \alpha_i - 2}{2}\right),$$

for $i = 0, 1, \ldots, k$.

For convenience set, when $a = a(i, j)$

$$\varphi(a) = \chi(|x_i - x_j| \geqslant 1).$$

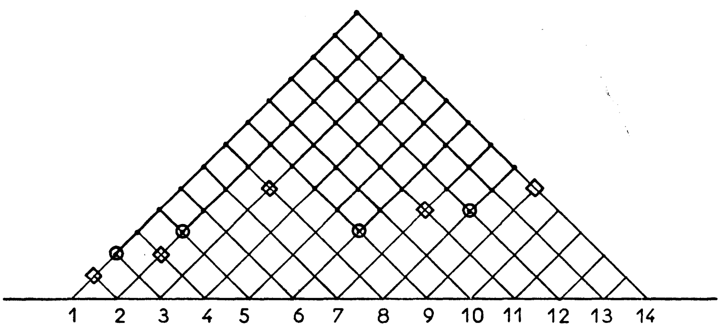

Fig. 4.

In figure 4 we have represented the case $n = 14$, $k = 4$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 2, 6, 8)$$
$$(\beta_1, \beta_2, \beta_3, \beta_4) = (3, 5, 9, 12).$$

The points of the rectangle $R_{a_1} \cup R_{a_2} \cup \cdots \cup R_{a_k}$ are represented by dots, the a's by circles and the b's by squares.

A look at this figure should make it clear that 4.6 will hold as soon as we choose $x_1, x_2, \ldots, x_n$ in such a way that

$$(4.7) \qquad \varphi(a_\nu) = 1 \qquad \text{for} \qquad \nu = 1, 2, \ldots, k$$

and

$$(4.8) \qquad \varphi(b_\nu) = 0 \qquad \text{for} \qquad \nu = 0, 1, \ldots, k$$

Let us set

$$x_1 = 0,\ x_2 - x_1 = \theta_1,\ x_3 - x_2 = \theta_2,\ \ldots .\ x_n - x_{n-1} = \theta_{n-1},$$

and endeavour to satisfy (4.7) and (4.8) with

$$\theta_1, \theta_2, \ldots, \theta_{n-1}$$

all positive.

The first condition gives

$$\theta_1 + \theta_2 + \cdots + \theta_{\beta_1 - 2} < 1. \tag{4.9}$$

Note that $\beta_1 \geqslant \alpha_1 + 1 \geqslant 2$. If $\beta_1 = 2$ (4.9) is trivially satisfied. If $\beta_1 \geqslant 3$ then we choose $\theta_1, \theta_2, \ldots, \theta_{\beta_1 - 2}$ all equal and such that

$$\theta_1 + \theta_2 + \cdots + \theta_{\beta_1 - 2} = 1/2 \tag{4.10}$$

The next condition will be satisfied if

$$\theta_{\alpha_1} + \theta_{\alpha_1+1} + \cdots + \theta_{\beta_1 - 1} = 1. \tag{4.11}$$

Note that the term $\theta_{\beta_1 - 1}$ does not appear in (4.10), thus we can choose it so that (4.11) is satisfied.

Next we need

$$\theta_{\alpha_1+1} + \theta_{\alpha_1+2} + \cdots + \theta_{\beta_2 - 1} < 1. \tag{4.12}$$

Here, since θ_{α_1} is missing and was contributing to (4.11), the sum of the terms already defined is less than one. So we have no difficulty in choosing the remaining terms (if any) all positive and such that (4.12) holds true.

The reader should convince himself that this process can be continued until all remaining $\theta_i's$ have been chosen.

This completes the proof of Theorem 4.2. Thus we can conclude that Theorems 1.2, 1.3 and 4.1 are all equivalent.

5. Inequalities for f *(x) and applications.

We shall start by proving Theorem I.3. To this end let Ψ, p satisfy I.2, I.3 with $\Psi(e^x)$ convex and set for convenience

$$\int_0^1 \int_0^1 \Psi\left(\frac{f(s) - f(t)}{p(s-t)}\right) ds\, dt = \mathrm{B}.$$

Let for each $0 < x < y \leqslant 1/2$

$$\Delta(x, y) = \{(s, t) : y \leqslant s \leqslant x + y,\ s - y \leqslant t \leqslant x\}$$

Since both this set and its reflection across the line $s = t$ are contained in $[0, 1] \times [0, 1]$, using I.6 we get

$$(5.1) \qquad \iint_{\Delta(x,y)} \Psi\left(\frac{f^*(s) - f^*(t)}{p(s-t)}\right) ds\, dt \leqslant 1/2\, \mathrm{B}.$$

Note that in $\Delta(x, y)$ we have

$$f^*(t) - f^*(s) \geqslant f^*(x) - f^*(y),$$
$$s - t \leqslant y.$$

Substituting in (5.1) we obtain

$$\frac{x^2}{2} \Psi\left(\frac{f^*(x) - f^*(y)}{p(y)}\right) \leqslant \mathrm{B}/2.$$

Thus, if Ψ^{-1} denotes the function inverse of Ψ we must have

$$(5.2) \qquad f^*(x) - f^*(y) \leqslant \Psi^{-1}\left(\frac{\mathrm{B}}{x^2}\right) p(y)$$

for all $0 < x < y \leqslant 1/2$.

We now construct an auxiliary sequence $\{x_n\}$ in $[0, 1]$ by setting $x_0 = 1/2$ and

$$(5.3) \qquad x_{n+1} = p^{-1}\left(\frac{p(x_n)}{2}\right) \quad \forall n \geqslant 0.$$

Let $0 < x < 1/2$ be given and let N be such that

$$(5.4) \qquad x_{\mathrm{N}+2} < x \leqslant x_{\mathrm{N}+1}.$$

We have

$$f^*(x) - f^*(1/2) = f^*(x) - f^*(x_{\mathrm{N}}) + \sum_{\nu=1}^{\mathrm{N}} f^*(x_\nu) - f^*(x_{\nu-1}).$$

Using the inequality (5.2) for each of these terms we get

(5.5)
$$f^*(x) - f^*(1/2) \leqslant \Psi^{-1}\left(\frac{\mathrm{B}}{x^2}\right) p(x_{\mathrm{N}}) + \sum_{\nu=1}^{\mathrm{N}} \Psi^{-1}\left(\frac{\mathrm{B}}{x_\nu^2}\right) p(x_{\nu-1})$$

Formula (5.3) gives [with (5.4)]

$$p(x_{\mathrm{N}}) = 2p(x_{\mathrm{N}+1}) = 4p(x_{\mathrm{N}+2}) \leqslant 4p(x),$$

and

$$\Psi^{-1}\left(\frac{B}{x_\nu^2}\right)p(x_{\nu-1}) = \Psi^{-1}\left(\frac{B}{x_\nu^2}\right)4(p(x_\nu) - p(x_{\nu+1}))$$
$$\leqslant 4\int_{x_{\nu+1}}^{x_\nu} \Psi^{-1}\left(\frac{B}{u^2}\right)dp(u).$$

Combining with (5.5)

$$f^*(x) - f^*(1/2) \leqslant 4\Psi^{-1}\left(\frac{B}{x^2}\right)p(x) + 4\int_{x_{N+1}}^{x_1} \Psi^{-1}\left(\frac{B}{u^2}\right)dp(u),$$

and this clearly implies the top part of I.9. To obtain the bottom part we just observe that

$$(-f)^* = -f^*(1-x).$$

We now show that Theorem I.3 implies Theorem I.2 at least when $\Psi(e^x)$ is convex.

