ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 23, nº 3 (1973), p. 125-154
<http://www.numdam.org/item?id=AIF_1973__23_3_125_0>

Ann. Inst. Fourier, Grenoble
23,3 (1973), 125-154

SPECTRAL SYNTHESIS AND THE POMPEIU PROBLEM

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1. Introduction.

In 1929 D. Pompeiu posed the following problem, which later became known in the literature as the *Pompeiu problem.* Let D be a bounded region in the xy-plane, and let Σ denote the group of all rigid motions of the plane. Suppose that f is a continuous function on the plane satisfying

$$\int_{\sigma(D)} \int f(x, y)\, dx\, dy = 0\ , \qquad \sigma \in \Sigma\ . \tag{1}$$

Does this imply $f(x, y) \equiv 0$? In his first paper on the subject [13], Pompeiu claimed that the answer is yes when D is a disk. However, this was later shown to be false ; in fact, the function $f(x, y) = \sin ax$, for a suitable choice of a, provides a counter-example. (See also Theorem 4.3 below.) Nevertheless, he did prove in [14] that the answer is yes if D is a square, under the further assumption that f tends to a limit at infinity. This restriction was later removed by C. Christov, who also showed that the answer is yes if D is a triangle or a parallelogram [2], [3]. Finally, a more abstract formulation of the problem was considered by L. Brown, F. Schnitzer and A. Shields in [1].

The Pompeiu problem may be phrased as follows. For which regions D is it true that (1) implies $f \equiv 0$? In this paper we present a solution to this problem, in terms of the Fourier-Laplace transform of the Lebesgue area measure on D (Theorem 4.1). Our results imply,

(*) Research supported in part by the National Science Foundation under grant number GP-20150.

in particular, that the answer to the Pompeiu problem is yes when D is any polygonal region (Theorem 5.9) or any convex set with at least one "corner" (Corollary 5.12).

Studying the Pompeiu problem leads to another problem of a similar nature, whose solution also appears in Theorem 4.1. We shall refer to this as the *Morera problem* because of its relationship to the classical theorem of Morera. Let $\{\Gamma\}$ denote a collection of rectifiable closed curves in the plane, and let f be continuous on $\mathbf{R}^2$. Suppose that, setting $\zeta = x + iy$,

$$\int_{\sigma(\mathrm{R})} f(\zeta)\, d\zeta = 0 \,, \qquad \sigma \in \Sigma,\ \Gamma \in \{\Gamma\} \,. \tag{2}$$

Does this imply that f is an entire holomorphic function of ζ ? Again, this implication is valid for some choices of $\{\Gamma\}$ and not for others. In [19] L. Zalcman observed that when D is a plane region whose boundary Γ is a rectifiable curve, then an affirmative answer to the Pompeiu problem for D implies, via Green's Theorem, an affirmative solution to the Morera problem for Γ. Our methods enable us to show that the converse is also true (Theorem 4.2). In addition, Zalcman gives, by methods similar to those of this paper, a solution to the Morera problem in the following interesting case. Let r_1 and r_2 be positive real numbers, and let $\{\Gamma\}$ consist of two circles of radius r_1 and r_2, respectively. Then (2) implies f is entire if and only if r_1/r_2 does not belong to a certain countable set (which may be identified). "Two circle theorems" of this general type were first considered by J. Delsarte [5], [6]. We refer the reader to [19] for an excellent account of the history of the Pompeiu and Morera problems and the related two circle theorems, as well as for further results along these lines.

The proof of our main result is based on the fundamental theorem of L. Schwartz on mean periodic functions of one variable. This seems to be a common ingredient in much of the work on these problems, particularly in the two circle theorems of Delsarte and Zalcman. In § 2 the necessary background material from the theory of distributions and mean periodic functions is assembled. Although all of these results are known, some of them do not seem to be readily available in the literature. Our main theorem (Theorem 3.1) is then proved in § 3 ; namely, it is shown that every closed, translation-invariant and rotation-invariant subspace of the space of continuous functions on $\mathbf{R}^2$ is spanned by the polynomial-exponential functions it contains. The proof is carried out by means of a sequence of reductions which finally allow an application of the theorem of L. Schwartz.

Theorem 4.1 follows easily from the results of § 3 ; the remainder of § 4 consists primarily of applications of this theorem. In particular, the two circle theorems of Delsarte and Zalcman are proved. § 5 is devoted to finding explicit examples of regions and curves which give affirmative answers to the Pompeiu and Morera problems, respectively. The paper concludes with § 6, where we indicate how much simpler it is to solve the Pompeiu problem if in (1) it is assumed that f is bounded.

