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Weakly semibounded boundary problems and sesquilinear forms


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WEAKLY SEMIBOUNDED
BOUNDARY PROBLEMS
AND SESQUILINEAR FORMS

by Gerd GRUBB

In this paper and its successor [8] we study boundary value problems for systems A of linear partial differential operators on a manifold \( \overline{\Omega} \) with boundary \( \Gamma \).

Let \( A \) be a \( q \times q \)-system of differential operators of order \( 2m \), let \( \rho u \) denote the Cauchy data \( \{ \gamma_0 u, \ldots, \gamma_{2m-1} u \} \) of \( u \) with respect to \( A \), and let \( B \) be a system of differential operators on \( \Gamma \); then \( A_B \) denotes the realization of \( A \) with domain

\[
D(A_B) = \{ u \in H^{2m}(\overline{\Omega}) | B_{\rho} u = 0 \}.
\]

(\( A \) and \( B \) actually operate on sections in vector bundles over \( \overline{\Omega} \), resp. \( \Gamma \).) The boundary condition \( B_{\rho} u = 0 \) is assumed to be normal in an appropriate sense. One is interested in the coerciveness inequalities

\[
(0.1) \quad \text{Re}(A u, u) \geq c_s \| u \|_s^2 - c_0 \| u \|_0^2, \quad u \in D(A_B),
\]

for \( s \in [0, m] \) (Sobolev norms); they all require the validity of a weaker inequality

\[
(0.2) \quad \text{Re}(A u, u) \geq -c \| u \|_m^2, \quad u \in D(A_B),
\]

which we call weak semiboundedness. It was shown in [6] how, for the case where \( A \) is scalar and elliptic, (0.1) with \( s = m \) (Gårding's inequality) is characterized by two conditions on \( A, B \): (i) a condition on the full operators \( B \) and \( A \) at \( \Gamma \), necessary and sufficient for (0.2); (ii) a condition on the principal symbols of \( A \) and \( B \), related to the condition by Agmon [1].
The present paper is devoted to a thorough study of (0.2) and its analogue for systems of « mixed order », without any _a priori_ assumptions on A; e.g. A may degenerate at \( \Gamma \). The results and notations will be applied to elliptic systems in [8], where we treat (0.1) and other properties along the lines of [5], [6].

In Chapter 1 we introduce notations, and set up a Green's formula and the « halfways » Green's formulae associating A with sesquilinear forms. Furthermore we define normal boundary conditions; here a class of triangular systems of differential operators on \( \Gamma \) play a central rôle.

In Chapter 2, (0.2) is characterized by an explicit condition on A and B, and it is proved that (0.2) is necessary and sufficient for the existence of a sesquilinear form \( a(u, \nu) \) on \( H^n(\Omega) \times H^n(\bar{\Omega}) \), for which

\[
(Au, \nu) = a(u, \nu), \quad \text{all } u, \nu \in D(A_a);
\]

(Theorem 2.4), this determines the boundary problems entering in variational theory. A number of alternative explicit conditions for (0.2) are given, in particular for the case where \( \Gamma \) is noncharacteristic for A; these will be of use in [8]. They are finally used to show that when the « total number of boundary conditions » equals \( mq \), then (0.2) holds precisely when the space of Dirichlet data for \( A_a \) equals the space of Dirichlet data for the formally adjoint realization \( A'_b \); and in that case \( A'_b \) is also weakly semibounded.

Chapter 3 treats the systems \( A = (A_{st})_{s,t=1,\ldots,q} \), where \( A_{st} \) is of order \( m_s + m_t \); \( \{m_1, \ldots, m_q\} \) denoting a set of not necessarily equal nonnegative integers. Let

\[
m = \max \{m_1, \ldots, m_q\}, \quad \text{and} \quad \bar{m} = m_1 + \cdots + m_q.
\]

For such systems, a workable definition of Green's formula and of normal boundary conditions does not seem to have been available (cf. [11, p. 241]), the trouble being, roughly speaking, that there are \( \bar{m} + mq \) Cauchy data, on which one usually wants to impose \( \bar{m} \) boundary conditions (less than half). We here present such definitions, and proceed to characterize the analogue of (0.2):

(0.3) \( \text{Re}(Au, u) \geq -c(\|u_1\|_{m_1}^2 + \cdots + \|u_q\|_{m_q}^2) \)
Weakenly semibounded boundary problems. The whole discussion in Chapter 2 is shown to generalize to these systems. (In particular, this determines the normal boundary problems to which de Figueiredo [9] can be applied in the study of coerciveness.)

As an extra benefit we find a Green's formula

\[(Au, \nu) - (u, A'\nu) = \langle xu, i\nu \rangle - \langle \beta^0 u, x'\nu \rangle\]

(valid for all smooth \(u\) and \(\nu\), where \(\beta^0 u\) consists of the \(m\) Dirichlet data of \(u\), and where, when \(\Gamma\) is noncharacteristic for \(A\), \(x\) and \(x'\) are surjective trace operators each consisting of \(m\) more data. Boundary conditions for \(A\) that can be expressed by differential operators on \(\beta^0 u\) and \(xu\) (the « reduced Cauchy data ») can be treated much like the 2\(m\)-order case. (For instance, it is possible to extend techniques of [11] and of [10] to such boundary conditions.) We show that the normal boundary conditions for which (0.3) holds, i.e. all normal boundary conditions arising in connection with sesquilinear forms, are indeed differential boundary conditions on \(\{\beta^0 u, xu\}\).

The author is grateful to G. Geymonat for having called our attention to the above systems of mixed order.

Plan of the paper.

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Chapter 3. — Systems of type \((m_\nu - m_\tau)_{\nu,\tau = 1, \ldots, q}\)
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3.3. Discussion of (3.26).
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Bibliography.
CHAPTER 1
NOTATIONS AND PRELIMINARIES

1.1. Green's formula.

Let \( \Omega \) be an \( n \)-dimensional compact riemannian manifold with boundary \( \Gamma \) and interior \( \Omega = \Omega \setminus \Gamma \). Let \( E \) be a \( \mathcal{C}^\infty \) complex hermitian vector bundle over \( \Omega \) with fiber dimension \( q \geq 1 \). Then the spaces of square integrable sections \( L^2(E), L^2(E|\Gamma) \), and the Sobolev spaces \( H^s(E), H^s(E|\Gamma) \) \( (s \in \mathbb{R}) \), \( H^s_0(E) \) \( (s \geq 0) \) may be defined (cf. e.g. [11]), and we denote inner products over \( \Omega \) by \( (\cdot,\cdot) \) and inner products (and dualities) over \( \Gamma \) by \( \langle \cdot,\cdot \rangle \). The space of \( \mathcal{C}^\infty \) sections with compact support in \( \Omega \) will be denoted \( \mathcal{C}_0^\infty(E) \).

We now introduce trace operators etc., essentially following Hörmander [10, p. 192-193]. Assume, as we may, that \( \Omega \) is imbedded in an \( n \)-dimensional riemannian manifold \( \Sigma \) without boundary, so that \( E \) is the restriction of a vector bundle over \( \Sigma \) on \( \Sigma \). Moreover, let \( n(x) \) denote the vector field consisting of the unit tangent vectors to the geodesics normal to \( \Gamma \) and oriented towards \( \Omega \); it is defined in a neighbourhood \( \Sigma_{t,\varepsilon} \) of \( \Gamma \) consisting of the points in \( \Sigma \) with geodesic distance \( -\varepsilon < t < \varepsilon \) from \( \Gamma, \varepsilon \) sufficiently small. Then one may choose a first order differential operator \( D_n \) in \( E \) whose symbol equals \( n(x), \xi \) for \( x \in \Sigma_{\varepsilon} \) \( (\xi \in \text{the cotangent bundle } T^*(\Sigma)) \); the so-called normal derivative. We then define the trace operators

\[
\gamma_k : u \mapsto (D_k u)|\Gamma, \quad k = 0, 1, 2, \ldots .
\]

(\textsuperscript{1}) When \( \Omega \) is an \( n \)-dimensional \( \mathcal{C}^\infty \) manifold, it may always be provided with an appropriate riemannian structure; we assume this has been done on beforehand, since we want to include the case \( \Omega \subseteq \mathbb{R}^n \). The compactness of \( \Omega \) is not used in any essential way; all estimates are local.
for \( u \in C^\infty(\tilde{E}) \) or \( C^\infty(E) \); recall that \( \gamma_k \) is continuous from \( H^s(E) \) into \( H^{r-k-\frac{1}{2}}(E|_\Gamma) \) for all \( s > k + \frac{1}{2} \) (cf. e.g. [11]).

First order differential operators \( P \) in \( \tilde{E} \) with principal symbol \( a(x) \cdot \xi \) satisfying \( a(x) \perp n(x) \) for \( x \in \Sigma_\varepsilon \), can be said to act along the parallel surfaces \( \Gamma_t \) of \( \Gamma \) (\( \Gamma_t \) consisting of the points in \( \Sigma \) with geodesic distance \( t \) from \( \Gamma \); \( t \in ]-\varepsilon, \varepsilon[ \), since for such operators, \( P\varphi|_{\Gamma_t} \) is independent of the choice of the extension \( \varphi \in C^\infty(\tilde{E}) \) of \( \varphi \in C^\infty(E|_{\Gamma_t}) \). We then denote \( P\varphi|_{\Gamma_t} = P\varphi \). Higher order operators acting along the \( \Gamma_t \) are obtained as sums of products of first order operators acting along \( \Gamma_t \).

For \( f \in C^\infty(\tilde{E}) \) denote by \( f^0 \) the section that equals \( f \) over \( \tilde{\Omega} \) and equals 0 over \( \Sigma \setminus \tilde{\Omega} \). Let \( \delta \) denote the distribution \( f \mapsto \int_{\Gamma_t} \gamma_0 f d\sigma \). Then one has the formulae

\[
(1.1) \quad D_n(f^0) = (D_n f)^0 - i\gamma_0 f \delta, \quad \text{and} \quad P(f^0) = (P f)^0,
\]

for \( f \in C^\infty(\tilde{E}) \), when \( P \) acts along the \( \Gamma_t \), \( |t| < \varepsilon \). We shall mainly use these formulae on the following forms:

\[
(1.2) \quad (D_n u, \nu) - (u, D_n \nu) = i \langle \gamma_0 u, \gamma_0 \nu \rangle, \quad u, \nu \in C^\infty(E),
\]

where \( D'_n \) is the formal adjoint of \( D_n \); note that \( D_n - D'_n \) is of order zero (for the symbol of \( D_n \) is real).

\[
(1.3) \quad (P u, \nu) - (u, P' \nu) = 0, \quad u, \nu \in C^\infty(E),
\]

where the formal adjoint \( P' \) of \( P \) again acts along the \( \Gamma_t \).

Let \( A \) be a \( C^\infty \) differential operator in \( E \) of order \( r > 0 \). In \( \Sigma_\varepsilon \) it may be decomposed uniquely

\[
(1.4) \quad A = \sum_{l=0}^{r} A_l D_n^l
\]

where the \( A_l \) are differential operators of order \( r - l \) acting along the \( \Gamma_t \), \( |t| < \varepsilon \); this is seen e.g. by induction from the first order case. Note that \( A_r \) is of order 0, so is locally multiplication with a \( q \times q \)-matrix; globally it may be viewed as a vector bundle morphism in \( E \). We shall identify zero order differential operators with morphisms in this way.
throughout the paper. Clearly, one has

**Remark 1.1.** — $\Gamma$ is non-characteristic for $A$ at a point $x \in \Gamma$ if and only if $A_r(x)$ is bijective.

Let $M$ be the set of integers

\[(1.5) \quad M = \{0, 1, \ldots, r - 1\},\]
then the Cauchy boundary operator $p$ for $A$ is defined as

\[(1.6) \quad p = \{\gamma_0, \ldots, \gamma_{r-1}\} = \{\gamma_k\}_{k \in M},\]
usually considered as a column vector. With $A'$ denoting the formal adjoint of $A$, we have Green's formula

**Lemma 1.2.** — For all $u$ and $\nu \in H^r(E)$,

\[(1.7) \quad (Au, \nu) - (u, A'\nu) = \langle \alpha p u, \rho \nu \rangle\]

where $\alpha = (\alpha_{jk})_{j, k \in M}$ is a system of differential operators $\alpha_{jk}$ in $E|\Gamma$ of orders $r - j - k - 1$, with

$$\alpha_{jk} = iA_{j+k+1} + \text{lower order operator}$$

for $r - j - k - 1 \geq 0$, and $\alpha_{jk} = 0$ for $r - j - k - 1 < 0$.

**Proof.** — It follows from (1.1) that for each $l$, each $f \in C^\infty(E)$

$$A_l D_n(f^0) = (A_l D_n f^0) - iA_l \sum_{k=0}^{l-1} D_n^{i-k}(\gamma_k f \delta),$$
i.e., with $u = f|_{\Gamma}$, $\nu \in C^\infty(E)$,

$$(u, (A_l D_n')\nu) = (A_l D_n' u, \nu) - i \sum_{k=0}^{l-1} \langle \gamma_k u, \gamma_0 (D_n')^{i-1-k} A_i \nu \rangle.$$

This gives

$$(A_l D_n' u, \nu) - (u, (A_l D_n')\nu) = i \sum_{k=0}^{l-1} \langle (A_l + S_{ik}) \gamma_k u, \gamma_{l-1-k} \nu \rangle,$$

where the $S_{ik}$ are differential operators of order $< r - l$ in $E|\Gamma$, stemming from commutation and taking adjoints. Collecting the terms we obtain (1.7).

$\alpha$ is of type $(-k, -r + 1 + j)_{j, k \in M}$ in the terminology of Hörmander [10, p. 135] (which we shall use throughout); i.e. it is continuous from $\prod_{k \in M} H^{s-k}(E|\Gamma)$ into $\prod_{j \in M} H^{s-r+1+j}(E|\Gamma)$ for
all \( \alpha \in \mathbb{R} \). Note that it is skew-triangular, the entries in the second diagonal being equal to the zero order operator \( A^0 \). By Remark 1.1, \( \alpha \) is thus invertible if and only if \( \Gamma \) is non-characteristic for \( A \); \( \alpha^{-1} \) is then also a skew-triangular system of differential operators.

1.2. The even order case; sesquilinear forms.

In this section we assume \( r = 2m \) (\( m \) integer \( \geq 1 \)), and establish an alternative version of (1.7), and the « halfways » Green’s formulae.

Define now the subsets \( M_0 \) and \( M_1 \) of
\[
M = \{0, 1, \ldots, 2m - 1\}
\]
by
\[
(1.8) \quad M_0 = \{0, \ldots, m - 1\}, \quad M_1 = \{m, \ldots, 2m - 1\},
\]
so \( M = M_0 \cup M_1 \).

The Cauchy boundary operator is split into the Dirichlet and the Neumann boundary operators \( \gamma \) and \( \nu \)
\[
(1.9) \quad \gamma = \{\gamma_k\}_{k \in M_0}, \quad \nu = \{\nu_k\}_{k \in M_1}, \quad \text{so} \quad \rho = \{\gamma, \nu\}.
\]
The matrix \( \alpha \) is split in four blocks
\[
(1.10) \quad \alpha = \begin{pmatrix} \alpha^{00} & \alpha^{01} \\ \alpha^{10} & 0 \end{pmatrix},
\]
where \( \alpha^{00} = (\alpha_{jk})_{j \in M_0, k \in M_1} \), clearly \( \alpha^{11} = 0 \). Then (1.7) takes the form
\[
(1.11) \quad (Au, \nu) - (u, A^0 \nu) = \langle \alpha^{00} \gamma u, \gamma \nu \rangle + \langle \alpha^{01} u, \gamma \nu \rangle + \langle \alpha^{10} \gamma u, \nu \nu \rangle,
\]
for \( u, \nu \in H^m(E) \).