To this end let

$$\int_0^{1/2} \Psi^{-1}\left(\frac{B}{u^2}\right)dp(u) < \infty.$$

We then have

$$\Psi^{-1}\left(\frac{B}{x^2}\right)p(x) \leqslant \int_0^x \Psi^{-1}\left(\frac{B}{u^2}\right)dp(u) \to 0$$

as $x \to 0$. Thus from I.9 we easily deduce

$$(5.6)\qquad f^*(0^+) - f^*(1^-) \leqslant 8\int_0^{1/2} \Psi^{-1}\left(\frac{B}{u^2}\right)dp(u).$$

The inequality (I.8) will be derived by a trick which here and in the following will be referred to as the *change of scale argument.*

For given $0 \leqslant x < y \leqslant 1$ we set

$$\tilde{f}(\tilde{s}) = f(x + \tilde{s}\Delta) \quad \Delta = y - x.$$

An easy calculation then shows that

$$\tilde{B} = \int_0^1\int_0^1 \Psi\left(\frac{\tilde{f}(\tilde{s}) - \tilde{f}(\tilde{t})}{p(\Delta(\tilde{s} - \tilde{t}))}\right)d\tilde{s}\,d\tilde{t}$$
$$= \frac{1}{\Delta^2}\int_x^y\int_x^y \Psi\left(\frac{f(s) - f(t)}{p(s-t)}\right)ds\,dt \leqslant \frac{B}{\Delta^2}.$$

Thus using 5.6. with f replaced by $\tilde{f}$, B by $\tilde{B}$ and $p(u)$ by $p(\Delta u)$ we obtain

$$\tilde{f}^*(0^+) - \tilde{f}^*(1^-) \leqslant 8\int_0^{1/2} \Psi^{-1}\left(\frac{B}{\Delta^2 u^2}\right) dp\,(\Delta u)$$
$$= 8\int_0^{\Delta/2} \Psi^{-1}\left(\frac{B}{u^2}\right) dp(u)$$

from which we deduce

$$\operatorname*{ess\,sup}_{s\in[x,y]} f(s) - \operatorname*{ess\,inf}_{s\in[x,y]} f(s) \leqslant 8\int_0^{|x-y|/2} \Psi^{-1}(B/u^2)\, dp(u).$$

This yields

$$\left|\frac{1}{h}\int_x^{x+h} f(t)\, dt - \frac{1}{h}\int_{y-h}^{y} f(t)\, dt\right| \leqslant 8\int_0^{|x-y|/2} \Psi^{-1}(B/u^2)\, dp(u)$$

for all $0 < h < |x-y|/2$.

Letting $h \to 0$, I.8 immediately follows whenever x and y are Lebesgue points of f.

We now prove a basic inequality from which Theorems I.4 to I.8 can be derived.

This can be stated as follows

LEMMA 5.1. — *Let* Ψ *and* p *satisfy* I.2, I.3 *with* $\Psi(e^x)$ *convex. Let* $\theta(s)$ *be monotone increasing and such that*

$$0 \leqslant \theta(s) \leqslant s$$

for all $0 \leqslant s \leqslant 1$ *then*

$$(5.7)\qquad \int_0^1 \Psi\left(\frac{f^*(\theta(s)) - f^*(s)}{p(s)}\right) \theta(s)\, ds \leqslant \frac{1}{2}\, I_{\Psi,p}(f).$$

Proof. — Since the region

$$R = \{(s,t) : 0 \leqslant s \leqslant 1,\ 0 \leqslant t \leqslant \theta(s)\}$$

and its reflection across the line $s = t$ are both in $[0,1] \times [0,1]$ I.6 yields

$$(5.8)\qquad \iint_R \Psi\left(\frac{f^*(t) - f^*(s)}{p(s-t)}\right) dt\, ds \leqslant \frac{1}{2}\, I_{\Psi,p}(f).$$

However, in R we have

$$f^*(t) - f^*(s) \geqslant f^*(\theta(s)) - f^*(s)$$

and

$$p(s-t) \leqslant p(s).$$

Substituting in (5.8), (5.7) immediately follows.

The inequality 5.7 in the case $\Psi(u) = |u|^\alpha$ ($\alpha > 1$) can be used to derive much sharper estimates for $f^*(x)$ than those given by Theorem I.3.

We can state them as follows

THEOREM 5.1. — *Let* p *satisfy* I.3 *and*

$$\left[\int_0^1 \int_0^1 \left|\frac{f(s)-f(t)}{p(s-t)}\right|^\alpha ds\, dt\right]^{1/\alpha} = \mathrm{B} < \infty. \tag{5.9}$$

Then, setting $\beta = \frac{\alpha}{\alpha-1}$, *for all* $0 < \delta < 1/2$ *we have*

(5.10)
$$\left.\begin{matrix} f^*(\delta) - f^*(1/2) \\ f^*(1/2) - f^*(1-\delta) \end{matrix}\right\} \leqslant \frac{\mathrm{B}}{\log 2}\left[\int_\delta^1 \left[\frac{p(s)}{s^{2/\alpha}}\right]^\beta \frac{ds}{s}\right]^{1/\beta}.$$

Proof. — We choose $\theta(s) = s/2$ in (5.7) and get

$$\int_0^1 \left[\frac{f^*(s/2)-f^*(s)}{p(s)}\right]^\alpha s\, ds \leqslant \mathrm{B}^\alpha.$$

Since $f^*(s)$ is non-increasing we have

$$\begin{aligned}[f^*(\delta) - f^*(1/2)] \log 2 &\leqslant \int_{\delta/2}^{\delta} f^*(s)\frac{ds}{s} - \int_{1/2}^1 f^*(s)\frac{ds}{s} \\ &= \int_{\delta/2}^{1/2} f^*(s)\frac{ds}{s} - \int_\delta^1 f^*(s)\frac{ds}{s} \\ &= \int_\delta^1 [f^*(s/2) - f^*(s)]\frac{ds}{s}.\end{aligned}$$