2. Spectral synthesis in $\mathcal{E}(\mathbf{R}^n)$ and $C(\mathbf{R}^n)$.

Let $\mathcal{E}(\mathbf{R}^n)$ denote the space of all infinitely differentiable functions on $\mathbf{R}^n$ with its usual topology, and $\mathcal{E}'(\mathbf{R}^n)$ its dual space, the space of distributions with compact support in $\mathbf{R}^n$ ([9], [18]). The pairing between $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{E}'(\mathbf{R}^n)$ is denoted by $T(f)$ for $f \in \mathcal{E}(\mathbf{R}^n)$ and $T \in \mathcal{E}'(\mathbf{R}^n)$, and for such f and T we denote by $T * f$ the convolution of T and f as an element of $\mathcal{E}(\mathbf{R}^n)$. The Borel measures μ with compact support in $\mathbf{R}^n$ will be identified with a subspace of $\mathcal{E}'(\mathbf{R}^n)$ via the formula $\mu(f) = \int_{\mathbf{R}^n} f(t)\, d\mu(t)$.

We shall make extensive use of the theory of Fourier-Laplace transforms of elements of $\mathcal{E}'(\mathbf{R}^n)$, so we include here a brief summary of these results. For details, see [18] or [9, Chapter 1, § 7]. For $T \in \mathcal{E}'(\mathbf{R}^n)$, the Fourier-Laplace transform of T is defined by

$$\hat{T}(z) = T(e^{iz.x}) \ ,$$

where $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $z \cdot x = z_1 x_1 + \cdots + z_n x_n$. The Paley-Wiener-Schwartz Theorem [9, p. 21] identifies the space $\hat{\mathcal{E}}'(\mathbf{R}^n) = \{\hat{T} : T \in \mathcal{E}'(\mathbf{R}^n)\}$ as the space of all entire functions F on $\mathbf{C}^n$ such that

$$|F(z)| \leqslant C(1 + |z|)^N e^{A|Im\, z|} \ , \qquad z \in \mathbf{C}^n, \tag{3}$$

for some constants A, C and N. Here $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ and $\mathrm{Im}\, z = (\mathrm{Im}\, z_1, \ldots, \mathrm{Im}\, z_n)$ is the imaginary part of z. $\hat{\mathcal{E}}'(\mathbf{R}^n)$ is equipped with the topology which makes the mapping $\mathrm{T} \to \hat{\mathrm{T}}$ a topological isomorphism when $\mathcal{E}'(\mathbf{R}^n)$ is given the strong topology as the dual of $\mathcal{E}(\mathbf{R}^n)$. Convergent sequences in $\hat{\mathcal{E}}'(\mathbf{R}^n)$ may be characterized as follows.

PROPOSITION 2.1. – ([7, Lemma 5.17]) *A sequence* $\{\mathrm{F}_j\}$ *in* $\hat{\mathcal{E}}'(\mathbf{R}^n)$ *converges to* $\mathrm{F} \in \hat{\mathcal{E}}'(\mathbf{R}^n)$ *if and only if*

i) $\mathrm{F}_j \to \mathrm{F}$ *uniformly on compact subsets of* $\mathbf{C}^n$, *and*

ii) *The inequality (3) holds for all the* F_j *with constants* A, C *and* N *independent of* j.

For further details on the topology of $\hat{\mathcal{E}}'(\mathbf{R}^n)$ we refer the reader to [7, Chapter 5, § V].

The main result we shall use concerning $\hat{\mathcal{E}}'$ is the following one, known as the "fundamental theorem of mean-periodic functions."

THEOREM 2.2. (L. Schwartz [17]). – *Let* I *be an ideal in* $\hat{\mathcal{E}}'(\mathbf{R})$ *whose functions have no common zeros in* $\mathbf{C}$. *Then there exists a sequence of functions in* I *converging to the constant function* 1.

Let $\mathrm{C}(\mathbf{R}^n)$ denote the space of all continuous functions on $\mathbf{R}^n$ with the usual topology of uniform convergence on compact sets. Although the Pompeiu and Morera problems were formulated in § 1 in terms of continuous functions, there is no difference, at least from the point of view taken here, if these problems are phrased in terms of infinitely differentiable functions. Similarly, the problem of spectral synthesis, which we shall discuss presently, may be phrased equivalently in either $\mathrm{C}(\mathbf{R}^n)$ or $\mathcal{E}(\mathbf{R}^n)$. This is all a consequence of the following well-known proposition, which is proved by the standard smoothing procedure.

PROPOSITION 2.3. – *Let* V *be a closed, translation-invariant subspace of* $\mathrm{C}(\mathbf{R}^n)$, *and let* $\mathrm{V}_1 = \mathrm{V} \cap \mathcal{E}(\mathbf{R}^n)$. *Then* V_1 *is dense in* V *in the topology of* $\mathrm{C}(\mathbf{R}^n)$.