**Definition 1.3.** — By a sesquilinear form \( a(u, \nu) \) on \( H^m(E) \) we shall understand an integro-differential form
\[
(1.12) \quad a(u, \nu) = \sum_{i \in I} (Q_i u, P_i \nu),
\]
where the \( Q_i \) and \( P_i \) are \( C^\infty \) differential operators in \( E \) of orders \( \leq m \), indexed by a finite index set \( I \); \( a(u, \nu) \) is defined and continuous for \( \{u, \nu\} \in H^m(E) \times H^m(E) \).
a(u, \nu) \text{ is said to be associated with } A \text{ if}

\begin{equation}
(1.13) \quad a(u, \nu) = (Au, \nu), \quad \text{for all } u, \nu \in C_0^\infty(E),
\end{equation}

i.e., if $A = \sum_{i \in I} P_i'Q_i$.

When $\Omega \subset \mathbb{R}^n$, (1.12) may of course be written in the usual way:

$$a(u, \nu) = \sum_{|\alpha|, |\beta|} a_{\alpha\beta} \partial^\alpha u \partial^\beta \nu.$$

**Lemma 1.4.** — Let $a(u, \nu)$ be a sesquilinear form on $H^m(E)$ associated with $A$. Then for all $u \in H^2m(E)$, $\nu \in H^m(E)$,

\begin{equation}
(1.14) \quad (Au, \nu) = a(u, \nu) + \langle \mathcal{A}^{01}u, \gamma\nu \rangle + \langle \mathcal{F}\gamma u, \gamma\nu \rangle
\end{equation}

where $\mathcal{F}$ is an $m \times m$-system of differential operators in $E|_\Gamma$, of type $(-k, -2m + 1 + j)_{j, k \in M_*}$.

**Proof.** — Applying Green's formula (1.7) to each $P_i'$ and (1.4) to each $Q_i$, we find

$$\langle Au, \nu \rangle - a(u, \nu) = \sum_{i \in I} \left[ (P_i'Q_i u, \nu) - (Q_i u, P_i\nu) \right]$$

$$= \sum_{i \in I} \langle P_i'Q_i u, \gamma\nu \rangle$$

$$= \langle \mathcal{R}_1u, \gamma\nu \rangle = \langle \mathcal{R}_1\gamma u, \nu\gamma \rangle + \langle \mathcal{R}_2\gamma u, \nu\gamma \rangle$$

where $\mathcal{R}_1$ is of type $(-k, -2m + 1 + j)_{j, k \in M_*}$ and $\mathcal{R}_2$ is of type $(-k, -2m + 1 + j)_{j, k \in M_*}$. In a similar way

$$\langle u, A'\nu \rangle - a(u, \nu) = \langle \gamma u, \mathcal{R}_3\nu \rangle + \langle \gamma u, \mathcal{R}_4\nu \rangle.$$

For any given $\varphi, \psi \in C^\infty(E|_\Gamma)^n$ there exist $u, \nu \in C_0^\infty(E)$ with $\nu u = \varphi$, $\gamma u = 0$, $\gamma \nu = \psi$. Inserting these, we get, by comparison with (1.11)

$$\langle \mathcal{R}_1\varphi, \psi \rangle = (Au, \nu) - (u, A'\nu) = \langle \mathcal{A}^{01}\varphi, \psi \rangle,$$

whence $\mathcal{R}_1 = \mathcal{A}^{01}$.

We shall now show that the operator $\mathcal{F}$ in (1.14) can take any value.

**Lemma 1.5.** — Let $\mathcal{F}$ be a first order differential operator in $E|_\Gamma$. Then there exists a sesquilinear form $s(u, \nu)$ on $H^1(E)$ such that

\begin{equation}
(1.15) \quad s(u, \nu) = \langle \mathcal{F}\gamma_0 u, \gamma_0\nu \rangle \quad \text{for } u, \nu \in H^1(E):\end{equation}
for any such form the operator \( S \) in \( E \) associated with \( s(u, v) \) is zero.

Proof. — Let \( \tilde{\mathcal{D}} \) be a first order operator in \( E \) that acts along the \( \Gamma_1 \) for \( |t| < \varepsilon \), so that \( \tilde{\mathcal{D}} \) acts like \( \mathcal{D} \) on \( \Gamma = \Gamma_0 \). Then by (1.2) and (1.3),

\[
\langle \mathcal{D} \gamma_0 u, \gamma_0 \nu \rangle = \langle \gamma_0 \tilde{\mathcal{D}} u, \gamma_0 \nu \rangle \\
= -i(D_n \tilde{\mathcal{D}} u, \nu) + i(\tilde{\mathcal{D}} u, D_n' \nu) \\
= -i(\tilde{\mathcal{D}} D_n u, \nu) - i([D_n, \tilde{\mathcal{D}}] u, \nu) + i(\tilde{\mathcal{D}} u, D_n' \nu) \\
= (iD_n u, \tilde{\mathcal{D}}' \nu - i[D_n, \tilde{\mathcal{D}}] u, \nu) + (i \tilde{\mathcal{D}} u, D_n' \nu),
\]

which is a sesquilinear form on \( H^1(E) \), since the commutator \([D_n, \tilde{\mathcal{D}}] = D_n \tilde{\mathcal{D}} - \tilde{\mathcal{D}} D_n \) is of first order. Since any form \( s(u, \nu) \) satisfying (1.15) vanishes for \( u \in C_0^\infty(E) \), the associated operator \( S \) in \( E \) is zero.

Proposition 1.6. — Let \( \mathcal{S} = (\mathcal{S}_{jk})_{j, k \in M_0} \) be a system of differential operators in \( E|_\Gamma \), of type \((-k, -2m + 1 + j))_{j, k \in M_0} \). Then there exists a sesquilinear form \( s(u, \nu) \) on \( H^m(E) \) so that

\[
(1.16) \quad s(u, \nu) = \langle \mathcal{S} \gamma_0 u, \gamma_\nu \rangle \quad \text{for} \quad u, \nu \in H^m(E),
\]

and the associated operator \( S \) in \( E \) is zero.

Proof. — The proof is reduced to the preceding case as follows:

Let \( \{j, k\} \in M_0 \times M_0 \). \( \mathcal{S}_{jk} \) is of order \( 2m - 1 - j - k \) and it may be written as a finite sum

\[
\mathcal{S}_{jk} = \sum_{i \in I} P_i Q_i R_i,
\]

where the \( P_i \) are of order \( m - 1 - j \), \( Q_i \) of order \( 1 \), \( R_i \) of order \( m - 1 - k \). Now, with notation as in the preceding proof,

\[
\langle \mathcal{S}_{jk} \gamma_0 u, \gamma_\nu \rangle = \sum_{i \in I} \langle P_i Q_i R_i \gamma_0 u, \gamma_\nu \rangle \\
= \sum_{i \in I} \langle Q_i R_i \gamma_0 u, P_i' \gamma_\nu \rangle \\
= \sum_{i \in I} \langle Q_i \gamma_0 R_i \gamma_0 \nu, \gamma_0 P_i' D_n' \nu \rangle,
\]
where $\hat{R}_n^k$ and $\hat{P}_n^k$ are differential operators in $E$ of order $m - 1$.

**Corollary 1.7.** — For any system $\mathcal{S}$ of type
\[( - k, - 2m + 1 + j)_{j, k \in \mathbb{N}_0},\]
of differential operators in $E|_{\Gamma}$, there exists a sesquilinear form $a(u, v)$ on $H^m(E)$ such that
\[(1.17)\quad (A u, v) = a(u, v) + \langle A_0^1 u, \gamma u \rangle + \langle \mathcal{S} \gamma u, \gamma v \rangle,
\]
for all $u \in H^{2m}(E), v \in H^m(E)$.

**Remark 1.8.** — The results of this section generalize immediately to the following situation: Let $A$ be of order $r = s + t$, $s$ and $t$ nonnegative integers. Let $M_{ot} = \{0, \ldots, t - 1\}$, $M_{1i} = \{t, \ldots, s + t - 1\}$, $M_{0s} = \{0, \ldots, s - 1\}$, $M_{1s} = \{s, \ldots, s + t - 1\}$, and set
\[\rho_{0s} = \{\gamma_k\}_{k \in \mathbb{N}_0}, \quad A^{0s,1t} = (A_{ik})_{i \in M_{ot}, k \in M_{1t}},\]
etc. Then (1.7) may be written
\[(Au, \nu) - (u, A' \nu) = \langle A_{0s}^{0t} \rho_{0t} u, \rho_{0s} \nu \rangle + \langle A_{1t}^{s1t} \rho_{1t} u, \rho_{0s} \nu \rangle + \langle A_{1t}^{s0t} \rho_{0t} u, \rho_{1t} \nu \rangle.
\]
By a sesquilinear form on $H^r(E) \times H^r(E)$ we understand an expression (1.12) where the $Q_i$ are of order $\leq t$ and the $P_i$ are of order $\leq s$, it is associated with $A$ when (1.13) holds. One finds that for such forms,
\[(1.18)\quad (Au, \nu) = a(u, \nu) + \langle A_{0s}^{0t} \rho_{0t} u, \rho_{0s} \nu \rangle + \langle \mathcal{S} \rho_{0t} u, \rho_{0s} \nu \rangle
\]
for all $u \in H^r(E), \nu \in H^r(E)$; where $\mathcal{S}$ can be any $s \times t$-system of differential operators in $E|_{\Gamma}$, of type
\[( - k, - r + 1 + j)_{j \in \mathbb{N}_0, k \in \mathbb{N}_0}.
\]
Note that $A_{0s}^{0t}$ is a quadratic submatrix of $A$, its second diagonal being contained in the second diagonal of $A$; so it is invertible if and only if $\Gamma$ is non-characteristic for $A$. 

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1.3. Triangular differential operators.

In this section we study a class of differential operators that are fundamental in our treatment of boundary conditions.

Let $N$ be a finite subset of $\mathbb{N} \cup \{0\}$, the non-negative integers; the number of elements in $N$ is denoted $|N|$. For each $j \in N$ there are given two hermitian vector bundles $F_j$ and $E_j$ over $\Gamma$, $F_j$ of fiber dimension $p_j \geq 0$ and $E_j$ of fiber dimension $q_j \geq 0$. (We shall use some elementary facts about vector bundles, for which we refer e.g. to Atiyah [3].) For each pair $(j, k) \in N \times N$ there is given a differential operator $B_{jk}$ from $E_k$ into $F_j$, of order $j - k$; the convention that differential operators of negative order are zero is used throughout; of course $B_{jk}$ is also zero if $p_j$ or $q_k$ is 0. The $B_{jk}$ form a matrix (or system) of differential operators $B = (B_{jk})_{j, k \in N}$ of type $(k, j)_{j, k \in N}$, i.e. $B$ is continuous from $\prod_{k \in M} H^{z-k}(E_k)$ to $\prod_{j \in M} H^{z-j}(F_j)$ for all $z \in \mathbb{R}$.

$B$ is triangular, since $B_{jk} = 0$ for $j < k$. We define its diagonal part $B_d$ and its subtriangular part $B_s$ by

\[(1.19)\quad B_d = (\delta_{jk} B_{jk})_{j, k \in N}, \quad B_s = B - B_d;\]

a matrix will be said to be subtriangular when the diagonal and all elements to one side of it are zero. The elements $B_{kk}$ in the diagonal are differential operators of order 0, so they may, as previously remarked, be regarded as vector bundle morphisms (from $E_k$ to $F_k$); $B_d$ is also a morphism, from $\bigoplus_{k \in N} E_k$ into $\bigoplus_{k \in N} F_k$.

**Proposition 1.9.** — Assume that $B_d$ is surjective (so in particular, $p_k \leq q_k$, all $k \in N$). Then the morphism

\[(1.20)\quad C_d = B_d^*(B_d B_d^*)^{-1}\]

is a right inverse of $B_d$. Moreover, the differential operator

\[(1.21)\quad C = C_d \sum_{k=0}^{|N|} (-B_s C_d)^k\]
is a right inverse of $B$; it is a system $C = (C_k)_{k \in \mathbb{N}}$ of type $(-k, -f)_{k \in \mathbb{N}}$, the $C_k$ being differential operators from $F_k$ into $E_f$. In particular, $B$ is surjective from $\prod_{k \in \mathbb{N}} H^{\alpha+k}(E_k)$ to $\prod_{j \in \mathbb{N}} H^{\alpha-j}(F_j)$ for all $\alpha \in \mathbb{R}$.

Proof. — The first statement follows from the corresponding statement for vector spaces. (Since $B_d$ is a diagonal matrix, one actually treats each $B_{kk}$ separately, and $C_d$ is a diagonal matrix with $C_{kk} = B_{kk}^*(B_{kk}B_{kk}^*)^{-1}$.) Now observe that $B_d C_d$ is a subtriangular differential operator in $\bigoplus_{f \in \mathbb{N}} F_f$ since $B_d$ is subtriangular and $C_d$ is diagonal. Thus $B_d C_d$ is nilpotent, its $|\mathbb{N}|$-th power being zero. Defining $C$ by (1.21), we then have

$$BC = (B_d + B_f)(C_d - C_d B_d C_d + C_d B_d C_d - ...)$$
$$= B_d C_d - B_d C_d B_d C_d + B_d C_d B_d C_d - ...$$
$$+ B_d C_d - B_d C_d B_d C_d + ...$$

$$= I,$$

since $B_d C_d = I$ (the identity in $\bigoplus_{f \in \mathbb{N}} F_f$). Clearly $C$ is a system of the described type; its continuity properties imply the surjectiveness of $B$.

Lemma 1.10. — Assumptions of Proposition 1.9. For each $k \in \mathbb{N}$, the kernel and image of the morphisms $B_{kk}$ resp. $B_{kk}^*$,

$$Z_k = \ker B_{kk} \quad \text{and} \quad R_k = \text{im } B_{kk}^*$$

are orthogonal subbundles of $E_k$, of dimension $q_k - p_k$ resp. $p_k$. Moreover,

$$C_{kk} B_{kk} = B_{kk}^*(B_{kk}B_{kk}^*)^{-1} B_{kk} = B_{kk}^* C_{kk} = (C_{kk}B_{kk})^*,$$

and it is the orthogonal projection of $E_k$ onto $R_k$; and $I - C_{kk}B_{kk}$ is the orthogonal projection of $E_k$ onto $Z_k$. Altogether, $C_d B_d = B_d^* C_d^*$ and is the orthogonal projection of $\bigoplus_{k \in \mathbb{N}} E_k$ onto $\text{im } B_d^* = \bigoplus_{k \in \mathbb{N}} R_k$, and $I - C_d B_d$ is the orthogonal projection of $\bigoplus_{k \in \mathbb{N}} E_k$ onto $\ker B_d = \bigoplus_{k \in \mathbb{N}} Z_k$. 


Proof. — Follows from the corresponding evident statements for vector spaces. (I denotes the identity in various spaces, its meaning should be clear from the context.)

When $B_d$ is surjective (so $B_d^*$ is injective) it follows from Proposition 1.9 that $B^*$ is injective with left inverse $C^*$. However, we do not have equality between $CB$ and $B^*C^*$ since $CB$ is lower triangular and $B^*C^*$ is upper triangular (unless of course $(CB)_z = 0$). We may define $\tilde{C} = B^*(BB^*)^{-1}$, which does satisfy $\tilde{CB} = B^*\tilde{C}^*$ in better analogy with Lemma 1.10. But $\tilde{C}$ will usually not be a differential operator but a \textit{pseudo-differential} operator, and not triangular, so we prefer to work with $C$ as right inverse of $B$. Note that the subtriangular part of $C$ is $C - C_d = C_d \sum_{k=1}^{[N]} (-B_zC_d)^k$.