But now we write

$$\int_\delta^1 [f^*(s/2) - f^*(s)]\frac{ds}{s} = \int_\delta^1 \left[\frac{f^*(s/2)-f^*(s)}{p(s)}\right] s^{2/\alpha}\frac{p(s)}{s^{2/\alpha}}\frac{ds}{s}.$$

Thus a use of Hölder's inequality gives

$$\begin{aligned}&[f^*(\delta) - f^*(1/2)] \log 2 \\ &\qquad \leqslant \left[\int_0^1 \left[\frac{f^*(s/2)-f^*(s)}{p(s)}\right]^\alpha s\, ds\right]^{1/\alpha} \left[\int_0^1 \left[\frac{p(s)}{s^{2/\alpha}}\right]^\beta \frac{ds}{s}\right]^{1/\beta}\end{aligned}$$

This clearly implies the top part of (5.10). The bottom part again follows because $(-f)^* = -f^*(1-x)$.

Theorems I.6 and I.7 are all immediate consequences of the inequalities in (5.10).

We need not do all this in detail here but we shall be contented with deriving the special case $\theta = 1/2$, $\alpha = 2$ of Theorem I.6. Namely the fact that the condition

$$\int_0^1 \int_0^1 \left[\frac{f(s)-f(t)}{s-t}\right]^2 ds\, dt \leqslant \mathrm{B}^2 < \infty$$

implies

$$\int_0^1 e^{\frac{(\log 2)^2}{2}\left[\frac{f(x)-f^*(1/2)}{\mathrm{B}}\right]^2} dx \leqslant 2\sqrt{2} \tag{5.11}$$

In this case we have $p(u) = u$, so (5.10) gives

$$f^*(x) - f^*(1/2) \leqslant \frac{\mathrm{B}}{\log 2}\left[\log\frac{1}{x}\right]^{1/2}.$$

Thus

$$e^{\frac{(\log 2)^2}{2}\left[\frac{f^*(x)-f^*(1/2)}{\mathrm{B}}\right]^2} \leqslant e^{1/2 \log 1/x} = \frac{1}{\sqrt{x}}.$$

Integrating over $(0, 1/2)$

$$\int_0^{1/2} e^{\frac{(\log 2)^2}{2}\left[\frac{f^*(x)-f^*(1/2)}{\mathrm{B}}\right]^2} dx \leqslant \sqrt{2}.$$

The bottom part of (5.10) yields the same bound for the integral over $(1/2, 1)$. This proves (5.11).

Theorems I.4 and I.8 are immediate consequences of the following general result

THEOREM 5.2. — *Let*

$$\mathrm{B} = \left[\int_0^1 \int_0^1 \left[\frac{f(s)-f(t)}{p(s-t)}\right]^\alpha ds\, dt\right]^{1/\alpha} < \infty, \tag{5.12}$$

where $\alpha > 1$ *and* p *in addition satisfies*

$$\int_0^1 \left[\frac{p(u)}{u^{2/\alpha}}\right]^\beta \frac{du}{u} < \infty$$

with $\beta = \frac{\alpha}{\alpha-1}$. *Then* $f(x)$ *is essentially continuous and for* *almost all* x, y *in* $[0, 1]$ *we have*

$$(5.13)\quad |f(x) - f(y)| \leqslant \frac{2B}{\log 2}\left[\int_0^{|x-y|}\left[\frac{p(u)}{u^{2/\alpha}}\right]^\beta \frac{du}{u}\right]^{1/\beta}$$

Proof. — This follows from (5.10) by the change of scale argument. We set as before for $0 \leqslant x < y \leqslant 1$

$$\tilde{f}(\tilde{s}) = f(x + \tilde{s}\Delta) \quad \Delta = y - x$$

and obtain from (5.12)

$$\int_0^1\int_0^1\left[\frac{\tilde{f}(\tilde{s}) - \tilde{f}(\tilde{t})}{p(\Delta(\tilde{s} - \tilde{t}))}\right]^\alpha d\tilde{s}\, d\tilde{t} \leqslant \frac{B^\alpha}{\Delta^2}.$$

From (5.10) we get

$$\tilde{f}(0^+) - \tilde{f}(1^-) \leqslant \frac{2B}{\Delta^{2/\alpha}\log 2}\left[\int_0^1\left[\frac{p(\Delta s)}{s^{2/\alpha}}\right]^\beta \frac{ds}{s}\right]^{1/\beta}$$

making the substitution $u = \Delta s$ (5.13) immediately follows.

Our final task is proving Theorem I.5.

We start by assuming that

$$\left[\int_0^1\int_0^1\left[\frac{|f(s) - f(t)|}{|s - t|^q}\right]^\alpha ds\, dt\right]^{1/\alpha} = B < \infty,$$

with

$$q \geqslant \frac{2}{\alpha} - 1.$$

and set

$$p = 1\Big/\left(\frac{2}{\alpha} - q\right).$$

From (5.10) with $\frac{p(s)}{s^{2/\alpha}} = s^{-1/p}$ we get

$$f^*(\delta) - f^*(1/2) \leqslant \frac{B}{\log 2}\left[\int_\delta^{1/2}\frac{ds}{s^{1+\beta/p}}\right]^{1/\beta}$$

or better

$$(5.14)\qquad f^*(\delta) - f^*(1/2) \leqslant \frac{B}{\log 2}\left(\frac{p}{\beta}\right)^{1/\beta}\frac{1}{\delta^{1/p}}.$$

The assertion that f is in weak L_p easily follows from this inequality.

To obtain the stronger inequality (I.12) we go back to (5.7) and set $\theta(s) = \gamma s$, where $\gamma > 0$ is to be determined. This gives

$$\int_0^1 [(f^*(\gamma s) - f^*(s))s^{1/p}]^\alpha \frac{ds}{s} \leqslant \frac{B^\alpha}{2\gamma}. \tag{5.15}$$

To simplify our calculations assume for a moment that $f^*(1/2) = 0$. We then have $f^*(s) \geqslant 0$ for $0 < s \leqslant 1/2$, thus setting $\delta_n = \gamma^n/2$ $(n = 0, 1, \ldots)$ and using the inequality

$$A^\alpha \leqslant B^\alpha + \alpha A^{\alpha-1}(A - B) \quad (A \geqslant B \geqslant 0)$$

for

$$A = f^*(\gamma s), \qquad B = f^*(\gamma s) - f^*(s)$$

we obtain

$$\int_{\delta_{n+1}}^{\delta_n} [f^*(\gamma s)]^\alpha s^{\alpha/p} \frac{ds}{s} \leqslant \int_{\delta_{n+1}}^{\delta_n} [f^*(\gamma s) - f^*(s)]^\alpha s^{\alpha/p} \frac{ds}{s} + \alpha \int_{\delta_{n+1}}^{\delta_n} [f^*(\gamma s)]^{\alpha-1} f^*(s) s^{\alpha/p} \frac{ds}{s}. \tag{5.16}$$

