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COMPLETENESS AND EXISTENCE OF BOUNDED BIHARMONIC FUNCTIONS ON A RIEMANNIAN MANIFOLD

by Leo SARIO

A.S. Galbraith has communicated to us the following intriguing problem: Does the completeness of a manifold imply, or is it implied by, the emptiness of the class $H^2_B$ of bounded nonharmonic biharmonic functions? Among all manifolds considered thus far in biharmonic classification theory (cf. Bibliography), those that are complete fail to carry $H^2_B$-functions, and one might suspect that this is always the case. We shall show, however, that there do exist complete manifolds of any dimension that carry $H^2_B$-functions. Moreover, there exist both complete and incomplete manifolds not permitting these functions, and, trivially, incomplete manifolds possessing them.

We attach a Bibliography of recent work in the field.

1. Let $C$ be the totality of complete Riemannian manifolds $M$, characterized by an infinite distance of any point of $M$ to the ideal boundary. Denote by $\mathcal{C}_N^{H^2_B}$ and $\mathcal{C}_N^{\bar{H}^2_B}$ the classes of $N$-manifolds, $N \geq 2$, for which $H^2_B = \emptyset$ or $\bar{H}^2_B \neq \emptyset$, respectively.

**Theorem 1.** $C \cap \mathcal{C}_N^{\bar{H}^2_B} \neq \emptyset$ for every $N$.

**Proof.** Take the $N$-cylinder

$$|x| < \infty, \quad |y| \leq 1, \quad i = 1, 2, \ldots, N - 1,$$

with each face $y_i = 1$ identified with $y_i = -1$, so as to obtain a covering space of the $N$-torus in the same manner as a conventional cylinder is a covering surface of the torus. Let $T$ be this $N$-cylinder with the Riemannian metric

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\[ ds^2 = \mu^{-2}(x) \, dx^2 + \mu^{-4/2(N-1)}(x) \sum_{i=1}^{N-1} dy_i^2 \]

where

\[ \mu(x) = (2 + x^2) \frac{1}{2} \log(2 + x^2). \]

To see that \( T \in C \), it suffices to show, in view of the symmetry, that \( \int_0^\infty \mu^{-1}(x) \, dx = \infty \). The verification is immediate:

\[ \int_0^\infty (2 + x^2)^{-\frac{1}{2}} \log^{-1} (2 + x^2) \, dx \geq \frac{1}{2} \int_0^\infty (2 + x)^{-1} \log^{-1}(2 + x) \, dx \]

\[ = \frac{1}{2} \int_0^\infty \log \log(2 + x) = \infty. \]

We introduce the function

\[ u(x) = \int_0^x \mu^{-3}(t) \int_0^t \mu(s) \int_0^s \mu^{-3}(r) \, dr \, ds \, dt. \]

The Laplace-Beltrami operator \( \Delta = d\delta + \delta d \) gives

\[ \Delta u = -g^{-\frac{1}{2}} (g^{xx} u')' = -\mu^{-1} (\mu \mu^2 u')' = -\int_0^x \mu^{-3}(r) \, dr \]

and

\[ \Delta^2 u = -\mu^{-1} (\mu \mu^2 (-\mu^{-3}))' = 0. \]

Thus \( u \) is nonharmonic biharmonic.

To see that \( u \) is bounded it suffices to show that it is so for \( x > 0 \). For all \( s > 0 \),

\[ \int_0^s \mu^{-3}(r) \, dr = \int_0^s (2 + r^2)^{-3/2} \log^{-3}(2 + r^2) \, dr = o(1), \]

and for all \( t > 0 \),

\[ \int_0^t \mu(s) \int_0^s \mu^{-3}(r) \, dr \, ds < c \int_0^t (2 + s^2)^{\frac{1}{2}} \log(2 + s^2) \, ds \]

\[ < 2c \int_0^t (2 + s) \log(2 + s) \, ds \]

\[ = c \left[ (2 + t)^2 \log(2 + t) \right. \]

\[ - \frac{1}{2} (2 + t)^2 + \text{const.} \].
Here and later $c$ is a constant, not always the same. We let $[\ ]$ stand for the expression in brackets and obtain

$$u(x) < c \int_0^x (2 + t^2)^{-3/2} \log^{-3}(2 + t^2) [\ ] \, dt.$$ 

The dominating term in the integrand is majorized by

$$\frac{1}{2} t^{-3} \log^{-3} t x (2 + t^2) \log(2 + t).$$

The integral from 1 to $x > 1$ is bounded, and consequently so is $u$ for all $x$.