We introduce the spaces (for $\alpha \in \mathbb{R}$):

\begin{align}
Z^\alpha(B) &= \left\{ \varphi \in \prod_{k \in \mathbb{N}} H^{\alpha - k - \frac{1}{2}}_z(E_k) | B\varphi = 0 \right\}; \\
Z(B) &= \bigcup_{\alpha \in \mathbb{R}} Z^\alpha(B); \\
R^\alpha(I - CB) &= (I - CB) \prod_{k \in \mathbb{N}} H^{\alpha - k - \frac{1}{2}}(E_k); \\
R(I - CB) &= \bigcup_{\alpha \in \mathbb{R}} R^\alpha(I - CB); \\
R^\alpha(B^*) &= B^* \prod_{j \in \mathbb{N}} H^{\alpha - j - \frac{1}{2}}(F_j); \\
R(B^*) &= \bigcup_{\alpha \in \mathbb{R}} R^\alpha(B^*); \\
Z^\alpha(I - B^*C^*) &= \left\{ \varphi \in \prod_{j \in \mathbb{N}} H^{\alpha - j - \frac{1}{2}}(F_j) | \varphi - C^*B^*\varphi = 0 \right\}; \\
Z(I - B^*C^*) &= \bigcup_{\alpha \in \mathbb{R}} Z^\alpha(I - B^*C^*). \tag{1.27}
\end{align}

These definitions apply similarly to $B_d$ and $C_d$, viewed as differential operators. We also have, with the notation (1.22)

\begin{align}
Z^\alpha(B_d) &= i_z \prod_{k \in \mathbb{N}} H^{\alpha - k - \frac{1}{2}}(Z_k), \\
Z(B_d) &= i_z \mathcal{D}' \left( \bigoplus_{k \in \mathbb{N}} Z_k \right), \\
R^\alpha(B_d^*) &= i_R \prod_{k \in \mathbb{N}} H^{\alpha - k - \frac{1}{2}}(R_k), \\
R(B_d^*) &= i_R \mathcal{D}' \left( \bigoplus_{k \in \mathbb{N}} R_k \right), \tag{1.29}
\end{align}

where $i_z$ and $i_R$ denote the injections $\bigoplus Z_k \subset \bigoplus E_k$ resp. $\bigoplus R_k \subset \bigoplus E_k$ (they may be omitted in less precise statements).
**Lemma 1.11.** — Assumptions of Proposition 1.9. For any \( \alpha \in \mathbb{R} \),

\begin{align*}
\tag{1.30} Z^\alpha(B) &= R^\alpha(I - CB), \\
\tag{1.31} R^\alpha(B^*) &= Z^\alpha(I - B^*C^*), \\
Z(B) &= R(I - CB), \\
R(B^*) &= Z(I - B^*C^*).
\end{align*}

**Proof.** — When \( B\varphi = 0, \varphi = \varphi - CB\varphi \). When \( \varphi = (I - CB)\psi, \ B\varphi = B\psi - BCB\psi = B\psi - B\psi = 0 \). This proves (1.30).

When \( \varphi = B^*\psi \),

\[(I - B^*C^*)\varphi = B^*\psi - B^*C^*B^*\psi = B^*\psi - B^*\psi = 0.\]

When \( (I - B^*C^*)\varphi = 0, \varphi = B^*(C^*\varphi) \). This proves (1.31).

**Lemma 1.12.** — Assumptions of Proposition 1.9. \( I - CB \) and \( I + C_dB_z \) are each others inverses. Moreover, for any \( \alpha \in \mathbb{R} \)

\begin{align*}
\tag{1.32} Z^\alpha(B) &= (I - CB_d)Z^\alpha(B_d), \\
Z(B) &= (I - CB_d)Z(B_d), \\
R^\alpha(B^*) &= (I + B^*_dC^*_d)R^\alpha(B^*_d), \\
R(B^*) &= (I + B^*_dC^*_d)R(B^*_d).
\end{align*}

**Proof.** — Since \( C_dB_z \) is subtriangular, \( I + C_dB_z \) has the inverse

\[ I - C_dB_z + (C_dB_z)^2 - \cdots = I - C_d \sum_{k=0}^{\lfloor N \rfloor} (-B_dC_d)^kB_z = I - CB_z, \]

cf. (1.21). Now

\[ B_d = B - B_z = B - BCB_z = B(I - CB_z), \]
\[ B = B_d + B_z = B_d + B_dC_dB_z = B_d(I + C_dB_z), \]

from which (1.32) immediately follows. (1.33) follows from the adjoint identities

\[ B^*_d = (I - B^*_dC^*_d)B^*, \quad B^* = (I + B^*_dC^*_d)B^*_d. \]

Combining this lemma with (1.28) and (1.29) we see how \( Z^\alpha(B) \) and \( R^\alpha(B^*) \) may be « parametrized » by full Sobolev spaces over bundles:

\begin{align*}
\tag{1.34} Z^\alpha(B) &= (I - CB_z)i_Z \sum_{k \in \mathbb{N}} H^{\alpha - k - \frac{1}{2}}(Z_k), \\
\tag{1.35} R^\alpha(B^*) &= (I + B^*_dC^*_d)i_R \sum_{k \in \mathbb{N}} H^{\alpha - k - \frac{1}{2}}(R_k),
\end{align*}
(similar statements for $Z(B)$ and $R(B^*)$), where $(I - C B_s) i_z$
and $(I + B_s C_s^*) i_n$ are injective differential operators. Note
that $R^s(B^*)$ may also be parametrized by

$$(1.36) \quad R^s(B^*) = B^* \sum_{j \in \mathbb{N}} H^{a-j-\frac{1}{2}}(F_j),$$

where $B^*$ is injective.

Surjectiveness of $B$ does not in general imply surjectiveness
of $B_d$. However, it does so in a special case:

**Lemma 1.13.** — Assume that $F_k = E_k$ for all $k \in \mathbb{N}$. Then
if $B$ is surjective from $\prod_{k \in \mathbb{N}} H^{a-k}(E_k)$ to $\prod_{j \in \mathbb{N}} H^{a-j}(E_j)$ for some
$a \in \mathbb{R}$, the diagonal part $B_d$ is an isomorphism (so $B$
is bijective for all $a$).

**Proof.** — We have

$$N = \{k_1, \ldots, k_p\} \quad \text{(where } 0 \leq k_1 < \ldots < k_p).$$

For $1 \leq q \leq p$ we denote by $B^q$ the submatrix of $B$
$(B^q)_{jk}, k \in \{k_1, \ldots, k_p\}$. Since $B$ is lower triangular and surjective,
all the submatrices $B^q$ are surjective. In particular, $B^1 = B_{k_1k_1}$
is a surjective morphism from $E_{k_1}$ to $E_{k_1}$, thus an isomor-
phism. We proceed by induction: Assume that $B^{q-1}$ has
bijective diagonal part; by Proposition 1.9, $B^{q-1}$ is bijective.
Let $\psi = \{0, \ldots, 0, \psi_{k_q} \}$ $(q$ elements), $\psi_{k_q} \in H^{a-k_q}(E_{k_q})$. Since
$B^q$ is surjective, there exists

$$\varphi^q = \{\varphi_{k_1}, \ldots, \varphi_{k_q}\} \in \prod_{r=1}^q H^{a-k_r}(E_{k_r}),$$

for which $B^q \varphi^q = \psi$. But since $B^{q-1}$ is injective,

$$\varphi_{k_1} = \cdots = \varphi_{k_{q-1}} = 0.$$

Then $\psi_{k_q} = B_{k_qk_q} \varphi_{k_q}$. This proves that $B_{k_qk_q}$ is a surjective
morphism from $E_{k_q}$ to $E_{k_q}$, and thus bijective. So $B^q$ has
bijective diagonal part.

**Remark 1.14.** — All calculations generalize immediately to
systems $B$, where the $B_{jk}$ with $j > k$ are pseudo-differential
operators of orders $j - k$, but where we still have that
the $B_{kk}$ are morphisms and the $B_{jk}$ with $j < k$ are zero.
1.4. Normal boundary conditions.

We shall now define the boundary value problems to be studied in Chapter 2: Let $\Lambda$ and $E$ be as in section 1.1. For each $j \in M = \{0, \ldots, r - 1\}$ there is given a hermitian bundle $F_j$ over $\Gamma$ of dimension $p_j > 0$. There is given a matrix $B = (B_{jk})_{j,k \in M}$ of differential operators $B_{jk}$ from $E|_\Gamma$ to $F_j$, of type $(-k, -j)_{j,k \in M}$ (as in section 1.3). Then $B$ defines the homogeneous boundary condition

$$B \rho u = 0 \tag{1.37}$$

or, equivalently: $\sum_{k \leq j} B_{jk} \gamma_k u = 0$ for all $j \in M$. We shall study the boundary value problem

$$Au = f, \quad B \rho u = 0,$$

or rather the realization $A_B$ of $A$ defined by

$$A_B : u \mapsto Au, \quad D(A_B) = \{u \in H^r(E) | B \rho u = 0\}. \tag{1.38}$$

The systems of boundary conditions usually studied can be put in the form (1.37); we have just grouped together the conditions of the same normal order (like Seeley [12]) and permitted the range space for each normal order $j$ to be a nontrivial bundle. Moreover, we have included zero bundles as ranges (those where $p_j = 0$) for convenience, so that we do not have to distinguish between $M$ and the set $J = \{j | p_j > 0\}$ that entered in the announcement of results [7]. For elliptic $\Lambda$ it is usually assumed that $\sum_{j \in M} p_j = mq$; we shall not assume that on beforehand.

**Definition 1.15.** — The boundary condition $B \rho u = 0$ — or the differential operator $B$ — will be said to be normal when $B_d = (\delta_{jk} B_{jk})_{j,k \in M}$ is a surjective vector bundle morphism. (Then in particular $p_j \leq q$ for all $j \in M$.)

The definition is a vector bundle version of that of Seeley [12] (cf. also Remark 2.2 below). It extends the well-known definition of Aronszajn and Milgram for scalar operators.
Remark 1.16. — Let us compare the present definition of normality with that of Geymonat [4, Definizione 2.2]. He considers the case where A and B are matrices of scalar differential operators (i.e., E and the F_j are trivial bundles), and his definition of normality requires that one can supplement B with \( rq - \sum_{j \in M} p_j \) rows to obtain a system 
\[
\hat{B} = (\hat{B}_{jk})_{j,k \in M}
\]
of \( q \times q \)-matrices, of type \((-k, -j)_{j,k \in M}\) and with bijective diagonal part. In our framework this means exactly that trivial bundles \( G_j \) of dimension \( q - p_j (j \in M) \) may be found, together with morphisms \( P_j : E|_{\Gamma} \to G_j \), such that the morphisms \( \hat{B}_j = B_j \oplus P_j : E|_{\Gamma} \to F_j \oplus G_j \) are isomorphisms. In comparison, the present definition of normality merely requires that the \( B_j \) be surjective (which is satisfied under Geymonat’s requirement); then if we let \( P_{0j} \) denote the orthogonal projections of \( E|_{\Gamma} \) onto \( Z_j = \ker B_j \), the \( \hat{B}_{0j} = B_j \oplus P_{0j} \) are isomorphisms of \( E|_{\Gamma} \) onto \( F_j \oplus Z_j \), for \( j \in M \). When both requirements are satisfied, \( P_{0j} \) defines an isomorphism of \( Z_j \) onto \( G_j \), for \( j \in M \). So, when \( E \) and the \( F_j \) are trivial and the \( B_j \) are surjective, Geymonat’s normality holds \textit{if and only if} the \( Z_j \) are trivial bundles. This is a global condition that will not in general be satisfied for surjective \( B_j \). (Example: Let 
\[
\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\},
\]
let \( E = \Omega \times \mathbb{R}^3 \), and let \( B_{ij} \) be the \( 1 \times 3 \)-matrix \( (x_1, x_2, x_3) \), for \( (x_1, x_2, x_3) \in \Gamma = S^2 \). Then \( \ker B_{ij} \) is the tangent bundle of \( S^2 \), which is nontrivial.) The present definition of normality is local and at the same time more general than Geymonat’s.

The « Lions-Magenes theory » of Geymonat [4] can easily be extended to the present normal boundary condition, on the basis of the Green’s formula

\[
(1.39) \quad (Au, \nu) - (u, A'\nu) = \langle \mathcal{A}(\hat{B}^0)_{-1} \hat{B}^0 \rho u, \rho \nu \rangle
\]

\[
= \langle B_{\rho u}, pr_F(\hat{B}^{0*})_{-1} \mathcal{A}^* \rho \nu \rangle + \langle P_{0\rho u}, pr_Z(\hat{B}^{0*})_{-1} \mathcal{A}^* \rho \nu \rangle,
\]

\( pr_F \) and \( pr_Z \) denoting the projections of \( \bigoplus_{j \in M} (F_j \oplus Z_j) \) onto \( \bigoplus_{j \in M} F_j \) resp. \( \bigoplus_{j \in M} Z_j \).
CHAPTER 2
WEAKLY SEMIBOUNDED REALIZATIONS
OF OPERATORS OF EVEN ORDER


Throughout this chapter we assume (with the notations of Chapter 1):

Assumption 2.1. — A is an arbitrary $C^\infty$ differential operator in $E$ of order $r = 2m$, $m$ integer $> 0$. $A_B$ is the realization defined by a normal boundary condition

(2.1) $B_{\phi} u = 0.$

We shall study the problem of determining those $B$ for which $A_B$ satisfies the inequality

(2.2) $\text{Re} \, e^{i\theta} (Au, u) \leq c \|u\|_s^2, \quad \text{all} \quad u \in D(A_B),$

for some $c > 0$, $\theta \in \mathbb{R}$. The inequality is always satisfied for $u \in C_0^\infty(E)$, so depends essentially on the boundary condition and the behaviour of $A$ at the boundary. However, it will be seen that it depends on the full operators $B$ and $A$ at $\Gamma$, not just on part of (e.g. the principal part of) their symbol. (2.2) is necessary (with $\theta = \pi$) for any of the « coerciveness inequalities »

(2.3) $\text{Re} \, (Au, u) \geq c_s \|u\|_s^2 - c_0 \|u\|_\phi^2, \quad u \in D(A_B),$

$s \in ]0, m], \quad \text{or just semiboundedness}$

(2.4) $\text{Re} \, (Au, u) \geq - c_0 \|u\|_\phi^2, \quad u \in D(A_B).$

These other properties will be treated in [8], under further assumptions on $A$. We shall here concentrate on the special aspects of (2.2), called weak semiboundedness for lack of a better name.

Remark 2.2. — The assumption of normality is partly justified by the observation of Seeley [12] that for elliptic boundary
value problems, Agmon's necessary and sufficient condition for the existence of a ray of minimal growth implies normality. When e.g. (2.4) holds, there are many rays of minimal growth. The investigations given below have led us to believe that, at least when $\Gamma$ is noncharacteristic for $A$ and $\sum_{j \in \mathbb{M}} p_j = mq$, normality is also necessary for (2.2) (cf. Remark 2.21).