Now, since $\forall\ u, v, \lambda \geqslant 0$

$$\alpha u v \leqslant \frac{\alpha - 1}{\lambda} u^\beta + \lambda v^\alpha$$

we get

$$\alpha \int_{\delta_{n+1}}^{\delta_n} [f^*(\gamma s)]^{\alpha-1} f^*(s) s^{\alpha/p} \frac{ds}{s} \leqslant \frac{\alpha - 1}{\lambda} \int_{\delta_{n+1}}^{\delta_n} [f^*(s)]^\alpha s^{\alpha/p} \frac{ds}{s}.$$

Substituting in (5.16), a change of variables gives

$$\frac{1}{\gamma^{\alpha/p}}\left(1 - \frac{\alpha - 1}{\lambda}\right) \int_{\delta_{n+2}}^{\delta_{n+1}} [f^*(s)]^\alpha s^{\alpha/p} \frac{ds}{s} \leqslant \int_{\delta_{n+1}}^{\delta_n} [f^*(\gamma s) - f^*(s)]^\alpha s^{\alpha/p} \frac{ds}{s} + \lambda \int_{\delta_{n+1}}^{\delta_n} [f^*(s)]^\alpha s^{\alpha/p} \frac{ds}{s}.$$

We now choose λ, γ so that

$$\rho = \frac{1}{\lambda \gamma^{\alpha/p}}\left(1 - \frac{\alpha - 1}{\lambda}\right) > 1$$

and set

$$\theta_n = \int_{\delta_{n+1}}^{\delta_n} [f^*(s)]^\alpha s^{\alpha/p} \frac{ds}{s}.$$

We have then

$$\rho\theta_{n+1} \leqslant \frac{1}{\lambda}\int_{\delta_{n+1}}^{\delta_n} [f^*(\gamma s) - f^*(s)]^\alpha s^{\alpha/p}\frac{ds}{s} + \theta_n.$$

Summing for $n = 0, 1, \ldots, N$ and using (5.15)

$$\rho[\theta_1 + \theta_2 + \cdots + \theta_N] \leqslant (\theta_0 + \theta_1 + \cdots + \theta_N) + \frac{1}{2\lambda}\frac{B^\alpha}{\gamma}.$$

Note now that from (5.14) we have

$$\theta_0 = \int_\gamma^{1/2} [f^*(s)s^{1/p}]^\alpha \frac{ds}{s} \leqslant \left(\frac{B}{\log 2}\right)^\alpha \left(\frac{p}{\beta}\right)^{\alpha/\beta} \log\frac{1}{2\gamma}.$$

Thus

$$(\rho - 1)(\theta_1 + \theta_2 + \cdots + \theta_N) \leqslant \left(\frac{B}{\log 2}\right)^\alpha \left(\frac{p}{\beta}\right)^{\alpha/\beta} \log\frac{1}{2\gamma} + \frac{B^\alpha}{2\lambda\gamma}$$

Passing to the limit as $N \to \infty$ we see that there is a constant $c(\alpha, p)$ such that

$$\int_0^{1/2} [f^*(s)s^{1/p}]^\alpha \frac{ds}{s} \leqslant B^\alpha c(\alpha, p).$$

In case $f^*(1/2) \neq 0$ working with $f(s)$ replaced by $f(s) - f^*(1/2)$ we obtain

$$\int_0^{1/2} [(f^*(s) - f^*(1/2))s^{1/p}]^\alpha \frac{ds}{s} \leqslant B^\alpha c(\alpha, p).$$

Of course, we must also have

$$\int_0^{1/2} [(f^*(1 - s) - f^*(1/2))s^{1/p}]^\alpha \frac{ds}{s} \leqslant B^\alpha c(\alpha, p).$$

Combining these last two inequalities with Minkowski's (I.12) is readily obtained.

Remark 5.1. — It is good to point out that the inequality (5.13) is always at least as good as the one we can derive from I.8, and in some cases it is definitely netter.

From Theorem I.2 we get that when $\Psi(u) = |u|^\alpha$ a sufficient condition for f to be continuous is

$$\varphi(\delta) = \int_0^\delta \frac{dp(u)}{u^{2/\alpha}} < \infty, \tag{5.17}$$

while from Theorem 5.2 we get the condition

$$(5.18) \qquad \Psi(\delta) = \left[\int_0^\delta \left[\frac{p(u)}{u^{2/\alpha}}\right]^\beta \frac{du}{u}\right]^{1/\beta} < \infty.$$

Now it is easy to see that in the case $p(u) = \dfrac{u}{\log 1/u}$, $\alpha = 2$ (5.17) fails to be satisfied while (5.18) holds true. Thus when $\Psi(u) = |u|^\alpha$ Theorem (5.2) predicts continuity in cases when Theorem I.2 does not. This fact was first discovered by C. Greenhall [9] who determined in the case $\alpha = 2$ the best possible estimates.

We do not know if a similar situation holds when $\Psi(u) = e^{cu^2}$ or $\Psi(u) = e|^u|$. If this were to be the case then some of the results obtained in [7] and [8] could be sharpened and the latter appears to be doubtful.

For a function f satisfying the inequality

$$\int_0^1 \int_0^1 \left[\frac{|f(s) - f(t)|}{p(s-t)}\right]^\alpha ds\, dt \leqslant B^\alpha$$

our two methods of estimating f^* give the modulus of continuity estimates

$$|f(x) - f(y)| \leqslant 4B\varphi\left(\frac{(x-y)}{2}\right)$$

$$|f(x) - f(y)| \leqslant \frac{2B}{\log 2}\psi(|x-y|).$$

That the latter is qualitatively always as good as the former when (5.17) holds follows from the simple inequality

$$\psi(\delta) \leqslant (\alpha/2)^\beta \varphi(\delta) \quad 0 < \delta < 1.$$

which was pointed out to us by D. Adams.

6. Applications to path smoothness of L_p-processes.

As we have mentioned earlier the results of this paper yield some new information about the path behaviour of Gaussian and non-Gaussian processes.

We shall not pursue this matter to the full here since it would lead us too far out of the present context. We just note that Theorem I.8 and Theorem 5.2 yield interesting improvements upon existing conditions for path smoothness of L_p-processes.

To this end let $\{\chi_t\}$ be a process on $[0, 1]$.