Let us now consider the problem of spectral synthesis in $\mathcal{E}(\mathbf{R}^n)$, following [17]. By a polynomial-exponential function we mean a function of the form $f(x) = p(x)\, e^{iz.x}$, $x \in \mathbf{R}^n$, where p is a polynomial and $z \in \mathbf{C}^n$. For V a closed, translation-invariant subspace of $\mathcal{E}(\mathbf{R}^n)$, let V_0 denote the closed linear space spanned by the polynomial-exponential functions in V. Thus $V_0 \subset V$. The problem of *spectral synthesis* is the question, "Must $V_0 = V$?"

The problem may be translated to a question about closed ideals in the ring $\hat{\mathcal{E}}'(\mathbf{R}^n)$ as follows. Given V and V_0 as above, let

$$V^\perp (\text{resp. } V_0^\perp) = \{T \in \mathcal{E}'(\mathbf{R}^n) : T(f) = 0, f \in V \text{ (resp., } f \in V_0)\}$$

and

$$I (\text{resp. } I_0) = \{\hat{T} : T \in V^\perp \text{ (resp. } T \in V_0^\perp)\} .$$

The following proposition, then, represents a rephrasing of the problem of spectral synthesis.

PROPOSITION 2.4. – I *and* I_0 *are closed ideals in* $\hat{\mathcal{E}}'(\mathbf{R}^n)$. *Moreover,* $I = I_0$ *if and only if* $V = V_0$.

Proof. – Clearly I is a closed subspace of $\hat{\mathcal{E}}'(\mathbf{R}^n)$. To prove I is an ideal, it suffices to show that, for every $x \in \mathbf{R}^n$ and $\hat{T} \in I$, the function $f(z) = e^{iz \cdot x}\, \hat{T}(z)$ belongs to I, since the exponential functions have dense linear span in $\hat{\mathcal{E}}'(\mathbf{R}^n)$. However, f is the Fourier transform of T_{-x}, the translate of T by $(-x)$. Since V is translation invariant, so is $V^\perp$, whence $f \in I$. Similarly, I_0 is a closed ideal in $\hat{\mathcal{E}}'(\mathbf{R}^n)$.

The last part of the proposition is a standard application of the Hahn-Banach Theorem. For $I = I_0$ if and only if $V^\perp = V_0^\perp$, which happens precisely when $V = V_0$, since V and V_0 are closed.

The ideal I_0 can be given a more precise description. For I an ideal in $\hat{\mathcal{E}}'(\mathbf{R}^n)$, let I_{loc} denote the ideal of all functions in $\hat{\mathcal{E}}'(\mathbf{R}^n)$ which belong to I locally. That is, $F \in I_{loc}$ if and only if for each $z_0 \in \mathbf{C}^n$ there is a neighborhood U of z_0, functions $F_1, \dots, F_m$ in I, and functions $G_1, \dots, G_m$ holomorphic on U, such that

$$F(z) = G_1(z)\, F_1(z) + \cdots + G_m(z)\, F_m(z), \qquad z \in U .$$

PROPOSITION 2.5. – $I_0 = I_{loc}$.

This proposition is well known, although we have been unable to find an exact statement of it in the literature. The inclusion $I_{loc} \subset I_0$ is not hard to verify. The other inclusion is more difficult, but the argument given in [10, pp. 199-200 (Steps 1 and 2)] or [12, pp. 282-

284] can be carried over with only minor changes. For the sake of completeness we include a proof.

Proof. – (1) $I_{loc} \subset I_0$. Let $\hat{T} \in I_{loc}$ and let $f(x) = p(x) e^{iz_0 \cdot x}$ belong to V_0. We must show $T(f) = 0$. There exist a neighborhood U of z_0, functions $\hat{T}_1, \ldots, \hat{T}_m$ in I, and functions $G_1, \ldots, G_m$ holomorphic on U, such that on U we have

$$\hat{T} = G_1 \hat{T}_1 + \cdots + G_m \hat{T}_m . \tag{4}$$

As usual, for any multi-order $\alpha = (\alpha_1, \ldots, \alpha_n)$ let D^α denote the differential operator

$$D_x^\alpha = i^{-(\alpha_1 + \cdots + \alpha_n)} \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} ,$$

let $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and for a polynomial $p(x) = \Sigma\, a_\alpha x^\alpha$ let

$$p(D) = \Sigma\, a_\alpha D^\alpha .$$

Then if f is as above we have

$$T(f) = p(D_z) [T(e^{iz \cdot x})]_{z=z_0} = p(D) [\hat{T}(z)]_{z=z_0} ; \tag{5}$$

hence we must prove that

$$p(D_z) [\hat{T}(z)]_{z=z_0} = 0 . \tag{6}$$

Now, for every $y \in \mathbf{R}^n$ $f_y \in V_0$, where

$$f_y(x) = f(x+y) = e^{iz_0 \cdot y} p(x+y) e^{iz_0 \cdot x} , \qquad x \in \mathbf{R}^n .$$