Since $r = 2m$, the notations of section 1.2 apply (cf. in particular (1.8), (1.9)). We split $B$ and its right inverse $C$ accordingly:

$$
B = \begin{pmatrix} B_0^0 & 0 \\ B_1^0 & B_1^{11} \end{pmatrix}, \quad B_0^{\delta} = (B_{jk})_{j \in \mathbb{M}_0^0, k \in \mathbb{M}_1} ; \\
C = \begin{pmatrix} C_0^0 & 0 \\ C_1^0 & C_1^{11} \end{pmatrix}, \quad C_0^{\delta} = (C_{jk})_{j \in \mathbb{M}_0^0, k \in \mathbb{M}_1} ;
$$

where $B_0^{11}$ and $C_0^{11}$ are zero since $j < k$ in $M_0 \times M_1$. Note that $C_0^{00}$ and $C_1^{11}$ are the right inverses by Proposition 1.9 to $B_0^{00}$ resp. $B_1^{11}$. The boundary condition (2.1) may now be formulated as

$$
(2.5) \quad B_0^{00} \gamma u = 0, \quad B_1^{10} \gamma u + B_1^{11} \nu u = 0.
$$

It is wellknown that $\varphi$ is surjective from $H^{2m}(E)$ onto $\prod_{k \in \mathbb{M}} H^{2m-k-\frac{3}{2}}(E|_{\Gamma})$. We therefore have two other formulations of (2.5) (recall notations (1.24), (1.25)):

**Lemma 2.3.** — A section $u \in H^{2m}(E)$ is in $D(A_B)$ if and only if $\{\gamma u, \nu u\}$ satisfies either of the equivalent conditions (i), (ii):

(i) $\gamma u \in Z^{2m}(B_0^{00})$, $\nu u + C_1^{10} B_1^{10} \gamma u \in Z^{2m}(B_1^{11})$;

(ii) $\gamma u \in R^{2m}(I - C_0^{00} B_0^{00})$, $\nu u + C_1^{11} B_1^{10} \gamma u \in R^{2m}(I - C_1^{11} B_1^{11})$.

**Proof.** — (i) is equivalent with (2.5) since $B_1^{11} C_1^{11} = I$.

(ii) is equivalent with (i) by Lemma 1.11.

We shall now prove the fundamental result

**Theorem 2.4.** — Let $A$ be a differential operator in $E$ of order $2m$, and $A_B$ the realization defined by a normal boundary
condition (2.5). The following statements (i)-(iv) are equivalent:

(i) There exist \( \theta \in \mathbb{R} \), \( c > 0 \) such that
\[
\text{Re } e^{i\theta}(Au, u) \leq c \|u\|_m^2, \quad \text{all } u \in D(A_B)
\]
(i.e., \( A_B \) is weakly semibounded).

(ii) The following identity holds
\[
(I - C^{00}B^{00})^* \mathcal{A}^{01}(I - C^{11}B^{11}) = 0.
\]

(iii) There exists a sesquilinear form \( a_B(u, \nu) \) on \( H^n(E) \) associated with \( A \), such that
\[
(Au, \nu) = a_B(u, \nu), \quad \text{all } u, \nu \in D(A_B).
\]

(iv) There exists \( c > 0 \) such that
\[
|\langle Au, \nu \rangle| \leq c \|u\|_m \|\nu\|_m, \quad \text{all } u, \nu \in D(A_B).
\]

Proof. — Clearly (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i), since \( a_B(u, \nu) \) is continuous on \( H^n(E) \times H^n(E) \). We shall now show that (i) \( \Rightarrow \) (ii). Let \( a(u, \nu) \) be any sesquilinear form on \( H^n(E) \) associated with \( A \). Then
\[
\langle Au, \nu \rangle = a(u, \nu) + \langle \mathcal{A}^{01}u, \nu \rangle + \langle \mathcal{S} \gamma u, \nu \rangle
\]
for some \( \mathcal{S} \) of type \( (-k, -2m+1+j)_{j,k \in M} \), cf. Lemma 1.4. By Lemma 2.3 we have
\[
\gamma u = (I - C^{00}B^{00}) \varphi_0,
\]
\[
\nu u = (I - C^{11}B^{11}) \varphi_1 - C^{11}B^{10} \gamma u,
\]
where \( \{\varphi_0, \varphi_1\} \) runs through
\[
\prod_{k \in M_0} H^{2m-k-\frac{1}{2}}(E|\Gamma) \times \prod_{k \in M_1} H^{2m-k-\frac{1}{2}}(E|\Gamma),
\]
This gives by insertion
\[
\langle \mathcal{A}^{01}u, \nu \rangle = \langle \mathcal{A}^{01}(I - C^{11}B^{11}) \varphi_1, \nu \rangle - \langle \mathcal{A}^{01}C^{11}B^{10} \gamma u, \nu \rangle,
\]
where also \( \mathcal{A}^{01}C^{11}B^{10} \) is of type \( (-k, -2m+1+j)_{j,k \in M} \). Then \( a(u, \nu), \langle \mathcal{S} \gamma u, \nu \rangle \) and \( \langle \mathcal{A}^{01}C^{11}B^{10} \gamma u, \nu \rangle \) are all continuous on \( H^n(E) \times H^n(E) \), so that (i) is equivalent with
the existence of \( \theta \in \mathbb{R}, c_1 > 0 \) for which

\[
\text{Re} \left( c_0^1(I - C^{11}B^{11}) \varphi_1, \gamma u \right) \leq c_1 \| u \|_m^2, \quad \text{all } u \in D(A_B).
\]

We now observe that for \( \omega \in C^\infty_0(E), u + \omega \in D(A_B) \) and \( \gamma(u + \omega) = \gamma u; \nu(u + \omega) = \nu u \). Then (2.12) implies

\[
\text{Re} \left( c_0^1(I - C^{11}B^{11}) \varphi_1, \gamma u \right) \leq c_1 \inf_{\omega \in C^\infty_0(E)} \| u + \omega \|_m^2
\]

\[
(2.13)
\]

by a well known theorem (cf. e.g. [11]). Inserting \( \gamma u = (I - C^{00}B^{00}) \varphi_0 \) and using the continuity of \( I - C^{00}B^{00} \) we conclude from (2.13)

\[
\text{Re} \left( c_0^1(I - C^{11}B^{11}) \varphi_1, (I - C^{00}B^{00}) \varphi_0 \right) = \text{Re} \left( (I - C^{00}B^{00})^* c_0^1(I - C^{11}B^{11}) \varphi_1, \varphi_0 \right)
\]

\[
\leq c_2 \sum_{k \in M_0} \| \varphi_{0k} \|_{m-k-\frac{1}{2}}^2, \quad \text{all } u \in D(A_B)
\]

valid for all the pairs \( \{ \varphi_0, \varphi_1 \} \). That can only hold if \( (I - C^{00}B^{00})^* c_0^1(I - C^{11}B^{11}) = 0 \).

Finally we show that (ii) \( \implies \) (iii). When (ii) holds, we have by (2.9), (2.11), using that also \( \gamma \nu \in \mathbb{R}^{2m}(I - C^{00}B^{00}) \)

\[
(Au, \nu) = a(u, \nu) + \left\langle (\mathcal{A} - c_0^1C^{11}B^{10})\gamma u, \gamma \nu \right\rangle,
\]

for \( u, \nu \in D(A_B) \). By Proposition 1.6 there exists a sesquilinear form \( s(u, \nu) \) on \( \mathbb{H}^{m}(E) \) satisfying

\[
s(u, \nu) = \left\langle (\mathcal{A} - c_0^1C^{11}B^{10})\gamma u, \gamma \nu \right\rangle, \quad \text{for } u, \nu \in \mathbb{H}^{m}(E).
\]

Let \( a_B(u, \nu) = a(u, \nu) + s(u, \nu) \). Then \( a_B(u, \nu) \) satisfies (iii). This completes the proof of the theorem.

Remark 2.5. — In the proof that (ii) implies (iii) we have in fact constructed \( a_B \) such that (2.7) (and thus also (2.8)) is valid for all \( u \in D(A_B), \) \( \text{all } \nu \in \mathbb{H}^{m}(E) \) with \( B^{00} \gamma \nu = 0 \).

Remark 2.6. — A somewhat analogous theory can be set up for operators \( A \) of arbitrary order \( r \), connecting the inequality (for an integer \( t \in [0, r] \))

\[
\| (Au, \nu) \| \leq c \| u \|_m \| \nu \|_{r-t},
\]

with sesquilinear forms on \( \mathbb{H}^t(E) \times \mathbb{H}^{r-t}(E) \), cf. Remark 1.8.
Remark 2.7. — It is also easy to prove without use of (iii) that (ii) implies (i). The equivalence of (i), (ii) and (iv) extends to the case where the $B_{jk}$ with $j > k$ are replaced by pseudo-differential operators $B_{jk}$ from $E|_{\Gamma}$ to $F_{j}$ of order $j - k$ (cf. Remark 1.14).

2.2. Discussion of (2.6).

We shall now look more closely at what (2.6) stands for.

**Lemma 2.8.** — The identity (2.6) is equivalent with each of the following statements (2.15)-(2.18)

\begin{align*}
(2.15) & \quad Z(B^{11}) \subseteq Z((1 - C^{00}B^{00})^*\alpha^{01}); \\
(2.16) & \quad \alpha^{01}Z(B^{11}) \subseteq R(B^{00}^*); \\
(2.17) & \quad (I - C^{11}B^{11})^*\alpha^{01}*(I - C^{00}B^{00}) = 0; \\
(2.18) & \quad \alpha^{01}Z(B^{00}) \subseteq R(B^{11}^*).
\end{align*}

**Proof.** — (2.6) may be written

\[ \alpha^{01}R(I - C^{11}B^{11}) \subseteq Z((I - C^{00}B^{00})^*) \]

which is equivalent with (2.15) and (2.16) by Lemma 1.11. (2.6) is equivalent with its adjoint equation (2.17), and thus with (2.18) by Lemma 1.11.

**Remark 2.9.** — Because of the continuity properties of the operators involved, each of the inclusions (2.15), (2.16) and (2.18) is equivalent with the inclusion between the spaces intersected with $\Pi H^{\varkappa - k - \frac{1}{2}}(E|_{\Gamma})$, any $\varkappa \in \mathbb{R}$ (the spaces $Z^\varkappa(\ldots)$, $R^\varkappa(\ldots)$ in (1.24)-(1.27)). Similar statements hold for the following results.

For any normal boundary condition we shall define the operator

\[ Q = (I - C^{00}B^{00})^*\alpha^{01}(I - C^{11}B^{11}), \]

it is an $m \times m$-system of differential operators in $E|_{\Gamma}$, of type $(- k, - 2m + 1 + j)_{j \in \mathbb{M}_k, k \in \mathbb{M}_r}$ just like $\alpha^{01}$; in particular it has zeroes below the second diagonal. Define the second-diagonal parts

\[ Q_{df} = (\delta_{j, 2m-1-k}Q_{jk})_{j \in \mathbb{M}_k, k \in \mathbb{M}_r}; \quad \alpha_{d}^{01} = (\delta_{j, 2m-1-k}\alpha_{jk})_{j \in \mathbb{M}_k, k \in \mathbb{M}_r}; \]
they are vector bundle morphisms. It is easily seen that

\[ (2.20) \quad Q_d = (I - C_d B_d^{00})^* \alpha_d^{01}(I - C_d B_d^{11}). \]

Since \( Q = 0 \) implies \( Q_d = 0 \), we get immediately

**Lemma 2.10.** — The identity (2.6) implies the following equivalent statements (2.21)-(2.24)

\[
\begin{align*}
(2.21) & \quad (I - C_d B_d^{00})^* \alpha_d^{01}(I - C_d B_d^{11}) = 0; \\
(2.22) & \quad Z(B_d^{11}) = Z((I - C_d B_d^{00})^* \alpha_d^{01}); \\
(2.23) & \quad \alpha_d^{01} Z(B_d^{11}) \subseteq R(B_d^{00}); \\
(2.24) & \quad \alpha_d^{01} Z(B_d^{00}) \subseteq R(B_d^{11}).
\end{align*}
\]

(2.21)-(2.24) actually express certain properties of the bundle \( \bigoplus_{j \in M} F_j \) in relation to \( \bigoplus_{j \in M} E_j \). Let us make this explicit in the case where \( \alpha_0 \) is invertible, i.e. \( \Gamma \) is noncharacteristic for \( A \):

**Theorem 2.11.** — Assume that \( \Gamma \) is noncharacteristic for \( A \). Then (2.6) implies that \( Z_j = \ker B_{jj} \) is isomorphic to a subbundle of \( F_{2m-1-j} \), for all \( j \in M \). In particular,

\[
\sum_{j \in M} p_j \geq mq.
\]

When furthermore \( \sum_{j \in M} p_j = mq \), then

\[
\begin{align*}
(2.25) & \quad Z(B_d^{11}) = (\alpha_d^{01})^{-1} R(B_d^{00}), \\
(2.26) & \quad Z(B_d^{00}) = (\alpha_d^{11})^{-1} R(B_d^{11}),
\end{align*}
\]

and \( Z_j \cong F_{2m-1-j} \) for all \( j \in M \).

**Proof.** — Since \( \alpha_d \) is skew-diagonal and invertible, (2.23) may be written

\[
Z(B_{jj}) \subseteq (\alpha_{2m-1-j,d})^{-1} R(B_{2m-1-j,d}^{*}, 2m-1-j), \quad \text{for all } j \in M_1.
\]

This is equivalent with the statement for bundles (cf. (1.22))

\[
Z_j \subseteq (\alpha_{2m-1-j,d})^{-1} B_{2m-1-j,d}^{*}, 2m-1-j F_{2m-1-j},
\]

where \( (\alpha_{2m-1-j,d})^{-1} B_{2m-1-j,d}^{*}, 2m-1-j \) is an injective morphism. This shows the first statement for \( j \in M_1 \); for \( j \in M_0 \) it follows similarly from (2.24).
Regarding dimensions, (2.23) and (2.24) imply
\[
(2.28) \sum_{j \in M_i} (q - p_j) \leq \sum_{j \in M_s} p_j, \quad \sum_{j \in M_s} (q - p_j) \leq \sum_{j \in M_i} p_j,
\]
respectively; both statements are equivalent with (2.25). When equality holds in (2.28), (2.23) and (2.24) represent inclusions between vector bundles of the same dimension, these must be identities, so (2.26) and (2.27) hold.

**Remark 2.12.** — When \( \Gamma \) is characteristic for \( A \), (2.6) may be satisfied with \( \sum_{j \in M} p_j < mq \), and (2.6) is in a sense less restrictive on \( B \). We refrain from a systematic treatment here.

Also the inclusions in Lemma 2.8 can now be improved, when \( \sum_{j \in M} p_j = mq \), and \( \Gamma \) is noncharacteristic.

**Theorem 2.13.** — Assume that \( \Gamma \) is noncharacteristic for \( A \), and that \( \sum_{j \in M} p_j = mq \). Then (2.6) is equivalent with each of the statements (2.29)-(2.32)

\[
(2.29) \quad Z(B_{11}) \subseteq (\alpha^{01})^{-1} R(B^{00*}), \\
(2.30) \quad Z(B_{11}) = (\alpha^{01})^{-1} R(B^{00*}), \\
(2.31) \quad Z(B_{11}) \supseteq (\alpha^{01})^{-1} R(B^{00*}), \\
(2.32) \quad B_{11} (\alpha^{01})^{-1} B^{00*} = 0.
\]

**Proof.** — We have from Lemma 2.8 that (2.6) is equivalent with (2.29). Clearly (2.31) and (2.32) are equivalent. Since (2.30) implies (2.29) and (2.31), it remains to show that (2.29) implies (2.30), and that (2.31) implies (2.30).

Assume (2.29). Since we are now dealing with differential operators and not just morphisms, the dimension argument in the previous proof is not directly applicable. We have however, using Theorem 2.11 and Lemma 1.12

\[
(2.33) \quad R(B^{00*}) = \alpha^{01} Z(B_{11}) \\
= \alpha^{01} (I - C_{11} B_{11}^*) Z(B_{11}) \\
= \alpha^{01} (I - C_{11} B_{11}^*) (\alpha^{01}_d)^{-1} R(B^{00*}) \\
= \alpha^{01} (I - C_{11} B_{11}^*) (\alpha^{01}_d)^{-1} (I - B^{00*} C^{00*}) R(B^{00*}) \\
= (I + K) R(B^{00*}),
\]
where $K$ is a subtriangular differential operator in $\bigoplus_{k \in \mathbb{M}_0} E|_\Gamma$, thanks to the subtriangular character of $B_{10}$ and $B_{00}^\ast$. Denoting $R(B_{00}^\ast)$ by $R$, we have found

\begin{equation}
R \supseteq (I + K)R.
\end{equation}

Now $K$ is a nilpotent operator on $\mathcal{D}'(E|_\Gamma)$. Moreover, since $I + K$ maps $R$ into $R$, $K$ itself maps $R$ into $R$. Then $(I + K)|_R$ has the inverse

$$I|_R - K|_R + (K|_R)^2 - \cdots + (-K|_R)^m,$$

so it maps $R$ onto $R$, and the inclusion in (2.34) must be the identity. Then also the inclusion in (2.33) is the identity, and we have proved (2.30).