In [7] we have proved the following result:

THEOREM 6.1. — *Suppose that*

$$\int_0^1 \int_0^1 \mathrm{E}\left(\left|\frac{\chi_s - \chi_t}{p(s-t)}\right|^2\right) ds\, dt < \infty, \tag{6.1}$$

where $p(u)$ *satisfies* I.3 *and*

$$\int_0^1 \frac{p(u)}{u^2}\, du < \infty \tag{6.2}$$

then $\{\chi_t\}$ *has a model* $\{\tilde{\chi}_t\}$ *with a.s. continuous paths. Indeed, for* $\{\tilde{\chi}_t\}$ *we can produce a random variable* $\mathrm{B}(\omega)$ *with finite expectation such that* $\forall s,\ t \in [0, 1]$

$$|\tilde{\chi}_s - \tilde{\chi}_t| \leqslant \sqrt{\mathrm{B}(\omega)} \int_0^{|s-t|} \frac{p(u)}{u^2}\, du.$$

A corollary of the above result is that if $\{\chi_t\}$ satisfies the condition

$$\mathrm{E}(|\chi_s - \chi_t|^2) \leqslant c \frac{|s-t|}{\left[\log \frac{2}{|s-t|}\right]^\lambda} \tag{6.3}$$

then, if $\lambda > 3$, $\{\chi_t\}$ has a model with a.s. continuous paths.

Indeed, it is easy to see that if we set

$$p(u) = \frac{|u|}{\left[\log \frac{2}{|u|}\right]^\gamma} \tag{6.4}$$

then (6.1) is satisfied as soon as

$$\gamma < \frac{\lambda - 1}{2} \tag{6.5}$$

while (6.2) requires

$$\gamma > 1.$$

Clearly such a γ can be found when $\lambda > 3$.

Using Theorem 5.2 and following step by step the arguments given in [7] we can improve Theorem 6,1 as follows :

THEOREM 6.2. — *Suppose* 6.1 *holds and*

$$\int_0^1 [p(u)]^2 \frac{du}{u^3} < \infty. \tag{6.6}$$

Then $\{\chi_t\}$ *has a model* $\{\tilde{\chi}_t\}$ *with a.s. continuous paths. Indeed, for* $\{\tilde{\chi}_t\}$ *we can produce a random variable* $B(\omega)$ *with finite expectation such that* $\forall s, t \in [0, 1]$ *we have*

$$|\tilde{\chi}_s - \tilde{\chi}_t| \leqslant \sqrt{B(\omega)}\left[\int_0^{|s-t|} [p(u)]^2 \frac{du}{u^3}\right]^{1/2}.$$

Again, if (6.3) holds and choose $p(u)$ as in (6.4) then (6.6) requires

$$\gamma > 1/2.$$

Combining with 6.5, we see that as soon as $\lambda > 2$ we can find γ so that both 6.1 and 6.6 are satified. We thus obtain as a corollary :

THEOREM 6.3. — *Suppose that*

$$E(|\chi_s - \chi_t|^2) \leqslant c \frac{|s-t|}{\left[\log \frac{2}{|s-t|}\right]^\lambda} \ \lambda > 2.$$

Then $\{\chi_t\}$ *has a model with a.s. continuous paths.*

To put this result in historical perspective, we point out that, quite independently from our work P. Bernard [1] showed that if

$$E(|\chi_s - \chi_t|^\alpha) \leqslant c \frac{|s-t|}{\left[\log \frac{2}{|s-t|}\right]^\lambda} \ (\alpha > 1) \tag{6.7}$$

then, if $\lambda > \alpha + 1$, $\{\chi_t\}$ has a model with a.s. continuous paths.

Thus we see that, at least when $\alpha = 2$, Bernard's result is a consequence of what we obtained in [7] and thus it is superseeded by our work here.

In [3] R. M. Dudley shows that the condition (6.7) with $\lambda = -1$ is *not* sufficient to guarantee path continuity and wonders where the best possible exponent λ lies between -1 and $\alpha + 1$.

Our Theorem I.8 indicates that

$$\lambda > \alpha$$

is sufficient to guarantee path continuity and we strongly suspect that this is best possible.

Indeed, setting

$$B(\omega) = \left[\int_0^1 \int_0^1 [\chi_t - \chi_s]^\alpha \left[\log \frac{2}{|s-t|}\right]^{\alpha\theta+\alpha/\beta} \frac{ds\,dt}{|s-t|^2}\right]^{1/\alpha};$$

and using (6.7) we easily get

$$E(B^\alpha(\omega)) \leqslant 2c \int_0^{1/2} \left[\log \frac{1}{\sigma}\right]^{\alpha\theta+\alpha-1+\lambda} \frac{d\sigma}{\sigma}.$$

Thus B^α will have a finite expectation as soon as

$$\alpha\theta + \alpha < \lambda$$

or better, as soon as

$$0 < \theta < \frac{\lambda - \alpha}{\alpha}.$$

Theorem I.8 then yields an estimate of the form

$$|\chi_t - \chi_s| \leqslant c_{\alpha,\theta} \frac{B(\omega)}{\left[\log \frac{2}{|s-t|}\right]^\theta}$$

for almost all $s, t \in [0, 1]$.

For our argument to be complete, we should go on to exhibit a model $\{\tilde{\chi}_t\}$ with continuous paths whenever $\lambda > \alpha$.

We are not doing this here, but we do not expect this further task to present serious difficulties.

In Dudley's paper we see that the condition $\lambda > \alpha$ was shown by Delporte [2] to be sufficient to guarantee path continuity for processes with a 2-dimensional time parameter.

However, we do not expect (see [5]) that the dimension is a factor as far as λ is concerned. From our experience in this subject, we conjecture that the weakest condition of type (6.7) for d-dimensional time processes is

$$E(|\chi_s - \chi_t|^\alpha) \leqslant c \frac{|s-t|^d}{\left[\log \frac{2}{|s-t|}\right]^\lambda}, \quad \lambda > \alpha.$$

In this connection we wish to point out for the record that the theorem of Marcus and Shepp [13] quoted by Dudley in Section 7.2 of [3] was conjectured by us in [7].

We close this section by pointing out that using our results here we can show that, even when $\lambda \leqslant \alpha$ a process $\{\chi_t\}$ satisfying 6.7 has paths with a considerable degree of smoothness.

In fact, some interesting conclusions can be derived using Theorems I.6, I.7.

As a matter of fact, a simple use of our fundamental inequality I.6 shows that a process $\{\chi_t\}$ which satisfies 6.7 with

$$\lambda > 0,$$

has paths which a.s. satisfy the intermediate value property.