Since every derivative of p is a limit in $\mathcal{E}(\mathbf{R}^n)$ of linear combinations of translates of p, it follows that $p^{(\alpha)}(x) e^{iz_0 \cdot x}$ belongs to V_0 for every α, where

$$p^{(\alpha)}(x) = i^{\alpha_1 + \cdots + \alpha_n} D^\alpha p(x) , \qquad x \in \mathbf{R}^n .$$

From the general Leibnitz formula [9, p. 10] we have for $1 \leqslant i \leqslant m$,

$$p(D) [G_i(z) \hat{T}_i(z)]_{z=z_0} = \Big(\sum_\alpha \frac{1}{\alpha!} D^\alpha G_i(z) p^{(\alpha)}(D) [\hat{T}_i(z)] \Big)_{z=z_0} . \tag{7}$$

But since $T_i \in V^\perp \subset V_0^\perp$, (5) with T replaced by T_i and p by $p^{(\alpha)}$ gives for all α,

$$p^{(\alpha)}(\mathrm{D})\,[\hat{\mathrm{T}}_i(z)]_{z=z_0} = 0\ , \qquad 1 \leqslant i \leqslant m\ .$$

Hence the left-hand side of (7) is zero for all i. Since (4) holds on U, we obtain (6).

(2) $\mathrm{I}_0 \subset \mathrm{I}_{loc}$. Let $\hat{\mathrm{T}} \in \mathrm{I}_0$ and fix $z_0 \in \mathrm{C}^n$. We must find a neighborhood U of z_0, functions $\hat{\mathrm{T}}_1, \ldots, \hat{\mathrm{T}}_m$ in I and functions $\mathrm{G}_1, \ldots, \mathrm{G}_m$ holomorphic on U such that (4) holds on U. By [10, Corollary 6.3.6, p. 153] it suffices to find formal power series $\mathrm{G}_1, \ldots, \mathrm{G}_m$ in $z - z_0$ such that (4) holds for some $\hat{\mathrm{T}}_1, \ldots, \hat{\mathrm{T}}_m \in \mathrm{I}$.

Consider the ideal I_{z_0} generated by I in the ring A_{z_0} of germs of functions holomorphic near z_0. I_{z_0} is finitely generated, since A_{z_0} is Noetherian, say be the germs at z_0 of the functions $\hat{\mathrm{T}}_1, \ldots, \hat{\mathrm{T}}_m$ in I. We shall find formal power series $\mathrm{G}_1, \ldots, \mathrm{G}_m$ about z_0 such that (4) holds for this choice of the $\hat{\mathrm{T}}_i$.

Equation (4) in formal power series may be written as the system of equations in $\mathrm{G}_1, \ldots, \mathrm{G}_m$ given by

$$\mathrm{D}^\alpha [\hat{\mathrm{T}}]_{z=z_0} = \mathrm{D}^\alpha \left[\sum_{i=1}^{m} \mathrm{G}_i \hat{\mathrm{T}}_i \right]_{z=z_0} \qquad \text{for all } \alpha\ .$$

By [10, Lemma 6.3.7, p. 153) this system has a solution in formal power series $\mathrm{G}_1, \ldots, \mathrm{G}_m$ if it is *compatible.* The compatibility means that if q is any polynomial, then

$$q(\mathrm{D}) \left[\sum_{i=1}^{m} \mathrm{G}_i \hat{\mathrm{T}}_i \right]_{z=z_0} = 0 \text{ for all choices of the } \mathrm{G}_i \Rightarrow q(\mathrm{D})\,[\hat{\mathrm{T}}(z)]_{z=z_0} = 0\ . \tag{8}$$

Now, the hypothesis of (8) implies that $q(\mathrm{D})\,[\hat{\mathrm{S}}(z)]_{z=z_0} = 0$ for all $\hat{\mathrm{S}} \in \mathrm{I}$, since $\hat{\mathrm{T}}_1, \ldots, \hat{\mathrm{T}}_m$ generate I_{z_0}. Thus (5) and the definition of I imply that the function $q(x)\, e^{iz_0 \cdot x}$ belongs to V, whence to V_0. But $\mathrm{T} \in \mathrm{V}_0^\perp$, so (5) gives the conclusion of (8), as asserted.