The proof that (2.31) implies (2.30) follows similarly from

$$Z(B_{11}) = (\mathcal{A}_0^1)^{-1} R(B_{00}^\ast)$$

$$= (\mathcal{A}_0^1)^{-1}(I + B_{00}^\ast C_{00}^\ast)\mathcal{A}_0^1(I + C_d B_{11})Z(B_{11})$$

$$= (I + K_d)Z(B_{11}).$$

**Corollary 2.14.** — When $\Gamma$ is noncharacteristic for $\Lambda$, and $\sum_{j \in \mathbb{M}} p_j = mq$, then (2.6) is equivalent with each of the statements (2.35)-(2.38)

\begin{align*}
(2.35) & \quad Z(B_{00}) \subseteq (\mathcal{A}_0^1)^{\ast^{-1}} R(B_{11}^\ast), \\
(2.36) & \quad Z(B_{00}) = (\mathcal{A}_0^1)^{\ast^{-1}} R(B_{11}^\ast), \\
(2.37) & \quad Z(B_{00}) \supseteq (\mathcal{A}_0^1)^{\ast^{-1}} R(B_{11}^\ast), \\
(2.38) & \quad B_{00} (\mathcal{A}_0^1)^{-1} B_{11}^\ast = 0.
\end{align*}

**Proof.** — Follows from Theorem 2.13, using that the identities are pairwise equivalent (adjoint).

Theorem 2.4 together with Corollary 2.14 prove Theorem 1 in [7].

**2.3. Existence and uniqueness of $B_{11}$ for given $B_{00}$.**

For the case where $\Gamma$ is noncharacteristic for $\Lambda$, and $\sum_{j \in \mathbb{M}} p_j = mq$, we shall consider the problem of how $B_{10}$ and $B_{11}$ may look, when $B_{00}$ is given, and $B$ shall satisfy (2.6).
(The question on how $B^{00}$ and $B^{10}$ depend on $B^{11}$ is treated similarly.)

We shall denote by $I^x$ the skew-unit matrix

$$I^x = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \ldots & 0 & 0 \end{bmatrix},$$

indexed according to its use. Denote $\bigoplus E|_{\Gamma}$ by $E^0$ and $\bigoplus E|_{\Gamma}$ by $E^1$. Given $F^0 = \bigoplus_{j \in M_0} F_j$, and $B^{00}$ going from $E^0$ to $F^0$.

1° Existence. — $B^{11}$ is required to have surjective diagonal part and satisfy (cf. Theorem 2.13)

$$Z(B^{11}) = (\alpha^{01})^{-1}R(B^{00*}).$$

Now we find by Lemmas 1.11-1.12

$$(\alpha^{01})^{-1}R(B^{00*}) = (\alpha^{01})^{-1}(I + B^{20}C^{00*})R(B^{00*})$$

$$= (\alpha^{01})^{-1}(I + B^{20}C^{00*})Z((I - C^{00}B^{00})^{*})$$

$$= Z((I - C^{00}B^{00})(I - B^{20}C^{00*})\alpha^{01}),$$

where $(I - C^{00}B^{00})^{*} = I - C^{00}B^{00}$ defines the orthogonal projection of $E^0$ onto $Z^0 = \bigoplus_{k \in M_0} Z_k$ ($Z_k = \ker B_{kk}$, cf. Lemma 1.10), let us denote it $pr_{Z^0}$. Then if we define

$$F_j = Z_{2m-1-j} \quad \text{for} \quad j \in M_1,$$

i.e. $F^1 = \bigoplus_{j \in M_1} F_j = I^xZ_0$ with $I^x = (\delta_{2m-1-j})_{j \in M_0, k \in M_0}$; and

$$B^{11} = I^x pr_{Z^0}(I - B^{20}C^{00*})\alpha^{01},$$

then $B^{11}$ is a differential operator from $E^1$ to $F^1$ of type $(-k, -j)_{j, k \in M_1}$ satisfying (2.40), and its diagonal part $B^{11}_{d} = I^x pr_{Z^0}\alpha^{01}_{d}$ is surjective. Also

$$\dim (\bigoplus_{j \in M_0} F_j) + \dim (\bigoplus_{j \in M_1} F_j) = mq$$

as required. $B^{10}$ can be any differential operator from $E^0$ to $F^1$ of type $(-k, -j)_{j, k \in M_0}$. 


Uniqueness. — Now let \( F_1 = 0 \) and \( B_{10} \) and \( B_{11} \) satisfying (2.6) etc. be given. By Theorem 2.11, \( F_j \cong Z_{2m-1-j} \) for \( j \in M_1 \), so \( F_1 = \Phi F_1 \) for some diagonal bijective morphism \( \Phi = (\Phi_{jk})_{j,k \in M_i} \). Moreover, we have \( Z(B_{11}) = Z(B_{11}) \).

Using the decomposition
\[
\mathcal{D}'(E^1) = Z(B_{11}) \oplus R(C_{11}) \quad \text{(direct sum)},
\]
written \( \varphi = \varphi_0 + \varphi_1 \), we find that
\[
B_{11} \varphi = B_{11}(\varphi_0 + \varphi_1) = B_{11} \varphi_1 = B_{11} C_{11} B_{11} \varphi_1
= B_{11} C_{11} B_{11} \varphi = \Phi^{-1} B_{11} C_{11} B_{11} \varphi
= \Phi \Psi B_{11} \varphi, \quad \text{for all } \varphi \in \mathcal{D}'(E^1),
\]
where \( \Psi = \Phi^{-1} B_{11} C_{11} \) is a bijective differential operator of type \((-k,-j)_{j,k \in M_i}\) in \( F_1 \); by Lemma 1.13 it has bijective diagonal part. Conversely, if \( \Psi \) is a bijective differential operator in \( F_1 \), then \( Z(\Phi \Psi B_{11}) = Z(B_{11}) \).

So there is existence and uniqueness of \( B_{11} \) up to isomorphisms. More precisely, we have found:

**Theorem 2.15.** — Assume that \( \Gamma \) is noncharacteristic for \( A \), and consider normal boundary conditions with \( \sum p_j = mq \).

Let \( \bigoplus F_j \) and \( B^{00} \) be given. Then the operators \( B_{10} \) and \( B_{11} \), for which (2.6) is satisfied, are characterized by

(i) \( \bigoplus F_j = \Phi \left( \bigoplus Z_{2m-1-k} \right) \), where \( \Phi = (\Phi_{jk})_{j,k \in M_i} \) is any diagonal vector bundle isomorphism.

(ii) \( B_{10} \) is any differential operator from \( \bigoplus E_{\Gamma} \) to \( \bigoplus F_j \) of type \((-k,-j)_{j \in M_o, k \in M_i} \).

(iii) \( B_{11} = \Phi \Psi B_{11} \), where \( \Psi \) is any differential operator in \( \bigoplus Z_{2m-1-k} \) with bijective diagonal part, and

\[
(2.42) \quad B_{11} = I^* \text{pr}_2(I - B^{00} C^{00}) \mathcal{A}^{01},
\]
as defined above.

\( \Psi \) is any differential operator in \( \bigoplus Z_{2m-1-k} \) with bijective diagonal part, and

\[
(2.42) \quad B_{11} = I^* \text{pr}_2(I - B^{00} C^{00}) \mathcal{A}^{01},
\]
as defined above.
2.4. The adjoint boundary condition.

We shall finally study the (formally) adjoint realization; it represents a boundary condition that may be determined by methods analogous to those of section 2.3. This leads to a more illuminating criterion for weak semiboundedness. The determination of the adjoint realization is independent of whether \( A \) is of even or odd order, so for that part, the order will again be denoted \( r \).

**Definition 2.16.** — The formally adjoint realization \((A_B)'\) of \( A_B \) is the operator sending \( \nu \) into \( A'\nu \), with domain

\[
D((A_B)') = \{ \nu \in H^r(E) | (Au, \nu) - (u, A'\nu) = 0 \text{ for all } u \in D(A_B) \}.
\]

We shall now show that there exists a differential operator \( B' \) so that \((A_B)'\) is the realization of \( A' \) determined by the boundary condition \( B'\varphi = 0 \).

**Proposition 2.17.** — \( D((A_B)') = D(A_B'), \) where

\[
B' = I^X \text{pr}_Z(I - B^*C^*)\mathcal{A}^*,
\]

here \( \text{pr}_Z \) denotes the orthogonal projection of \( \bigoplus_{k \in M} E|_\Gamma \) onto \( \bigoplus_{k \in M} Z_k \), and \( I^X = (\delta_{2m-1-j, k})_j, k \in M \). \( B' \) is a system \((B'_j)_j, k \in M \) of differential operators \( B'_j \) from \( E|_\Gamma \) to \( Z_{2m-1-j} \) (for \( j, k \in M \)), of type \((- k, - j)_j, k \in M \). When \( \Gamma \) is noncharacteristic for \( A, B' \) is normal.

**Proof.** — When \( u \in D(A_B) \), \( \rho u \) runs through

\( Z^*(B) = R^r(I - CB) \),

so

\[
(Au, \nu) - (u, A'\nu) = \langle \mathcal{A}(I - CB)\varphi, \rho\nu \rangle = \langle \varphi, (I - CB)^*\mathcal{A}^*\rho\nu \rangle.
\]

Thus

\[
D((A_B)') = \{ \nu \in H^r(E) | (I - CB)^*\mathcal{A}^*\rho\nu = 0 \} = \{ \nu \in H^r(E) | \rho\nu \in Z^*(I - CB)^*\mathcal{A}^* \}.
\]
Like in (2.41) we find
\[ Z^\tau((I - CB)^* \alpha^*) = (\alpha^*)^{-1} R^\tau(B^*) = (\alpha^*)^{-1}(I + B_d^* C_d) R^\tau(B_d^*) \]
\[ = Z^\tau((I - C_d B_d)^* (I - B_d^* C^*) \alpha^*) \]
\[ = Z^\tau(B'), \]
with \( B' \) defined by (2.44), since \((1 - C_d B_d)^* = 1 - C_d B_d\) is the orthogonal projection of \( \bigoplus_{k \in \mathbb{M}} E|\Gamma \) onto \( \bigoplus_{k \in \mathbb{M}} Z_k \), cf. Lemma 1.10. (As in section 2.3 we insert \( \mathcal{Y} \) to get the correct type.) The diagonal part is \( B'_d = \mathcal{Y} \) pr \( \alpha^*_d \), which is surjective when \( \alpha \) is invertible (i.e., when \( \Gamma \) is noncharacteristic).

Next, we consider the uniqueness question. Given a bundle \( \bigoplus_{j \in \mathbb{M}} F_j \) and a normal system \( \tilde{B}' = (\tilde{B}'_{jk})_{k \in \mathbb{M}} \) of differential operators \( \tilde{B}'_{jk} \) from \( E|\Gamma \) to \( F_j \), of type \(( -k, -j) \), \( k \in \mathbb{M} \), for which \( (A_B)' = A_B' \). This means that
\[ (2.45) \quad Z^\tau(\tilde{B}') = Z^\tau(B') \]
in view of Proposition 2.17. Now \( Z^\tau(\tilde{B}') = R^\tau(I - \tilde{C}' \tilde{B}') \), so (2.45) implies
\[ B'(I - \tilde{C}' \tilde{B}') = 0. \]

In particular, the diagonal part must be zero, so we conclude
\[ Z(B'_d) \supset R(I - \tilde{C}' \tilde{B}'_d) = Z(\tilde{B}'_d). \]
Assume in the rest of this proof that also \( B' \) is normal; then an analogous argument gives the opposite inclusion, so in fact
\[ Z(\tilde{B}'_d) = Z(B'_d). \]

This gives for the bundles:
\[ \bigoplus_{j \in \mathbb{M}} F'_j \cong \bigoplus_{j \in \mathbb{M}} (E|\Gamma \ominus \ker B'_{jj}) = \bigoplus_{j \in \mathbb{M}} (E|\Gamma \ominus \ker B'_{jj}) \cong \bigoplus_{j \in \mathbb{M}} Z_{2m-1-j}, \]
the range space for \( B' \). So \( \bigoplus_{j \in \mathbb{M}} F'_j = \Phi \left( \bigoplus_{j \in \mathbb{M}} Z_{2m-1-j} \right) \) for some diagonal bijective morphism. It is finally seen from (2.45), like in the proof of Theorem 2.15, that \( \tilde{B}' = \tilde{B}' C' B' \) where \( \tilde{B}' C' \) is a differential operator from \( \bigoplus_{k \in \mathbb{M}} Z_{2m-1-k} \) to \( \bigoplus_{j \in \mathbb{M}} F'_j \).
with bijective diagonal part. We have proved

**Theorem 2.18.** — Assume that $B'$, defined by (2.44), is
normal. A normal system $B' = (B'_{jk})_{j,k \in M}$ of differential
operator $B'_{jk}$ from $E|\Gamma$ to $F'$, of type $(-k, -j)_{j,k \in M}$, satisfies
\[ D((A_B)'') = D(A'_B') \]
if and only if

(i) $F'_j \cong Z_{2m-1-j}$ for each $j \in M$;

(ii) $B' = \Psi B'$, where $\Psi$ is a bijective differential operator
from $\bigoplus_{k \in M} Z_{2m-1-k}$ to $\bigoplus_{j \in M} F'$, of type $(-k, -j)_{j,k \in M}$.

Now let $r = 2m$. We then have, with obvious notations,

\[
(2.47) \quad B' = \begin{bmatrix} I^X & 0 \\ 0 & I^X \end{bmatrix} \begin{bmatrix} pr_{Z_{00}} & 0 \\ 0 & pr_{Z_{10}} \end{bmatrix} \times
\begin{bmatrix} I - B_{00}^0 C_{00}^0 & S_1 \\ 0 & I - B_{11}^1 C_{11}^1 \end{bmatrix} \begin{bmatrix} \alpha_{00}^0 & \alpha_{10}^0 \\ \alpha_{01}^0 & 0 \end{bmatrix} =
\begin{bmatrix} I^X & S_2 \\ 0 & I^X \end{bmatrix} \begin{bmatrix} pr_{Z_{00}}(I - B_{11}^1 C_{11}^1) \alpha_{01}^0 \\ 0 \end{bmatrix}
\]

$(S_1$ and $S_2$ not worth calculating). Thus, using Lemma 1.12,

\[
(2.48) \quad Z(B'^0) = Z(I^X \begin{bmatrix} pr_{Z_{00}}(I - B_{11}^1 C_{11}^1) \alpha_{01}^0 \end{bmatrix} = (\alpha_{01}^0)^{-1} (I + B_{11}^1 C_{11}^1) Z(I - B_{11}^1 C_{11}^1) = (\alpha_{01}^0)^{-1} R(B_{11}^1).)
\]

This formula is valid whether $\alpha_{01}^0$ is invertible or not, if we
by $(\alpha_{01}^0)^{-1}$ understand the mapping of a set into its inverse
image by $\alpha_{01}^0$. Connecting this with Lemma 2.8 (in particular
(2.18), cf. also Theorem 2.4), we find

**Theorem 2.19.** — $A_B$ is weakly semibounded (or, equivalently, satisfies (2.6)) if and only if

\[ Z^{2m}(B'^0) \subset Z^{2m}(B'^0), \]

that is, if and only if $\gamma D(A_B) \subset \gamma D((A_B)'')$.