To see how this comes about, note that a non-increasing function $f^*(s)$ which satisfies

$$\int_0^1 \int_0^1 \Psi\left(\frac{f^*(s) - f^*(t)}{p(s-t)}\right) ds\, dt < \infty$$

cannot have any jump in (0, 1) when

$$\text{(6.8)} \qquad \int_0^1 \Psi\left(\frac{\varepsilon}{p(\sigma)}\right) \sigma\, d\sigma = \infty \quad \forall \varepsilon > 0.$$

When $\Phi(u) = |u|^\alpha$ and

$$\text{(6.9)} \qquad p(u) = \frac{|u|^{2/\alpha}}{\left[\log \frac{2}{|u|}\right]^\gamma}$$

this condition reduces to

$$\int_0^1 \left[\log \frac{2}{\sigma}\right]^{\gamma\alpha} \frac{d\sigma}{\sigma} = \infty$$

or better

$$\gamma > -1/\alpha$$

Now we observe that when $\{\chi_t\}$ satisfies (6.7) we have

$$\text{(6.10)} \qquad \mathrm{E}\left(\int_0^1 \int_0^1 \left|\frac{\chi_s - \chi_t}{p(s-t)}\right|^\alpha ds\, dt\right) < \infty$$

as soon as

$$\int_0^1 \frac{\sigma}{\left[\log \frac{2}{\sigma}\right]^\lambda} \frac{d\sigma}{p^\alpha(\sigma)} < \infty.$$

When $p(u)$ is given by (6.9) this reduces to

$$\int_0^1 \left[\log \frac{2}{\sigma}\right]^{-\lambda+\alpha\gamma} \frac{d\sigma}{\sigma} < \infty$$

or better

$$\gamma < \frac{\lambda - 1}{\alpha}.$$

Thus we immediately conclude that when

$$-1/\alpha < \frac{\lambda - 1}{\alpha},$$

there is a $p(u)$ such that both 6.8 (with $\Psi(u) = |u|^\alpha$) and (6.10) are satisfied.

This implies in particular that for a process $\{\chi_t\}$ satisfying (6.7) with $\lambda > 0$ the paths have a.s. monotone rearrangements χ_s^* with no jumps in (0, 1).

To obtain the full result announced earlier we need only apply the change of scale argument introduced in Section 5.

Thus we can state

THEOREM 6.4. — *If*

$$\mathrm{E}(|\chi_s - \chi_t|^\alpha) \leq c \frac{|s-t|}{\left[\log \frac{2}{|s-t|}\right]^\lambda}$$

with $\lambda > 0$. *Then with probability one a path* χ_t *takes in any subinterval of* (0, 1) *every value between its ess sup and its ess inf.*

7. Conditions implying constancy. Uniform convergence results for Fourier series and further remarks.

For certain choices of Ψ and p the only f's satisfying

$$(7.1) \quad I_{\Psi,p}(f) = \int_0^1 \int_0^1 \Psi\left(\frac{f(x)-f(y)}{p(x-y)}\right) dx\, dy < \infty$$

are essentially the constants.

When this happens let us, for convenience, say that our condition (7.1) « *implies constancy* ».

From Theorem (5.2) we see that

$$(7.2) \qquad \int_0^1 \int_0^1 \left|\frac{f(x)-f(y)}{|x-y|^{\theta}}\right|^{\alpha} dx\, dy = B^{\alpha} < \infty$$

gives (when $\theta > 2/\alpha$)

$$(7.3) \quad |f(x)-f(y)| \leqslant \frac{B}{\log 2}\frac{1}{[(\theta-2/\alpha)\beta]^{1/\beta}}|x-y|^{\theta-2/\alpha}.$$

This implies constancy as soon as $\theta > 1 + 2/\alpha$.

However, it is well known and easy to show using elementary Fourier analysis that (7.2) with $\alpha = 2$ implies constancy as soon as $\theta \geqslant 3/2$.

For some reasons we do not fully understand (7.3) (when $\alpha = 2$) is *best possible* when $1 \leqslant \theta < 3/2$ and it is just jibberish when $\theta \geqslant 3/2$.

Nevertheless, if we modify our estimating procedure, our methods can be used to derive constancy conditions which agree with all the previously known ones.

Indeed, we can show that (7.2) when $\alpha \geqslant 1$ implies constancy as soon as

$$(7.4) \qquad \theta \geqslant 1 + 1/\alpha.$$

The basic estimate from which this result can be derived can be stated as follows :

THEOREM 7.1. — *Let* Ψ, p *satisfy* I.2, I.3 *and suppose that* Ψ *is convex. Let*

$$\lambda_0 = \sup\left\{\lambda \geqslant 0 : \int_0^1 \Psi\left(\lambda\frac{\sigma}{p(\sigma)}\right) d\sigma < \infty\right\}.$$

Then if $f^*(x)$ *is non-increasing in* $[0, 1]$ *and*

$$\int_0^1 \int_0^1 \Psi\left(\frac{f^*(x) - f^*(y)}{p(x-y)}\right) dx\, dy < \infty$$

we have

$$f^*(0^+) - f^*(1^-) \leqslant \lambda_0. \tag{7.5}$$

Proof. — Let $0 < \delta < 1$ and

$$R_\delta = \{(s, t) : 0 \leqslant t \leqslant 1 - \delta;\ 0 \leqslant s - t \leqslant \delta\}$$

Making the substitution $s = \sigma + t$ we get

$$\iint_{R_\delta} \Psi\left(\frac{f^*(s) - f^*(t)}{p(s-t)}\right) ds\, dt = \int_0^{1-\delta} \int_0^\delta \Psi\left(\frac{f^*(t) - f^*(t+\sigma)}{p(\sigma)}\right) d\sigma\, dt$$

Using the convexity of Ψ we derive

$$\int_0^\delta \Psi\left(\frac{1}{1-\delta} \int_0^{1-\delta} \frac{f^*(t) - f^*(t+\sigma)}{p(\sigma)}\, dt\right) d\sigma \leqslant \frac{1}{1-\delta} I_{\Psi, p}(f^*). \tag{7.6}$$

Since $f^*(t)$ is non-increasing and $\sigma \leqslant \delta$

$$\int_0^{1-\delta} [f^*(t) - f^*(t+\sigma)]\, dt = \int_0^\sigma f^*(t)\, dt - \int_{1-\delta}^{1-\delta+\sigma} f^*(t)\, dt \geqslant [f^*(\delta) - f^*(1-\delta)]\sigma$$

Substituting in (7.6)

$$\int_0^\delta \Psi\left(\frac{f^*(\delta) - f^*(1-\delta)}{(1-\delta)} \frac{\sigma}{p(\sigma)}\right) d\sigma \leqslant \frac{1}{1-\delta} I_{\Psi, p}(f^*).$$