Thus the problem of spectral synthesis in $\mathcal{E}(\mathbf{R}^n)$ may be reformulated as the problem of determining for which closed ideals I in $\hat{\mathcal{E}}'(\mathbf{R}^n)$ it is true that $\mathrm{I} = \mathrm{I}_{loc}$. In this framework the fundamental theorem of L. Schwartz cited earlier may be stated as follows.

THEOREM 2.2'. – *For every closed ideal* I *in* $\hat{\mathcal{E}}'(\mathbf{R})$, $\mathrm{I} = \mathrm{I}_{loc}$.

3. Ideals invariant under rotations.

By Theorem 2.2′ and Proposition 2.3 every closed, translation-invariant subspace of $C(\mathbf{R})$ or $\mathcal{E}(\mathbf{R})$ is spanned by the polynomial-exponential functions it contains. The analogue of Theorem 2.2′ is not known for $n > 1$. However, in this section we prove that $\mathrm{I} = \mathrm{I}_{loc}$ when $n > 1$ and I is rotation invariant.

Notation. – Let $\mathbf{T}$ denote the group of all rotations of $\mathbf{R}^2$ or $\mathbf{C}^2$, i.e., all transformations of the form

$$\tau = \tau_\theta : \quad \begin{aligned} z_1' &= z_1 \cos\theta + z_2 \sin\theta \\ z_2' &= - z_1 \sin\theta + z_2 \cos\theta \ , \qquad -\pi < \theta \leqslant \pi \ . \end{aligned}$$

Thus $\mathbf{T}$ is a subgroup of Σ, the group of rigid motions of $\mathbf{R}^2$.

THEOREM 3.1. – *Let* I *be a closed ideal in* $\hat{\mathcal{E}}'(\mathbf{R}^2)$ *such that* $f \circ \tau \in \mathrm{I}$ *whenever* $f \in \mathrm{I}$ *and* $\tau \in \mathbf{T}$. *Then* $\mathrm{I} = \mathrm{I}_{loc}$.

Remark. – The same theorem is true for $\hat{\mathcal{E}}'(\mathbf{R}^n)$, $n > 0$, if the group of rotations $\mathbf{T}$ is replaced by the group $SO(n)$ of orthogonal transformations of $\mathbf{R}^n$ with determinant $+1$. The applications of Theorem 3.1 in § 4 can also be modified accordingly.

The theorem will be proved by reducing it to Theorem 2.2. We first make some preliminary reductions.

Let

$$\mathrm{J} = \{ f \in \hat{\mathcal{E}}'(\mathbf{R}^2) : f\mathrm{I}_{loc} \subset \mathrm{I} \} \ .$$

It is routine to check that J is an ideal in $\hat{\mathcal{E}}'(\mathbf{R}^2)$, and J is closed since I is closed. Since I is invariant under the rotations τ, it follows that I_{loc} and J are also invariant under rotations.

LEMMA 3.2. – *If f is an entire function on $\mathbf{C}^2$ such that $f(\tau(w)) = 0$ for all $\tau \in \mathbf{T}$ and some fixed $w = (w_1, w_2) \in \mathbf{C}^2$, with $w_1^2 + w_2^2 \neq 0$, then*

$$f(z) = (z_1^2 + z_2^2 - w_1^2 - w_2^2)\, g(z) \ , \qquad z \in \mathbf{C}^2 \ , \qquad (9)$$

where g is an entire function on $\mathbf{C}^2$. Furthermore, if $f \in \hat{\mathcal{E}}'(\mathbf{R}^2)$ then so is g.

Proof. – The entire function of θ

$$h(\theta) = f(w_1 \cos\theta + w_2 \sin\theta, \ -w_1 \sin\theta + w_2 \cos\theta)$$

vanishes for all real θ by hypothesis. Hence $h(\theta) = 0$ for all complex θ, so $f(z) = 0$ on the set

$$\{(w_1 \cos\theta + w_2 \sin\theta, \ -w_1 \sin\theta + \cos\theta) : \theta \in \mathbf{C}\} = \\ = \{(z_1, z_2) : z_1^2 + z_2^2 = w_1^2 + w_2^2\} .$$

Now, if $w_1^2 + w_2^2 \neq 0$ then the function $p(z) = z_1^2 + z_2^2 - w_1^2 - w_2^2$ is an irreductible entire function on $\mathbf{C}^2$, whence $p(z)^{-1} f(z)$ is an entire function. Thus (9) must hold.

It is easy to check that when $f \in \hat{\mathcal{E}}'(\mathbf{R}^2)$ an estimate of the form (3) also holds for g.