For the noncharacteristic case this has the consequence

**Corollary 2.20.** — Let $\Gamma$ be noncharacteristic for $A$. If
$A_B$ is weakly semibounded, then $(A_B)'$ is weakly semibounded
if and only if $\sum_{j \in M} p_j = mq$. When $\sum_{j \in M} p_j = mq$, (2.6) is equivalent with
\[
\text{(2.50)} \quad Z(B^{00}) = Z(B'^{00}),
\]
i.e., $\gamma D(A_B) = \gamma D(A'_B)$.

Proof. — Let $A_B$ be weakly semibounded. Then by Theorem 2.11, $\sum_{j \in M} p_j \geq mq$. When $\sum_{j \in M} p_j > mq$, the fiber dimension of the range space for $B'$ is
\[
\sum_{j \in M} (q - p_{2m-1-j}) = 2mq - \sum_{k \in M} p_k < mq,
\]
so, by Theorem 2.11, $A'_B$ cannot be weakly semibounded.

Now assume $\sum_{j \in M} p_j = mq$. Then by Theorem 2.13 and (2.48), (2.6) is equivalent with (2.50). Since $(A'_B)' = A_B$, and (2.50) is symmetric in $\{A, B\}$ and $\{A', B'\}$, (2.50) must also be necessary and sufficient for the weak semiboundedness of $A'_B$.

Remark 2.21. — It was shown in [6, Theorem 3.4] for the case where $A$ is elliptic, that also for general realizations $\tilde{A}$ of $A$, $\gamma D(\tilde{A}) \subseteq \gamma D(\tilde{A}^*)$ (closure in $\prod_{k \in M_0} H^{-k-\frac{1}{2}}(E|_\Gamma)$) is necessary for weak semiboundedness (the above proof was inspired from that theory). This is the reason for our conjecture that normality is necessary under much more general circumstances than those accounted for in Remark 2.2; for the lack of normality tends to enlarge $Z(B^{00})$ and diminish $\gamma D((A_B)'$).
CHAPTER 3

SYSTEMS OF TYPE \((m_i, -m_i), i=1, \ldots, q\).

3.1. Green's formulae.

In this chapter we describe how the results of Chapters 1 and 2 extend to systems \(A\) that are of "mixed order", and of a symmetric type.

Let \(\{m_1, \ldots, m_q\}\) be a set of nonnegative integers, and let \(A = (A_{st})_{s,t=1, \ldots, q}\) be a \(q \times q\)-matrix of differential operators on \(\Omega\), of type \((m_i, -m_i), i=1, \ldots, q\); i.e., \(A_{st}\) is of order \(m_s + m_t\). Among the systems of this type are the strongly elliptic systems, cf. [2]. Denote

\[
N = \{1, \ldots, q\}, \quad \bar{m} = \max_{i \in N} m_i, \quad \bar{m} = m_1 + \cdots + m_q,
\]

and assume \(m > 0\).

For such systems one usually studies boundary conditions of the following kind: There is given a set of \(p\) integers \(\{\nu_1, \ldots, \nu_p\}\) and a \(p \times q\)-matrix of differential operators \(B = (B_{st})_{s=1, \ldots, p; t=1, \ldots, q}\) on \(\Omega\), of type \((m_i, \nu_s), t=1, \ldots, q;\)

it defines the boundary value problem (3.2)-(3.3)

\[
(3.2) \quad Au = f, \quad \text{i.e.,} \quad \sum_{t=1}^q A_{st}u_t = f_t, \quad s = 1, \ldots, q;
\]

\[
(3.3) \quad \gamma_0 Bu = 0, \quad \text{i.e.,} \quad \sum_{t=1}^q \gamma_0 B_{st}u_t = 0, \quad s = 1, \ldots, p;
\]

here \(f = \{f_1, \ldots, f_q\}, u = \{u_1, \ldots, u_q\}\).

(3.3) determines a realization \(A_B\) by

\[
A_B : u \mapsto Au, \quad D(A_B) = \{u \in \prod_{t \in N} H^{m_t+m}(\Omega) | \gamma_0 Bu = 0\}.
\]
We shall say that $A_{\bar{\gamma}}$ is weakly semibounded if there exist $c > 0$, $\theta \in \mathbb{R}$ such that

\[ \text{Re} \ e^{i\theta} \langle Au, u \rangle \leq c \sum_{i \in \mathbb{N}} \| u_i \|_{m_i}^2, \quad \text{for all} \quad u \in D(A_{\bar{\gamma}}); \]

like in Chapter 2, the inequality depends only on $A$ and $\mathcal{B}$ at $\Gamma$, but involves the exact operators, not just for instance principal symbols. When $A$ is elliptic, one usually assumes $p = \bar{m}$; we do not assume that on beforehand.

If for some $s$, $m_s - \mu_s < 0$ for all $t$, then the terms $\mathcal{B}_{st}$ are all zero, so $\sum_{t=1}^q \gamma_0 \mathcal{B}_{st} u_t = 0$ is trivially satisfied; we can therefore assume that $\mu_s \leq \max m_t = m$ for all $s$. We shall furthermore assume that the $\mu_s$ are $\geq -m + 1$, thereby we exclude boundary conditions of very high order, just like boundary conditions of order $\geq 2m$ were excluded in Chapter 2. Our boundary conditions still include those that arise in connection with sesquilinear forms as in Guedes de Figueiredo [9].

To apply our techniques we shall set up a Green’s formula and reformulate (3.3), such that differential operators on $\Gamma$ of the same order are grouped together.

Let $\{s, t\} \in \mathbb{N} \times \mathbb{N}$. We have by Lemma 1.2 for $u_t$, $\nu_s \in C^\infty(\overline{\Omega})$

\[ [I_{st} = ](A_{st} u_t, \nu_s) - (u_t, A_{st}' \nu_s) = \sum_{j', k' = 0}^{m_s + m_t - 1} \alpha_{st, j' k'} \gamma_{j' k'} u_t, \gamma_{j' k'} \nu_s, \]

where $\alpha_{st, j' k'}$ is of order $m_s + m_t - j' - k' - 1$. Set $j = j' - m_s + m$ and $k = k' - m_t + m$, and set

$\tilde{\alpha}_{st jk} = \alpha_{st, j+m_s-m, k+m_t-m}$

for

\[ j \in \{- m_s + m, \ldots, 2m - 1\} \]

and

\[ k \in \{- m_t + m, \ldots, 2m - 1\}, \]

where we put $\tilde{\alpha}_{st jk} = 0$ for $j \geq m_t + m$ or $k \geq m_s + m$. Then $\tilde{\alpha}_{st jk}$ is of order

\[ m_s + m_t - (j + m_s - m) - (k + m_t - m) - 1 = 2m - j - k - 1 \]
for all \( j, k \). (Here, for \( j \geq m_t + m \) or \( k \geq m_s + m \), the order is negative in accordance with \( \tilde{\partial}_{stjk} = 0 \). We have actually just augmented the usual boundary matrix by some zero rows and columns.) Now

\[
I_{st} = \sum_{j=-m_s+m}^{2m-1} \sum_{k=-m_t+m}^{2m-1} \langle \tilde{\partial}_{stjk} \gamma_{k+m_t-m} u_t, \gamma_{j+m_s-m} v_s \rangle.
\]

For \( u, v \in \sum_{t \in \mathbb{N}} C^\infty(\Omega) \) one has

\[
(3.4) \quad (Au, v) - (u, A'v) = \sum_{s, t \in \mathbb{N}} I_{st},
\]

and we shall now regroup the terms in \( \sum_{s, t \in \mathbb{N}} I_{st} \). Define as usual

\[
(3.5) \quad M = \{0, \ldots, 2m-1\}, \quad M_0 = \{0, \ldots, m-1\}, \quad M_1 = \{m, \ldots, 2m-1\}.
\]

For each \( k \in M \), define

\[
(3.6) \quad N_k = \{t \in \mathbb{N} | k + m_t - m \geq 0\},
\]

and denote \( |N_k| = q_k \). Clearly,

\[
(3.7) \quad \emptyset \neq N_0 \subset \ldots \subset N_{m-1} \subset N_m = N_{m+1} = \ldots = N_{2m-1} = N,
\]

using that \( N_m = \{t | m_t \geq 0\} = \mathbb{N} \) (and all \( N_k \) equal \( \mathbb{N} \) if and only if all \( m_t \) equal \( m \) as in Chapter 2). Note that \( q_0 = \cdots = q_{2m-1} = q \). Moreover, it is easily seen that

\[
(3.8) \quad q_0 + \cdots + q_{m-1} = m.
\]

Denote the trivial bundles \( \Gamma \times \prod_{t \in \mathbb{N}_k} \mathcal{G} \) by \( E_k \).

Now define for each \( \{ j, k \} \in M \times M \) the \( N_j \times N_k \) matrix \( \tilde{\partial}_{jk} \) by

\[
(3.9) \quad \tilde{\partial}_{jk} = (\tilde{\partial}_{stjk})_{s \in N_j, t \in N_k};
\]

it is a differential operator from \( E_k \) to \( E_j \) of order \( 2m - j - k - 1 \). Altogether the \( \tilde{\partial}_{jk} \) from a system

\[
(3.10) \quad \tilde{\partial} = (\tilde{\partial}_{jk})_{j, k \in M}
\]

of type \( (-k, -2m + 1 + j)_{j, k \in M} \).
Introduce the vector valued trace operators $\beta_k$, for $k \in M$, by

$$\beta_k u = \{Y^k_{-m+k} u_t\}_{t \in N};$$

they are continuous and surjective from $\prod_{t \in N} H^{\alpha+k+m}(\Omega)$ onto $H^{\alpha+2m-k+1}(E_k)$, for $\alpha + 2m > k + \frac{1}{2}$, respectively. Altogether they form a vector (of vectors) $\beta$:

$$\beta u = \{\beta_k u\}_{k \in M}.$$

Actually, $\beta u$ consists of a rearrangement of the traces $\{\gamma_0 u_t, \gamma_1 u_t, \ldots, \gamma_{m_m-m}-u_t; \ldots; \gamma_0 u_q, \gamma_1 u_q, \ldots, \gamma_{m_q-m-1} u_q\}$, and it has a total number of $\sum_{k \in M} (m_t + m) = \sum_{k \in M} q_k$ elements.

These are exactly all that enter in (3.4). It is thus reasonable to call $\beta u$ the Cauchy data of $u$. We now find

$$\sum_{s, t \in N} I_{st} = \sum_{j, k \in M} \sum_{s, t \in N} \sum_{j, k \in M} \langle \tilde{a}_{stjk} Y_{k+m-t-m} u_t, \gamma_{j+m-s-m} \gamma \rangle$$

$$= \sum_{j, k \in M} \sum_{j, k \in M} \langle \tilde{a}_{stjk} Y_{k+m-t-m} u_t, \gamma_{j+m-s-m} \gamma \rangle$$

$$= \sum_{j, k \in M} \langle \tilde{a}_{stjk} \beta_k u, \beta_x \rangle = \langle \tilde{a}_x u, \beta_x \rangle,$$

and have hereby proved Green's formula

$$(3.13) \quad (Au, v) - (u, A^\prime v) = \langle \tilde{a}_x u, \beta_x \rangle,$$

it is valid for all $u, v \in \prod_{t \in N} H^{m_t+m}(\Omega)$. (A similar formula holds when the functions $u_t$ are replaced by sections in bundles, we omit this aspect for simplicity.)

Let us now consider the case where $\Gamma$ is noncharacteristic for $A$. This means that the $N \times N$-matrix $A^0$, whose entries are the functions $a_{st,m_t+m}$ stemming from the decompositions $A_{st} = \sum_{i=0}^{m_t+m} a_{st,i} D_i^n$, is bijective. The elements $\tilde{a}_{j,2m-1-j}$ are $N_j \times N_{2m-1-j}$-submatrices of $A^0$, so that when the $m_t$ are not all equal, $\tilde{a}$ can never be invertible (in view of (3.7), cf. also below). This is the main reason for the trouble with setting up e.g. a Lions-Magenes theory for boundary value problems for systems of mixed order. However,
it is possible to treat a particular class of boundary value problems, as indicated below.

Define (cf. (3.5))

\[(3.14) \quad \beta^0 u = \{\beta_k u\}_{k \in M}, \quad \beta^1 u = \{\beta_k u\}_{k \in M},\]

and note that \(\beta^0 u\) is a rearrangement of the Dirichlet data \(\{\gamma_0 u_1, \ldots, \gamma_{m-1} u_1; \ldots; \gamma_0 u_q, \ldots, \gamma_{m-1} u_q\}\);

\(\beta^0 u\) has \(\sum_{k \in M} q_k = \tilde{m}\) entries and \(\beta^1 u\) has \(mq\) entries. With the usual decomposition of \(\tilde{\alpha}\)

\[
\tilde{\alpha} = \begin{bmatrix} \tilde{\alpha}^{00} & \tilde{\alpha}^{01} \\ \tilde{\alpha}^{10} & 0 \end{bmatrix}, \quad \tilde{\alpha}^{\delta \epsilon} = (\tilde{\alpha}_{jk})_{j \in M, k \in M},
\]

(3.13) takes the form

\[(3.15) \quad (Au, \nu) - (u, A'\nu) = \langle \tilde{\alpha}^{00} \beta^0 u, \beta^0 \nu \rangle + \langle \tilde{\alpha}^{01} \beta^1 u, \beta^0 \nu \rangle + \langle \tilde{\alpha}^{10} \beta^0 u, \beta^1 \nu \rangle.\]

Because of (3.7), the second diagonal in \(\tilde{\alpha}^{01}\), resp. \(\tilde{\alpha}^{10}\), consists of \(N_j \times N\)-submatrices, resp. \(N \times N_k\)-submatrices, of \(A^0\). Then we obtain, by application of Proposition 1.9 to \(I^x \tilde{\alpha}^{01}\) and to \(I^x (\tilde{\alpha}^{10})^*\) (cf. (2.39)):

**Theorem 3.1.** — When \(\Gamma\) is noncharacteristic for \(A\), \(\tilde{\alpha}^{01}\) is surjective with a right inverse \(\mathcal{O}^{01}\) of type

\[(- 2m + 1 + k, - j)_{j \in M, k \in M},\]

and \(\tilde{\alpha}^{10}\) is injective with a left inverse \(\mathcal{O}^{10}\) of type

\[(- 2m + 1 + k, - j)_{j \in M, k \in M}.\]

It will be seen below that these properties suffice to generalize the results of Chapter 2 in a very satisfactory way. Moreover, (3.15) may be viewed as a Green's formula for some special boundary operators, for Theorem 3.1 clearly implies

**Corollary 3.2.** — Define \(x\) and \(x'\) by

\[(3.16) \quad xu = \tilde{\alpha}^{01} \beta^1 u, \quad x'u = - \tilde{\alpha}^{10} \beta^1 u - \tilde{\alpha}^{00} \beta^0 u.\]
Then we have for all \( u, v \in \prod_{i \in \mathbb{N}} H^{m_i + m}(\Omega) \)

\[
(3.17) \quad (Au, v) - (u, A'v) = \langle Xu, \beta^0 u \rangle - \langle \beta^0 u, Xu' \rangle,
\]

where, if \( \Gamma \) is noncharacteristic for \( A \), \( \{\beta^0, x\} \) and \( \{\beta^0, x'\} \) are surjective continuous mappings of \( \prod_{i \in \mathbb{N}} H^{\alpha_i + m_i + m}(\Omega) \) onto \( \prod_{k \in M_k} H^{2m_k - \frac{1}{2}}(E_k) \times \prod_{k \in M_k} H^{2m_k + \frac{1}{2}}(E_k) \) for all \( \alpha > -\frac{1}{2} \).