In view of the definition of λ_0 we must have

$$f^*(\delta) - f^*(1-\delta) \leqslant (1-\delta)\lambda_0.$$

Passing to the limit as $\delta \to 0$ (7.5) follows.
We thus obtain the following interesting

Corollary 7.1. — *If* p *satisfies* I.3 *and*

$$\int_0^1 \left|\frac{\sigma}{p(\sigma)}\right|^\alpha d\sigma = \infty \tag{7.7}$$

then the condition

$$\int_0^1 \int_0^1 \left| \frac{f(x) - f(y)}{p(x-y)} \right|^\alpha dx\, dy < \infty$$

for $\alpha \geqslant 1$ *implies constancy.*

Proof. — Clearly (7.7) implies $\lambda_0 = 0$ in this case. Thus Theorems I.1 and 7.1 combined give

$$\text{ess sup } f - \text{ess inf } f = f^*(0^+) - f^*(1^-) = 0 \quad \text{Q.E.D.}$$

It is to be noted that Theorem (7.1) can also be used to derive some continuity results. For instance Theorem (7.1) has the following curious

Corollary 7.2. — *A measurable function on* $[0, 1]$ *is essentially Lipschitzian if and only if there is a* $B > 0$ *such that*

$$(7.8) \qquad \int_0^1 \int_0^1 \exp \left\{ \frac{|f(x) - f(y)|}{B|x-y|} \log \frac{1}{|x-y|} \right\} dx\, dy < \infty.$$

Proof. — The necessity of (7.8) is trivial. For the sufficiency we resort to the change of scale argument and set for

$$0 \leqslant x < y \leqslant 1$$
$$\tilde{f}(\tilde{s}) = f(x + \Delta \tilde{s}) \quad \Delta = y - x.$$

We then have

$$\int_0^1 \int_0^1 e^{\frac{|\tilde{f}(\tilde{s}) - \tilde{f}(\tilde{t})|}{B\tilde{p}(\tilde{s}-\tilde{t})}} d\tilde{s}\, d\tilde{t} = \int_x^y \int_x^y e^{\frac{|f(s)-f(t)|}{Bp(s-t)}} \frac{ds\, dt}{\Delta^2}.$$

provided $\tilde{p}(u) = p(\Delta u)$.

When $p(u) = \dfrac{|u|}{\log \frac{1}{|u|}}$, using Theorem 7.1 we get

$$\tilde{f}^*(0^+) - \tilde{f}^*(1^-) \leqslant \sup \left\{ \lambda : \int_0^1 e^{\lambda \frac{\sigma}{B\Delta\sigma} \log \frac{1}{\Delta\sigma}} d\sigma < \infty \right\}$$
$$\leqslant \sup \left\{ \lambda : \int_0^1 \left[\frac{1}{\Delta\sigma} \right]^{\frac{\lambda}{B\Delta}} d\sigma < \infty \right\} = B\Delta.$$

This immediately gives

$$|f(x) - f(y)| \leqslant B|x - y|$$

for almost all $x, y \in [0, 1]$. Q.E.D.

Our methods can be used to derive information about the behaviour of partial sums of classical Fourier series. This stems from the following observations.

Note first that if $f(x)$ is measurable on $(-\infty, +\infty)$ and periodic of period 2π then setting

$$A^\alpha = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+\delta) - f(x)|^\alpha \, dx \frac{d\delta}{[p(\delta)]^\alpha},$$

$$B^\alpha = \int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} \left|\frac{f(x) - f(y)}{p(x-y)}\right|^\alpha dx\, dy$$

we have

$$(7.9) \qquad A^\alpha \leqslant B^\alpha \leqslant 2A^\alpha + \frac{2^{\alpha-1}}{(p(\pi))^\alpha} 6\pi \|f\|_\alpha^\alpha$$

Provided of course $p(u) = p(-u) \downarrow 0$ as $|u| \downarrow 0$ in $[-4\pi, 4\pi]$.

This given using M. Riesz's theorem we immediately derive

$$\left[\int_{-\pi}^{\pi} |s_n(x+\delta, f) - s_n(x, f)|^\alpha \, dx\right]^{1/\alpha} \leqslant c_\alpha \left[\int_{-\pi}^{\pi} |f(x+\delta) - f(x)|^\alpha \, dx\right]^{1/\alpha}$$

where we have set as customary

$$s_n(x, f) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu x}, \quad c_\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\nu t} f(t)\, dt.$$

Combining with (7.9) we deduce that

$$\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} \left[\frac{s_n(x, f) - s_n(y, f)}{p(x-y)}\right]^\alpha dx\, dy \leqslant \left\{2A^\alpha + \frac{2^{\alpha-1}}{(p(\pi))^\alpha} 6\pi\|f\|_\alpha^\alpha\right\} c_\alpha^\alpha.$$

Applying Theorem 5.2 we get

$$|s_n(x, f) - s_n(y, f)| \leqslant \left[2A^\alpha + \frac{2^{\alpha-1}}{(p(\pi))^\alpha} 6\pi\|f\|_\alpha^\alpha\right]^{1/\alpha} \frac{c_\alpha}{\log 2}\left[\int_0^{|x-y|} \left[\frac{p(u)}{u^{2/\alpha}}\right]^\beta \frac{du}{u}\right]^{1/\beta}.$$

Thus we can state

THEOREM 7.2. — *If f is periodic of period 2π and*

$$(7.10)\quad A^\alpha = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x+\delta) - f(x)|^\alpha \, dx \frac{d\delta}{[p(\delta)]^\alpha} < \infty$$

with $p(u)$ satisfying I.3 *and*

$$(7.11)\qquad \int_0^\pi \left[\frac{p(u)}{u^{2/\alpha}}\right]^\beta \frac{du}{u} < \infty \quad \left(\beta = \frac{\alpha}{\alpha - 1}\right)$$

then the partial sums $\{s_n(x, f)\}$ of the Fourier series of f converge uniformly. Indeed they are equicontinuous and there is a constant c depending only on A, $\|f\|_\alpha$ *and α such that $\forall x, y$:*

$$|s_n(x, f) - s_n(y, f)| \leqslant c\left[\int_0^{|x-y|} \left[\frac{p(u)}{u^{2/\alpha}}\right]^\beta \frac{du}{u}\right]^{1/\beta}.$$

It is interesting to note that when $1 < \alpha \leqslant 2$ (7.10) with (7.11) can be shown by classical methods to imply the absolute convergence of the Fourier series of f. However, when $\alpha > 2$ this is not the case. Indeed, it is not difficult to verify that

$$f(x) = \sum_{n\neq 0}^{\infty} \frac{\pm 1}{n} e^{inx}$$

satisfies the condition (7.10) with any $\alpha > 2$, for almost all changes of sign.

For sure it must be possible to put together a classical Fourier analytical proof of the full result in Theorem 7.2. That is a proof that doesn't use our basic inequality (I.6).