LEMMA 3.3. – *The functions in* J *have no common zeros.*

Proof. – Let $w \in \mathbf{C}^2$. We shall find a function $g \in \mathrm{J}$ such that $g(w) \neq 0$. Suppose first that $w_1^2 + w_2^2 \neq 0$. Let $p(z)$ be as in the proof of Lemma 3.2 and let $n \geqslant 0$ be the largest integer such that $p(z)^n$ divides every $f \in \mathrm{I}$. Then $p(z)^n$ also divides every $h \in \mathrm{I}_{loc}$. That is, for all $h \in \mathrm{I}_{loc}$ there exists $\widetilde{h} \in \hat{\mathcal{E}}'(\mathbf{R}^2)$ such that $h(z) = p(z)^n \, \widetilde{h}(z)$. Choose $f \in \mathrm{I}$ such that $p(z)^{n+1}$ does not divide f, and write $f(z) = p(z)^n g(z)$ where $g \in \hat{\mathcal{E}}'(\mathbf{R}^2)$. Then for $h \in \mathrm{I}_{loc}$,

$$gh = f\widetilde{h} \in \mathrm{I} \ ;$$

hence $g \in \mathrm{J}$. Also, $g(\tau(w)) \neq 0$ for some $\tau \in \mathbf{T}$, for if not then $p(z)$ would divide g by Lemma 3.2. But then $p(z)^{n+1}$ would divide $f(z)$, contradicting the choice of f. If τ is chosen so that $g(\tau(w)) \neq 0$, then $g \circ \tau$ is a function in J not vanishing at w.

Now assume $w_1^2 + w_2^2 = 0$ but $w \neq 0$. If $w_1 = iw_2$, analogs of Lemma 3.2 and the argument above, with $p(z)$ replaced by the function $z_1 - iz_2$, imply that w is not a common zero of J. Similarly, if $w_1 = -iw_2$ one obtains the same result by replacing $p(z)$ by $z_1 + iz_2$.

Finally, we consider the case $w = 0$. Let j and k be the greatest integers such that $(z_1 + iz_2)^j$ and $(z_1 - iz_2)^k$ divide every function in I, and set

$$q(z) = (z_1 + iz_2)^j \, (z_1 - iz_2)^k \ .$$

Let

$$I_0 = q^{-1} I = \{q^{-1} f : f \in I\} .$$

It is a routine matter to check that I_0 is a closed ideal in $\hat{\mathcal{E}}'(\mathbf{R}^2)$ invariant under rotations and that $q(I_0)_{loc} = I_{loc}$.

We claim that if (0,0) is a common zero of I_0 it is an isolated common zero. Indeed, if w is a common zero of I_0 with $w_1^2 + w_2^2 = \alpha^2 \neq 0$, then $(\alpha, 0)$ is also a common zero of I_0 by Lemma 3.2. Hence, since the set of such numbers α clearly cannot have 0 as a limit point, it follows that if (0,0) is not an isolated common zero of I_0 there must exist a nonzero common zero w of I_0 such that $w_1^2 + w_2^2 = 0$. But since neither $z_1 + iz_2$ nor $z_1 - iz_2$ divides every function in I_0, by the analogs of Lemma 3.2 mentioned earlier, such a w cannot exist. Thus (0,0) is at worst an isolated common zero of I_0.

The proof of [11, Theorem 4.5], applied to the case of I_0 as an ideal in $\mathcal{E}'(\mathbf{R}^2)$, asserts that if (0,0) is an isolated common zero of I_0, then there exists $g \in \hat{\mathcal{E}}'(\mathbf{R}^2)$ such that $g(0,0) \neq 0$ and $g(I_0)_{loc} \subset I_0$. In fact, $g \in J$. For if $h \in I_{loc}$, then $h = q\widetilde{h}$ for some $\widetilde{h} \in (I_0)_{loc}$, so

$$gh = gq\widetilde{h} \in qI_0 = I .$$

Thus $g \in J$, and the proof is complete.

Proof of Theorem 3.1. – We shall show that $1 \in J$, which will complete the proof.