We shall call \( \{\beta^0 u, Xu\} \) (resp. \( \{\beta^0 u, Xu'\} \)) the reduced Cauchy data of \( u \) with respect to \( A \) (resp. \( A' \)). (The \( \dot{\xi}^{00} \)-term may of course be distributed in other ways). Boundary conditions for \( A \) that can be expressed as normal conditions on the reduced Cauchy data (i.e., « factor through \( \dot{\xi}^{01} \) ») can be treated much like those of Chapter 2; in particular one may set up a Lions-Magenes theory (details will be given elsewhere). Note that the Dirichlet operator \( \beta^0 \) belongs to this class. One of the main theorems of this chapter (Theorem 3.11) will be that the boundary conditions defining weakly semibounded realizations are indeed conditions on the reduced Cauchy data. A few more comments on this class are given in section 3.4.

We conclude this section by establishing the « halfways » Green’s formulae. By a sesquilinear form on \( \prod_{i \in \mathbb{N}} H^{m_i}(\Omega) \) we shall understand an integro-differential form

\[
(3.18) \quad a(u, \nu) = \sum_{s, t \in \mathbb{N}} \sum_{i \in I(s, t)} (Q_{st} u, P_{st} \nu),
\]

where the \( Q_{st} \) and \( P_{st} \) are differential operators on \( \overline{\Omega} \) of order \( \leq m_t \), resp. \( \leq m_s \), and the \( I(s, t) \) are finite index sets. \( a(u, \nu) \) is defined and continuous on \( \prod_{i \in \mathbb{N}} H^{m_i}(\Omega) \times \prod_{i \in \mathbb{N}} H^{m_i}(\Omega) \), and it is associated with \( A \) if and only if

\[
(3.19) \quad A_{st} = \sum_{i \in I(s, t)} P'_{st} Q_{st}, \quad \text{all} \quad s, t \in \mathbb{N}.
\]

Applying Remark 1.8 to each \( A_{st} \) and collecting the terms one finds, just as in section 1.2

**Theorem 3.3.** — When \( a(u, \nu) \) is a sesquilinear form on \( \prod_{i \in \mathbb{N}} H^{m_i}(\Omega) \) associated with \( A \), then for all \( u \in \prod_{i \in \mathbb{N}} H^{m_i + m}(\Omega) \),
all \( \nu \in \prod_{i \in \mathbb{N}} H^{m_i}(\Omega) \)

(3.20) \((Au, \nu) = a(u, \nu) + \langle \tilde{\mathcal{E}} \partial^1 u, \partial^0 \nu \rangle + \langle \mathcal{S} \partial^0 u, \partial^0 \nu \rangle; \)

where \( \mathcal{S} = (\mathcal{S}_j)_{j, k \in \mathbb{M}_e} \) is a system of differential operators from \( E_k \) to \( E_j \), of type \((-k, -2m + 1 + j)_{j, k \in \mathbb{M}_e}\). Conversely, for any such system \( \mathcal{S} \) there exists a sesquilinear form \( a(u, \nu) \) on \( \prod_{i \in \mathbb{N}} H^{m_i}(\Omega) \), fitting together with \( A \) in (3.20).

### 3.2. Normal boundary conditions; weakly semibounded realizations.

We shall now reformulate (3.3). By (1.4) we have

\[
\gamma_0 \mathcal{B}_t u_t = \sum_{i=0}^{m_t - \mu_s - 1} \mathcal{B}_{st, i} \gamma_t u_t,
\]

where each \( \mathcal{B}_{st, i} \) is a differential operator in \( \Gamma \) of order \( m_t - \mu_s - l \). Letting \( k = l - m_t + m \), we can write

\[
\gamma_0 \mathcal{B}_t u_t = \sum_{k=-m_t+m}^{2m-1} \mathcal{B}_{st, k+m_t-m} \gamma_{k+m_t-m} u_t,
\]

where we have added on some zero terms (recall \(-m \leq -\mu_s \leq m - 1\)), and \( \mathcal{B}_{st, k+m_t-m} \) is of order \( m - \mu_s - k \). Introduce the index sets

\[
L = \{1, \ldots, p\}, \quad L_j = \{s \in L \mid m - \mu_s = j\} \quad \text{for} \quad j \in \mathbb{M},
\]

and denote \( |L_j| = p_j \). Clearly, \( L \) equals the disjoint union \( \bigcup_{j \in \mathbb{M}} L_j \), and \( \sum_{j \in \mathbb{M}} p_j = p \). Denote the trivial bundles \( \Gamma \times \prod_{j \in L_j} \mathbb{C} \) by \( F_j \) (with \( F_j \) being the zero bundle \( \Gamma \times \{0\} \) when \( L_j = \emptyset \)). For each \( \{j, k\} \in \mathbb{M} \times \mathbb{M} \) we now define the \( p_j \times q_k \)-matrix

\[
B_{jk} = (\mathcal{B}_{st, k+m_t-m})_{s \in L_j, t \in L_k}
\]

it is a differential operator from \( E_k \) to \( F_j \) of order \( j - k \). Altogether the \( B_{jk} \) form a system \( \mathcal{B} = (B_{jk})_{j, k \in \mathbb{M}} \) of type
With $\beta$ as defined above, the boundary condition (3.3) may then be written in the form

$$B\beta u = 0,$$

(or, equivalently, $\sum_{k \leq j} B_{jk}\beta_k u = 0$ for all $j \in M$).

Our considerations in the following will be valid also when the $F_j$ are nontrivial vector bundles. From now on we study boundary conditions (3.21) where $B = (B_{jk})_{j,k \in M}$ goes from $\bigoplus_{k \in M} E_k$ to $\bigoplus_{j \in M} F_j$ and is of type $(-k, -j)_{j,k \in M}$, and the $F_j$ are any bundles over $\Gamma$ of dimension $p_j$. Definition 1.15 can now be generalized:

**Definition 3.4.** — The boundary condition $B\beta u = 0$ — or the operator $B$ — will be said to be normal when the diagonal part $B_{jj} = (\delta_{jk} B_{jk})_{j,k \in M}$ is a surjective morphism. (Then in particular $p_j \leq q_j$ for all $j \in M$).

Assume from now on that the boundary condition is normal. We split $B$ in blocks as usual

$$B = \begin{bmatrix} B^{00} & 0 \\ B^{10} & B^{11} \end{bmatrix}, \quad B^{\delta\xi} = (B_{jk})_{j \in M, k \in M},$$

and the considerations in section 1.3 now apply to $B$, $B^{00}$ and $B^{11}$, which have the right inverses $C$, $C^{00}$ resp. $C^{11}$. (3.21) may be written

$$B^{00}\beta^0 u = 0, \quad B^{10}\beta^0 u + B^{11}\beta^1 u = 0.$$

Define $Z^2(B^{00}) = \left\{ \varphi \in \prod_{k \in M} H^{2-k-\frac{1}{2}}(E_k)|B^{00}\varphi = 0 \right\}$, etc., then Lemma 2.3 generalizes to the present case.

**Theorem 3.5.** — Let $B = (B_{jk})_{j,k \in M}$ be a system of differential operators from $E_k$ to $F_j$, of type $(-k, -j)_{j,k \in M}$, defining a normal boundary condition $B\beta u = 0$. Let $A_B$ be the realization of $A$ defined by

$$A_B : u \mapsto Au, \quad D(A_B) = \left\{ u \in \prod_{i \in N} H^{m_i+n}(\Omega)|B\beta u = 0 \right\}.$$
The following statements (i)-(iv) are equivalent:

(i) There exist $\theta \in \mathbb{R}$, $c > 0$ such that

\begin{equation}
\text{Re} e^{i\theta} (Au, u) \leq c \sum_{i \in \mathbb{N}} \|u_i\|_{m_i}^2, \quad \text{all} \quad u \in D(A_B)
\end{equation}

(i.e., $A_B$ is weakly semibounded).

(ii) The following identity holds

\begin{equation}
(I - C^0 B^{00})^* \tilde{\alpha}^{01} (I - C^1 B^{11}) = 0.
\end{equation}

(iii) There exists a sesquilinear form $a_B(u, \nu)$ on $\prod_{i \in \mathbb{N}} H^{m_i}(\overline{\Omega})$ associated with $A$, such that

\begin{equation}
(Au, \nu) = a_B(u, \nu), \quad \text{all} \quad u, \nu \in D(A_B).
\end{equation}

(iv) There exists $c > 0$ such that

\begin{equation}
|\langle Au, \nu \rangle| \leq c \left( \sum_{i \in \mathbb{N}} \|u_i\|_{m_i}^2 \right)^{\frac{1}{2}} \left( \sum_{i \in \mathbb{N}} \|\nu_i\|_{m_i}^2 \right)^{\frac{1}{2}}, \quad \text{all} \quad u, \nu \in D(A_B).
\end{equation}

Proof. — With the notations introduced above, the proof goes in complete analogy with the proof of Theorem 2.4 ($\gamma$ and $\nu$ being replaced by $\beta^0$ and $\beta^1$, $\tilde{\alpha}^{01}$ replaced by $\tilde{\alpha}^{01}$).

The Remarks 2.5 and 2.7 extend immediately to the present case. Remark 2.6 extends as follows: For systems $A = (A_{st})_{s,t \in \mathbb{N}}$ of type $(m_t - \ell_t)_{s,t \in \mathbb{N}}$ where $\{m_t\}_{t \in \mathbb{N}}$ and $\{\ell_t\}_{t \in \mathbb{N}}$ are sets of nonnegative integers, one can set up Green's formulae generalizing (3.15) and (3.20), just like the formulae in Remark 1.8 generalize (1.11) and (1.14), and one can again define Cauchy data for $A$ and for $A'$, and normal boundary conditions. Then the inequality

\[ |\langle Au, \nu \rangle| \leq c \left( \sum_{i \in \mathbb{N}} \|u_i\|_{m_i}^2 \right)^{\frac{1}{2}} \left( \sum_{i \in \mathbb{N}} \|\nu_i\|_{m_i}^2 \right)^{\frac{1}{2}} \]

may be set in relation to sesquilinear forms on

\[ \prod_{i \in \mathbb{N}} H^{m_i}(\overline{\Omega}) \times \prod_{i \in \mathbb{N}} H^{s_i}(\overline{\Omega}), \]

generalizing Theorem 3.5. We refrain from details in order to limit notations. [Not all systems of mixed order are of this type.]
3.3. Discussion of (3.26).

All the statements in sections 2.2 and 2.3, that do not depend on whether $\mathfrak{a}^{01}$ is invertible, extend immediately. We shall use the convention that when an operator $S$ is not invertible, $S^{-1}$ denotes the mapping sending sets into their inverse images by $S$, i.e. $S^{-1}$ is viewed as a relation.

Lemmas 2.8 and 2.10 thus generalize to

**Lemma 3.6.** — The identity (3.26) is equivalent with each of the following statements

\begin{align*}
(3.29) & \quad Z(B^{11}) \subset Z((1 - C^{00}B^{00})^*\mathfrak{a}^{01}), \\
(3.30) & \quad Z(B^{11}) \subset (\mathfrak{a}^{01})^{-1}R(B^{00*}), \\
(3.31) & \quad Z(B^{00}) \subset (\mathfrak{a}^{01*})^{-1}R(B^{11*}),
\end{align*}

and it implies each of the equivalent statements

\begin{align*}
(3.32) & \quad Z(B_d^{11}) \subset Z((1 - C_d^{00}B_d^{00})^*\mathfrak{a}_d^{01}), \\
(3.33) & \quad Z(B_d^{11}) \subset (\mathfrak{a}_d^{01})^{-1}R(B_d^{00*}), \\
(3.34) & \quad Z(B_d^{00}) \subset (\mathfrak{a}_d^{01*})^{-1}R(B_d^{11*}).
\end{align*}

Here, the right sides in (3.29) and (3.30) are identical, by Lemma 1.11. When (3.29) holds, we already have

$$B^{10}\beta^0u + B^{11}\beta^1u = 0 \iff \beta^1u + C^{11}B^{10}\beta^0u \in Z(B^{11})$$

$$\iff \beta^1u + C^{11}B^{10}\beta^0u \in Z((1 - C^{00}B^{00})^*\mathfrak{a}^{01})$$

$$\iff (1 - C^{00}B^{00})^*\mathfrak{a}^{01}\beta^1u + (1 - C^{00}B^{00})^*\mathfrak{a}^{01}C^{11}B^{10}\beta^0u = 0,$$

showing that our boundary condition implies a condition on the reduced Cauchy data. A more precise statement will be obtained in Theorem 3.11 below.

We shall now show how Theorems 2.11 and 2.13 may be generalized.

**Theorem 3.7.** — Assume that $\Gamma$ is noncharacteristic for $A$. Then (3.26) implies that $Z_j = \ker B_{jj}$ is a subbundle of $(\mathfrak{a}_{2m-1-j,j})^{-1}B_{2m-1-j}F_{2m-1-j}$ for $j \in M_1$, resp. is isomorphic to a subbundle of $F_{2m-1-j}$ for $j \in M_0$. In particular,

$$\sum_{j \in M} p_j \geq \sum_{k \in M_0} q_k \quad [= \bar{m}].$$
When furthermore $\sum_{j \in M} p_j = \sum_{k \in M_0} q_k$,

\begin{align*}
(3.36) & \quad Z(B^{11}) = (\tilde{\alpha}_d^{01})^{-1} R(B_d^{00*}), \\
(3.37) & \quad Z(B_d^{00}) = (\tilde{\alpha}_d^{01*})^{-1} R(B_d^{11});
\end{align*}

so $Z_j = (\tilde{\alpha}_{2m-1-j}^{01})^{-1} B_{2m-1-j, 2m-1-j}^{*} F_{2m-1-j}$ for $j \in M_1$, and $Z_j = (\tilde{\alpha}_j^{*})^{-1} B_{2m-1-j, 2m-1-j}^{*} F_{2m-1-j} \cong F_{2m-1-j}$ for $j \in M_0$.

Proof. — When $\Gamma$ is noncharacteristic, then $\tilde{\alpha}_d^{01}$ is a surjective morphism from $\bigoplus_{k \in M_1} E_k$ (of dimension $mq$) to $\bigoplus_{j \in M_0} E_j$ (of dimension $\sum q_j$), and $\tilde{\alpha}_d^{01*}$ is an injective morphism in the other direction. Then (3.33) resp. (3.34) imply the first statement of the theorem; in particular we have for the dimensions in (3.33) resp. (3.34)

\begin{align*}
(3.38) & \quad \sum_{j \in M_1} (q - p_j) + mq - \sum_{j \in M_0} q_j + \sum_{j \in M_0} p_j; \\
(3.39) & \quad \sum_{j \in M_0} (q - p_j) \leq \sum_{j \in M_1} p_j.
\end{align*}

Each of these inequalities is equivalent with (3.35). Now assume there is identity in (3.35) and thus in (3.38) and (3.39). Then (3.33) resp. (3.34) are inclusions between vector bundles of the same dimension, so they are identities, and we have proved (3.36) and (3.37).

To prove the analogue of Theorem 2.13 we shall first extend some considerations from section 2.3. Denote, for $\varepsilon = 0, 1$, $E^\varepsilon = \bigoplus_{k \in M_1} E_k$, $Z^\varepsilon = \bigoplus_{k \in M_1} Z_k$ and $F^\varepsilon = \bigoplus_{k \in M_1} F_k$. Then $I - C_d^{00}B_d^{00}$ defines the orthogonal projection of $E^0$ onto $Z^0$, denote it $\text{pr}_{Z^0}$.