This completes the proof of Theorem 1.

2. The following simple example, valid for $N > 3$, is perhaps also of interest. Let

$$T : |x| < \infty, \quad |y| \leq \pi, \quad |z_i| \leq 1, \quad i = 1, \ldots, N - 2,$$

with the metric

$$ds^2 = dx^2 + e^{-x} \, dy^2 + e^{(2e^x - x)/(N-2)} \sum_{i=1}^{N-2} dz_i^2,$$

the opposite faces again identified by pairs. Clearly $T \in C$.

The function

$$u = \cos y$$

belongs to $H^2 B$. In fact,

$$\Delta u = -e^{-e^x + x} (e^{e^x - x}) (-\cos y)$$

$$= e^x \cos y,$$

and

$$\Delta^2 u = -e^{-e^x + x} [(e^{e^x - x})' \cos y + e^{e^x - x} e^x (-\cos y)] = 0.$$

Thus $T \in C \cap \widetilde{\mathcal{C}}_{H^2 B}$.

3. The reason that we are only interested in nonharmonic biharmonic functions is, of course, that completeness is known not
to exclude bounded harmonic functions (Nakai-Sario [6]). For \( N \geq 3 \), we insert here a simple proof of this fact.

Take the \( N \)-cylinder

\[ T: |x| < \infty, \quad |y| \leq 1, \quad |z_i| \leq 1, \quad i = 1, \ldots, N-2, \]

with the metric

\[ ds^2 = dx^2 + e^{2x^2} dy^2 + \sum_{i=1}^{N-2} dz_i^2. \]

Trivially \( T \in \mathcal{C} \). The function

\[ h(x) = \int_0^x e^{-t^2} \, dt \]

is harmonic,

\[ \Delta h = -e^{-x^2} (e^{x^2} e^{-x^2})' = 0. \]

It also is bounded and, in fact, even Dirichlet finite:

\[ D(h) = c \int_{-\infty}^\infty e^{-2x^2} e^{x^2} \, dx < \infty. \]

4. We return to nonharmonic biharmonic functions.

**Theorem 2.** \( - C \cap \mathcal{H}^N_{2B} \neq \emptyset \) for every \( N \).

**Proof.** – The Euclidean \( N \)-space \( \mathbb{E}^N \in \mathcal{C} \). Every biharmonic function \( u \) has an expansion in spherical harmonics \( S_{nm} \)

\[ u = \sum_{u=0}^\infty \sum_{m=1}^{m_n} (a_{nm} r^{n+2} + b_{nm} r^n) S_{nm}. \]

If \( u \in \mathcal{H}^2B \), then

\[ \int_{|x|=r} u S_{nm} \, d\omega = c(a_{nm} r^{n+2} + b_{nm} r^n) \]

is bounded in \( r \), hence \( a_{nm} = b_{nm} = 0 \) for all \( n \), except for \( b_{01} \). Therefore \( u \) is constant.

5. In view of \( u = r^2 \in \mathcal{H}^2B \) on the Euclidean \( N \)-ball, we have trivially \( \tilde{C} \cap \mathcal{H}^N_{2B} \neq \emptyset \) for every \( N \), with \( \tilde{C} \) the totality of incomplete Riemannian manifolds. It remains to show:
THEOREM 3. - $C \cap \Theta^N_{H^2B} \neq \emptyset$ for every $N$.

Proof. — Let $E^N_\alpha$ be the $N$-space $0 < r < \infty$ with the metric
\[ ds = r^\alpha |dx| , \]
$\alpha$ a constant. It is known (Sario-Wang [19, 21]) that if $N \geq 4$, $E^N_\alpha \in \Theta_{H^2B}$ for every $\alpha$; $E^2_\alpha \in \Theta_{H^2B}$ if and only if $\alpha \neq -1 + n/2$, $n = 1, 2, \ldots$; $E^3_\alpha \in \Theta_{H^2B}$ if and only if $\alpha \neq -1 \mp \left[ \frac{1}{2} n(n + 1) \right]^{1/2}$.

On the other hand, $E^N_\alpha \in \mathcal{C}$ for every $\alpha$, hence the theorem.

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