For $f \in \hat{\mathcal{E}}'(\mathbf{R}^2)$ define

$$\widetilde{f}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \circ \tau_\theta(z)\, d\theta , \qquad z \in \mathbf{C}^2 ,$$

and

$$\widetilde{J} = \{\widetilde{f} : f \in J\} .$$

Since J is rotation invariant and closed, we have $\widetilde{J} \subset J$. And $\widetilde{f} \circ \tau = \widetilde{f}$ for all $\widetilde{f} \in \widetilde{J}$ and $\tau \in \mathbf{T}$, so $\widetilde{f}$ is a function of $z_1^2 + z_2^2$. More precisely, there is an even entire function $F_f(\zeta)$ of one complex variable such that

$$\widetilde{f}(z_1, z_2) = F_f(\sqrt{z_1^2 + z_2^2}) ;$$

namely,

$$F_f(\zeta) = \widetilde{f}(\zeta, 0) = \widetilde{f}(0, \zeta), \qquad \zeta \in \mathbf{C} . \tag{10}$$

Since $\widetilde{f} \in \hat{\mathcal{E}}'(\mathbf{R}^2)$, (10) shows that $F_f \in \hat{\mathcal{E}}'(\mathbf{R})$.

Let

$$J_1 = \{F_f : f \in J\}.$$

Then, as one may easily check, J_1 is an ideal in the space of even functions in $\hat{\mathcal{E}}'(\mathbf{R})$. The functions in J_1 have no common zeros, as may be seen as follows. Since the functions in J have no common zeros, it follows that for every $R > 0$ and every $\varepsilon > 0$ there exists $f \in J$ such that $|f(z) - 1| < \varepsilon$ if $|z| \leqslant R$. Indeed, let A denote the uniform closure of the polynomials over the set $\Delta = \{z : |z| \leqslant R\}$. A is a commutative Banach algebra with identity, and it is well known that since Δ is convex, whence polynomially convex, the maximal ideal space of A may be identified with Δ in the canonical way [8, p. 67]. It is easy to see that the closure $\overline{J}$ of $\{f|\Delta : f \in J\}$ in A is an ideal in A contained in no maximal ideal. Thus $\overline{J} = A$; in particular, there is a function $f \in J$ such that $|f(z) - 1| < \varepsilon$, $z \in \Delta$. Then also $|\widetilde{f}(z) - 1| < \varepsilon$, $z \in \Delta$, so $|F_f(\zeta) - 1| < \varepsilon$ if $|\zeta| < R$. In particular, $\varepsilon < 1$ gives $F_f(\zeta) \neq 0$ if $|\zeta| < R$.

Finally we may apply Theorem 2.2 to the ideal generated by J_1 in $\hat{\mathcal{E}}'(\mathbf{R})$ to obtain a sequence $\{F_n\}$ in the ideal generated by J_1 converging to 1 in $\hat{\mathcal{E}}'(\mathbf{R})$. Each F_n must be of the form

$$\sum_{j=1}^{m} h_j(\zeta)\, G_j(\zeta) ,$$

where each $h_j \in \hat{\mathcal{E}}'(\mathbf{R})$ and $G_j \in J_1$. But then

$$\sum_{j=1}^{m} h_j(\zeta)\, G_j(\zeta) + \sum_{j=1}^{m} h_j(-\zeta)\, G_j(-\zeta) = \sum_{j=1}^{m} (h_j(\zeta) + h_j(-\zeta))\, G_j(\zeta)$$

belongs to J_1, since each G_j is even and J_1 is an ideal in the space of even functions in $\hat{\mathcal{E}}'(\mathbf{R})$. In other words, $H_n(\zeta) = \frac{1}{2}(F_n(\zeta) + F_n(-\zeta))$ belongs to J_1, and clearly $H_n \to 1$ in $\hat{\mathcal{E}}'(\mathbf{R})$. Thus, referring to Proposition 2.1, we see that there are functions $\widetilde{f}_n \in \widetilde{J}$ converging to 1 in $\hat{\mathcal{E}}'(\mathbf{R}^2)$. Since $\widetilde{J} \subset J$ and J is closed, we have $1 \in J$, which completes the proof.

Combining Theorem 3.1 with the discussion in § 2 we obtain

THEOREM 3.4. – *Every translation-invariant and rotation-invariant closed subspace of either* $C(\mathbf{R}^2)$ *or* $\mathcal{E}(\mathbf{R}^2)$ *is spanned by the polynomial-exponential functions it contains.*

4. The Pompeiu, Morera, and mean-value properties.

Let us say that a family $\mathcal{J} \subset \mathcal{E}'(\mathbf{R}^2)$ has the *Pompeiu property* if for $f \in \mathcal{E}(\mathbf{R}^2)$, $T(f \circ \sigma) = 0$ for all $T \in \mathcal{J}$ and $\sigma \in \Sigma$ implies $f = 0$. Thus the results of Pompeiu and Christov cited in § 1 state that area measure on a triangle or parallelogram has the Pompeiu property. We shall say $\mathcal{J}$ has the *Morera property* if for $f \in \mathcal{E}(\mathbf{R}^2)$, $T(f \circ \sigma) = 0$ for all $T \in \mathcal{J}$ and $\sigma \in \Sigma$ if and only if f is an entire holomorphic function of $\zeta = x + iy$. We shall apply Theorem 3.4 to characterize those families $\mathcal{J} \subset \mathcal{E}'(\mathbf{R}^2)$ which have the Pompeiu and Morera properties and those whose annihilators in the above sense consist only of harmonic functions. As was pointed out in § 1, verification of the Pompeiu and Morera properties are related in some cases.