**Proposition 3.8.** — Define the system $B^{11} = (B_{jk})_{j,k \in M_1}$ of differential operators $B_{jk}$ from $E_k$ into $Z_{2m-1-j}$, of type $(-k, -j)_{j,k \in M_1}$, by

\begin{equation}
(3.40) \quad B^{11} = I^\times \text{pr}_{Z^0}(I - B_{00}^{00*}C_{00}^{00*})\tilde{\alpha}_d^{01},
\end{equation}

its diagonal part $I^\times \text{pr}_{Z^0}\tilde{\alpha}_d^{01}$ is surjective when $\Gamma$ is noncharacteristic for $A$. Then

\begin{equation}
(3.41) \quad (\tilde{\alpha}_d^{01})^{-1} R(B_d^{00*}) = Z((I - C_d^{00}B_d^{00})^*\tilde{\alpha}_d^{01}) = Z(B^{11}).
\end{equation}
Proof. — A direct generalization of the existence proof in section 2.3.

Theorem 3.9. — Assume that $\Gamma$ is noncharacteristic for $\Lambda$, and that $\sum_{j \in M} p_j = \overline{m}$. Then (3.26) is equivalent with each of the following statements (3.42)-(3.44)

(3.42) $Z(B^{11}) \subseteq Z(\overline{B}^{11})$,
(3.43) $Z(B^{11}) = Z(\overline{B}^{11})$,
(3.44) $Z(B^{11}) \supseteq Z(\overline{B}^{11})$,

for $\overline{B}^{11}$ defined in Proposition 3.8.

Proof. — In view of (3.41) and Lemma 3.6, (3.26) is equivalent with (3.42). Since (3.43) implies (3.42) and (3.44), we have to show (3.42) $\implies$ (3.43), and (3.44) $\implies$ (3.43). By use of Theorem 3.7 we find, assuming (3.42),

(3.45) $Z(B^{11}_d) = (\tilde{\alpha}_d^{\circ})^{-1} R(B_d^{00}) \\
\quad = Z((I - C_d^{00} B_d^{00}) \tilde{\alpha}_d^{\circ}) = Z(\overline{B}^{11}_d)$.

Then (3.42) implies, by use of Lemma 1.12,

$Z(\overline{B}^{11}) \supseteq Z(B^{11}) = (I - C^{11} B^{11}_d) Z(\overline{B}^{11}_d) \\
\quad = (I - C^{11} B^{11}_d) Z(B^{11}_d) = (I - C_d^{11} B_d^{11})(I + C_d^{11} B_d^{11}) Z(\overline{B}^{11}) \\
\quad = (I + K) Z(\overline{B}^{11})$,

where $K$ is subtriangular. Now the argument in the proof of Theorem 2.13 applies, showing that the inclusion must be the identity, and we have proved (3.43).

(3.44) implies (3.43) in a similar way.

Part of Corollary 2.14 is immediately generalized (and the remaining part will come out as a corollary at the end of section 3.4):

Corollary 3.10. — Assumptions of Theorem 3.9. (3.26) is equivalent with (3.46) and (3.47)

(3.46) $Z(B^{00}) \subseteq (\tilde{\alpha}^{01*})^{-1} R(B^{11*})$
(3.47) $Z(B^{00}) = (\tilde{\alpha}^{01*})^{-1} R(B^{11*})$. 
With Theorem 3.9 we can now show that weakly semibounded realizations represent boundary conditions on the reduced Cauchy data.

**Theorem 3.11.** — Assume that $\Gamma$ is noncharacteristic for $A$, and that $A_B$ is the realization of a normal boundary condition

\[ (3.48) \quad B^0 u = 0, \]

with $\sum_{j \in M} p_j = \bar{m}$. If $A_B$ is weakly semibounded, then there exists a differential operator $\hat{B}^{11} = (\hat{B}_{jk})_{j \in M, k \in M_0}$ from $\bigoplus_{k \in M_0} E_k$ to $\bigoplus_{j \in M} F_j$, of type $(-2m + 1 + k, - j)_{j \in M, k \in M_0}$ and with surjective second-diagonal part, such that

\[ (3.49) \quad B^{11} = \hat{B}^{11} \tilde{\alpha}^{01}. \]

Hereby (3.48) is equivalent with

\[ (3.50) \quad B^{00} \beta^0 u = 0, \quad B^{10} \beta^0 u + \hat{B}^{11} \kappa u = 0. \]

**Proof.** — By Theorem 3.9, we have

\[ Z(B^{11}) = Z(\bar{B}^{11}); \]

moreover, we have by Theorem 3.7 that $F^1 = \Phi I^X Z^0$, with $I^X = (\delta_{j,2m-1-k})_{j \in M, k \in M_0}$ and $\Phi$ a diagonal vector bundle isomorphism. Now

\[ \mathcal{D}'(E^1) = Z(\bar{B}^{11}) + R(\bar{C}^{11}) \]

so that, using the argument in section 2.3,

\[ (3.51) \quad B^{11} = B^{11} \bar{C}^{11} \bar{B}^{11} \]

\[ = B^{11} \bar{C}^{11} I^X \text{pr}_{Z^0} (I - B^{00}_* C^{00}* ) \tilde{\alpha}^{01} = \hat{B}^{11} \tilde{\alpha}^{01}, \]

where $\hat{B}^{11} = B^{11} \bar{C}^{11} I^X \text{pr}_{Z^0} (I - B^{00}_* C^{00}* )$ is a differential operator from $E^0$ to $F^1$ of type

\[ (-2m + 1 + k, -1)_{j \in M, k \in M_0}. \]

Note that

\[ (3.52) \quad \hat{B}^{11} = \Phi \Psi I^X \text{pr}_{Z^0} (I - B^{00}_* C^{00} * ), \]

where $\Psi = \Phi^{-1} B^{11} \bar{C}^{11}$ is a bijective differential operator in $\bigoplus_{k \in M, k \in M_0}$ of type $(-k, - j)_{j, k \in M_0}$, so that by Lemma 1.13
its diagonal part is a vector bundle isomorphism. Thus the second-diagonal part of $\hat{B}^{11}$,

$$\hat{B}^{11}_d = \Phi \Psi_d \mathcal{I}^x \text{pr}_Z,$$

is a surjective morphism.

This proof not only gives the existence of $\hat{B}^{11}$, it also serves to discuss the characterization of $B^{11}$, when $B^{00}$ is given and $B$ shall satisfy (3.26). For we then have by (3.51)-(3.52)

$$B^{11} = \Phi \Psi \overline{B}^{11};$$

and on the other hand, any such operator (with $\Phi$ a diagonal vector bundle isomorphism and $\Psi$ a bijective differential operator in $\bigoplus_{k \in M_1} Z_{2m-1-k}$ of type $(-k, -j)$) satisfies $Z(B^{11}) = Z(\overline{B}^{11})$. So Theorem 2.15 carries over word for word.

**Theorem 3.12.** — The complete analogue of Theorem 2.15 holds.

### 3.4. The adjoint boundary condition.

Define the formally adjoint realization $(A_B)'$ as the operator sending $\nu$ into $A'\nu$ and with domain

$$(3.53) \quad D((A_B)') = \{\nu \in \prod_{i \in N} H^{m_i+m_i}(\overline{\Omega}) | (A u, \nu) - (u, A' \nu) = 0 \text{ for all } u \in D(A_B)\}.$$  

Like in section 2.4, we easily find that

$$D((A_B)') = \{\nu \in \prod_{i \in N} H^{m_i+m_i}(\overline{\Omega}) | (I - CB)^* c^* \beta \delta \nu = 0 \}$$

so defining

$$(3.54) \quad B' = \mathcal{I}^x \text{pr}_Z (I - B^* C^*) \tilde{\alpha}^*$$

we have

$$(3.55) \quad D((A_B)') = D(A_B).$$

However, $B'$ need not be normal; in fact (cf. (2.47))

$$B' = \begin{bmatrix} B^{00} & 0 \\ B'^{10} & B'^{11} \end{bmatrix}. $$
where
\[
B^{00} = I \times \text{pr}_Z(I - B_{11}^0 C_{11}^0)\tilde{c}_{01}^0;
\]
\[
B'^{11} = I \times \text{pr}_Z(I - B_{20}^0 C_{00}^0)\tilde{c}_{10}^0,
\]
where we can only be sure that $B'^{11}$ has surjective diagonal part (when $\Gamma$ is noncharacteristic), cf. Theorem 3.1. But we have in any case, as in section 2.4,
\[
Z(B'^{00}) = Z((I - C_{11}^{11} B_{11}^{11})^* \tilde{c}_{01}^0) = (\tilde{c}_{01}^0)^{-1} R(B^{11}^*),
\]
which can be applied to Lemma 3.6 and Corollary 3.10, giving

**Proposition 3.13.** — $A_B$ is weakly semibounded if and only if
\[
\tag{3.56}
Z^{2m}(B^{00}) \subseteq Z^{2m}(B'^{00}),
\]
that is, if and only if $\beta^0 D(A_B) \subseteq \beta^0 D(A'_B')$. When $\Gamma$ is noncharacteristic and $\sum_{j \in M} p_j = \bar{m}$, (3.56) is equivalent with
\[
\tag{3.57}
Z^{2m}(B^{00}) = Z^{2m}(B'^{00}),
\]
i.e., $\beta^0 D(A_B) = \beta^0 D(A'_B')$.

Now $A_B$ and $A'_B'$ are no longer analogous, so the trick of Corollary 2.20 cannot be applied to prove weak semiboundedness of $A'_B'$. (Computations seem unmanageable.) We shall circumvent this by using that we are in fact dealing with boundary conditions on the reduced Cauchy data $\{\beta^0 u, \chi u\}$ (boundary conditions where $B^{11}$ factors through $\tilde{c}_{01}$), in view of Theorem 3.11. For such conditions, it is easy to repeat the whole theory in a simpler version based on the Green's formula
\[
(Au, \nu) = a(u, \nu) + \langle \chi u, \beta^0 \nu \rangle + \langle \mathcal{E} \beta^0 u, \beta^0 \nu \rangle,
\]
cf. (3.20). This gives

**Theorem 3.14.** — Assume that $\Gamma$ is noncharacteristic for $A$. Let $A_B$ be the realization defined by a boundary condition
\[
\tag{3.58}
B^{00} \beta^0 u = 0, \quad B^{10} \beta^0 u + \tilde{B}^{11} \chi u = 0,
\]
where the differential operators $B^{00}$, $B^{10}$ and $\tilde{B}^{11}$ go from
\[ \bigoplus_{k \in M_0} E_k \text{ to } \bigoplus_{j \in M_1} F_j \text{ and } \bigoplus_{j \in M_1} F_j \text{ (respectively), and are of types } (-k, -j), k \in M_0, \text{ and } (-k, -j), j \in M_1, \text{ respectively. Assume that the boundary condition is normal in the sense that the diagonal part } B_{d^{00}}^0 \text{ and the second-diagonal part } B_{d^{11}}^1 \text{ are surjective morphisms, and assume that } \sum_{j \in M} p_j = m. \text{ Let } \widetilde{I^Xe^{11}} \text{ denote the right inverse of } \tilde{B}_{d^{11}}^1 \text{ according to Proposition 1.9. Then } A_B \text{ is weakly semibounded if and only if each of the following equivalent conditions hold:}

\begin{align*}
(I - C^{00}B^{00})^* & (I - \tilde{C}^{11}\tilde{B}^{11}) = 0; \\
Z(\tilde{B}^{11}) & \subseteq R(B^{00}); \\
Z(\tilde{B}^{11}) & = R(B^{00}); \\
\tilde{B}^{11}B^{00} & = 0; \\
Z(B^{00}) & \subseteq R(\tilde{B}^{11}); \\
Z(B^{00}) & = R(\tilde{B}^{11}); \\
Z(B^{00}) & \supseteq R(\tilde{B}^{11}).
\end{align*}

It is used in the proof that (3.58) is equivalent with

\begin{align*}
\beta^0 u & \in R(I - C^{00}B^{00}); \\
xu + \tilde{C}^{11}B^{10}\beta^0 u & \in R(I - \tilde{C}^{11}\tilde{B}^{11}).
\end{align*}

By Green's formula (3.17):

\[(Au, \nu) - (u, A'\nu) = \langle xu, \beta^0\nu \rangle - \langle \beta^0 u, x'\nu \rangle\]

we now see, using (3.66), that \( \nu \in \mathcal{D}((A_B)^*) \) if and only if

\[0 = \langle (I - \tilde{C}^{11}\tilde{B}^{11})\varphi_1 - \tilde{C}^{11}B^{10}(I - C^{00}B^{00})\varphi_0, \beta^0\nu \rangle - \langle (I - C^{00}B^{00})\varphi_0, x'\nu \rangle\]

for all smooth \( \varphi_0, \varphi_1 \), i.e. if and only if

\[ (I - \tilde{C}^{11}\tilde{B}^{11})^*\beta^0\nu = 0 \]

\[ (I - C^{00}B^{00})^*B^{10}\star\tilde{C}^{11}\star\beta^0\nu + (I - C^{00}B^{00})^\star x'\nu = 0. \]

This may be reformulated by the usual technique to a normal boundary condition

\[ \tilde{B}^{100}\beta^0\nu = 0, \quad \tilde{B}^{110}\beta^0\nu + \tilde{B}^{111}x'\nu = 0, \]
where $B'_{00}$, $B'_{10}$ and $B'_{11}$ go from $\bigoplus_{k \in \mathcal{M}_0} E_k$ to $\bar{F}_0$, $\bar{F}_1$ and $\bar{F}_1$ (respectively), of types as in Theorem 3.14, $B'_{00}$ and $B'_{11}$ having surjective diagonal resp. second-diagonal part; here $\bar{F}_0 = \bigoplus_{j \in \mathcal{M}_0} \ker B_{2m-1-j, j}$ and $\bar{F}_1 = \bigoplus_{j \in \mathcal{M}_1} \ker B_{2m-1-j, 2m-1-j}$. So $(A_B)' = A'_B$, the realization of $A'$ defined by the boundary condition (3.67). Moreover, $A_B$ is the formally adjoint realization to $A'_B$ in the analogous way. Observing that

$$Z^m(B'_{00}) = \rho_0 D((A_B)') = Z^m((I - \tilde{C}_{11}B_{11}^*)) = R^m(B_{11}^*),$$

we may write (3.64) as

$$(3.68) \quad Z^m(B'_{00}) = Z^m(B'_{00});$$

and since $A_B$ and $(A_B)' = A'_B$, now enter in a symmetric way, we can conclude that (3.68) must also imply weak semi-boundedness of $(A_B)'$. So we have proved

**Theorem 3.15.** — When $\Gamma$ is noncharacteristic for $A$ and $\sum_{j \in \mathcal{M}} p_j = \bar{m}$, then $A_B$ is weakly semibounded if and only if $(A_B)'$ is weakly semibounded.

The formulation analogous to Corollary 2.20 is also valid. Let us finally note that the adjoint equation to (3.62),

$$(3.69) \quad B'^{00}B_{11}^* = 0,$$

and its equivalent statement

$$(3.70) \quad Z(B'^{00}) \supset R(B_{11}^*),$$

provide the analogues of the last two statements in Corollary 2.14: When $B_{11} = \tilde{B}_{11}\tilde{\alpha}_{01}^*$ as in Theorem 3.11, then $B_{11}^* = \tilde{\alpha}_{01}^*B_{11}^*$, where $\tilde{\alpha}_{01}^*$ is invertible, so (3.70) and (3.69) are equivalent with

$$(3.71) \quad Z(B'^{00}) \supset (\tilde{\alpha}_{01}^*)^{-1}R(B_{11}^*),$$

resp.,

$$(3.72) \quad B'^{00}(\tilde{\alpha}_{01}^*)^{-1}B_{11}^* = 0,$$

the perhaps simplest version of (3.26). Corollary 3.10 can now be completed with (3.71) and (3.72).
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