However, so far we have been unable to find one.

The details we have omitted in this section along with some other material on Fourier Series will be the subject of a forthcoming publication by the first named author.

Before closing we would like to make a few additional remarks.

In [5] we have considered the functionals

$$B_{\Psi, p}(f) = \sup_I \int_I \int_I \Psi\left(\frac{f(x) - f(y)}{p(|I|)}\right) dx\, dy$$

where Ψ and p satisfy (I.2), (I.3) and Ψ is convex. Here I is a subinterval of $[0, 1]$ and $|I|$ denotes its length.

Now, the following theorem holds :

THEOREM 7.3. — *Under the above hypctheses if* $f^*(x)$ *is non-increasing in* $[0, 1]$ *and*

$$B_{\Psi,p}(f^*) \leqslant B < \infty \tag{7.12}$$

then for $0 < x \leqslant 1/2$

$$\left.\begin{array}{l}\dfrac{1}{x}\displaystyle\int_0^x f^*(t)\,dt - f^*(1/2)\\ f^*(1/2) - \dfrac{1}{x}\displaystyle\int_{1-x}^1 f^*(t)\,dt\end{array}\right\} \leqslant 4\int_x^{1/2} \Psi^{-1}\left(\frac{B}{u^2}\right) dp(u) + 4\Psi^{-1}\left(\frac{B}{x^2}\right) p(x). \tag{7.13}$$

Proof. — Setting, as we did at the beginning of Section 5, $x_0 = 1/2$ and

$$x_{n+1} = p^{-1}\left(\frac{p(x_n)}{2}\right), \tag{7.14}$$

from (7.12) we get

$$\frac{1}{x_n x_{n+1}} \int_0^{x_n} \int_0^{x_{n+1}} \Psi\left(\frac{f^*(x) - f^*(y)}{p(x_n)}\right) \leqslant \frac{B}{x_n x_{n+1}}.$$

Using the convexity of Ψ and (7.14) we immediately deduce that

$$\begin{aligned}\frac{1}{x_{n+1}} \int_0^{x_{n+1}} f^*(x)\,dx - \frac{1}{x_n}\int_0^{x_n} f^*(y)\,dy &\leqslant \Psi^{-1}\left(\frac{B}{x_n x_{n+1}}\right) p(x_n)\\ &\leqslant 4\int_{x_{n+2}}^{x_{n+1}} \Psi^{-1}\left(\frac{B}{u^2}\right) dp(u).\end{aligned}$$

Proceeding as we did at the beginning of Section 5 we easily derive for all $0 < x \leqslant 1/2$

$$\begin{aligned}\frac{1}{x}\int_0^x f^*(t)\,dt - 2\int_0^{1/2} f^*(t)\,dt \leqslant 4\int_0^{1/2} \Psi^{-1}\left(\frac{B}{u^2}\right) dp(u)\\ + 4\Psi^{-1}\left(\frac{B}{x^2}\right) p(x).\end{aligned}$$

This clearly implies one half of (7.13). The other half is obtained by replacing $f^*(x)$ with $-f^*(1-x)$.

This given, one is tempted to formulate (in analogy to I.6) the inequality

$$B_{\Psi,p}(f^*) \leqslant CB_{\Psi,p}(f) \tag{7.15}$$

for some constant C.

It turns out that when $\Psi(u) = |u|$ and $p(u) = |u|^2$, (7.15) is indeed true with $C = 10$.

Now, in this case, it is not difficult to show that the condition

$$B_{\Psi,p}(f) = \sup_I \frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)|\, dx\, dy < \infty$$

is equivalent to $f \in \mathrm{BMO}$. Thus, (7.15) and (7.13) combined lead to an interesting new proof of the John-Nirenberg theorem [11].

Unfortunately, it can be shown that when $\Psi(u) = |u|$ and $p(u) = |u|^\alpha$ $1 < \alpha < 2$, (7.15) is false for any C.

Nevertheless, it is our belief that there is a functional $A_{\Psi,p}(f)$ which reduces to $B_{\Psi,p}(f)$ when $\Psi(u) = |u|$ and $p(u) = |u|^2$ which satisfies

$$B_{\Psi,p}(f^*) \leqslant CA_{\Psi,p}(f) \tag{7.16}$$

for some C.

In fact, when $\Psi(u) = |u|$ and $p(u) = |u|^{2-1/\alpha}(1 < \alpha \leqslant \infty)$ we can take

$$A_{\Psi,p}(f) = \sup_{\{I_k\}} \frac{\sum_k |I_k|^{-1} \int_{I_k} \int_{I_k} |f(x) - f(y)|\, dx\, dy}{\left[\sum_k |I_k|\right]^{1/\beta}} \tag{7.17}$$

where we have set $\beta = \dfrac{\alpha}{\alpha - 1}$ and the sup is taken over all sequences of disjoint intervals $\{I_k\}$ such that

$$\bigcup_k I_k \subset [0, 1].$$

This given, note that if f is in weak L_α, i.e., if

$$m\{x : |f(x)| \geqslant \lambda\} \leqslant \frac{C}{\lambda^\alpha} \quad \forall \lambda > 0 \tag{7.18}$$

then

$$\sum_k \frac{1}{|I_k|} \int_{I_k} \int_{I_k} |f(x) - f(y)| \, dx \, dy \leqslant 2 \sum_k \int_{I_k} |f(x)| \, dx,$$

and clearly (7.18) implies

$$\int_{\sum_k I_k} |f| \, dx \leqslant c \Big(\sum_k |I_k| \Big)^{1/\beta}$$

Thus in this case the functional in (7.17) is finite. Viceversa, if this functional is finite, (7.16) and (7.13) immediately yield (7.18) back again.

We therefore obtain the following interesting result.

THEOREM 7.4. *A function $f(x)$ integrable in $[0, 1]$ is in weak L_α if and only if there is a constant* B *such that*

$$\sum_k \frac{1}{|I_k|} \int_{I_k} \int_{I_k} |f(x) - f(y)| \, dx \, dy \leqslant B \Big(\sum_k |I_k| \Big)^{\alpha/\alpha-1}$$

for all sequences $\{I_k\}$ of disjoint intervals contained in $[0, 1]$.

It is not difficult to see that this result is a little stronger than one of the theorems proved by John and Nirenberg in [11].

We wish to point out also that using the Martingale techniques of [6] a proof of Theorem (7.4) can be obtained quite directly and without any dimension restrictions.

It would be of interest to find out what $A_{\Psi,p}(f)$ should be for general Ψ and p in order that (7.16) holds true.

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A. Garsia,
P. O. Box 295,
Rancho Santafé, Calif. (USA).

E. Rodemich.