For $T \in \mathcal{E}'(\mathbf{R}^2)$ and $\tau \in \mathbf{T}$, let T_τ be the distribution defined by $T_\tau(f) = T(f \circ \tau^{-1})$, $f \in \mathcal{E}(\mathbf{R}^2)$. A subset $\mathcal{J}$ of $\mathcal{E}'(\mathbf{R}^2)$ will be called rotation invariant if $T_\tau \in \mathcal{J}$ for all $T \in \mathcal{J}$ and $\tau \in \mathbf{T}$.

Notation. – For $\alpha \in \mathbf{C}$, let

$$M_\alpha = \{z = (z_1, z_2) \in \mathbf{C}^2 : z_1^2 + z_2^2 = \alpha\},$$

and write $M_0 = M_0^+ \cup M_0^-$, where

$$M_0^\pm = \{z : z_2 = \pm i z_1\}.$$

Remark. – M_0^+, M_0^- and the M_α are all invariant under $\mathbf{T}$, and it follows as in the proof of Lemma 3.2 that if f is an entire function on $\mathbf{C}^2$ such that $f \circ \tau(z_0) = 0$ for all $\tau \in \mathbf{T}$ and some element z_0 of, say, M_α, then $f \equiv 0$ on M_α.

THEOREM 4.1. – *Let $\mathcal{J}$ be a subset of $\mathcal{E}'(\mathbf{R}^2)$ and set*

$$Z = \cap\{\hat{T}^{-1}(0) : T \in \mathcal{J}\}.$$

Suppose $M_\alpha \not\subset Z$ for all $\alpha \neq 0$.

i) *$\mathcal{J}$ has the Pompeiu property if and only if $0 \notin Z$.*

ii) *$\mathcal{J}$ has the Morera property if and only if $M_0^+ \subset Z$ and*

$$[(z_2 - iz_1)^{-1}\, \hat{T}(z)]_{z=0} \neq 0$$

for some $T \in \mathcal{J}$.

iii) *The following conditions on* $\mathcal{J}$ *are equivalent.*

a) *For* $f \in \mathcal{E}(\mathbf{R}^2)$, $T(f \circ \sigma) = 0$ *for all* $T \in \mathcal{J}$ *and* $\sigma \in \Sigma$ *if and only if* f *is harmonic on* $\mathbf{R}^2$.

b) $M_0 \subset Z$ *and* $[(z_1^2 + z_2^2)^{-1}\,\hat{T}(z)]_{z=0} \neq 0$ *for some* $T \in \mathcal{J}$.

If, on the other hand, $M_\alpha \subset Z$ *for some* $\alpha \neq 0$, *then there exists a nonharmonic exponential function* f *such that* $T(f \circ \sigma) = 0$ *for all* $T \in \mathcal{J}$ *and* $\sigma \in \Sigma$.

Proof. – Let $\mathcal{J}^*$ be the rotation-invariant subset of $\mathcal{E}'(\mathbf{R}^2)$ generated by $\mathcal{J}$, and set $Z^* = \cap\{\hat{T}^{-1}(0) : T \in \mathcal{J}^*\}$. Since Z^* is rotation invariant, it follows from the preceding remark that Z^* is a union of sets M_α for certain nonzero values of α, perhaps along with $\{(0,0)\}$, M_0^+ and/or M_0^-. By Theorem 3.4 the space

$$V = \{f \in \mathcal{E}(\mathbf{R}^2) : T(f \circ \sigma) = 0 \quad \text{for all} \quad T \in \mathcal{J} \quad \text{and} \quad \sigma \in \Sigma\}$$

is spanned by the polynomial-exponential functions it contains. Since the exponential functions in V are those of the form $f_z(x, y) = \exp i(z_1 x + z_2 y)$ with $z = (z_1, z_2) \in Z^*$, i) now follows from the fact that if a polynomial-exponential function is in V, then the corresponding exponential function is also in V.

To prove ii) and iii) first notice that f_z is harmonic if and only if $z \in M_0$ and holomorphic in $\zeta = x + iy$ if and only if $z \in M_0^+$. Thus $Z = M_0^+$ if $\mathcal{J}$ has the Morera property and $Z = M_0$ if $\mathcal{J}$ satisfies condition a) of iii). Hence in the first case each element of $\mathcal{J}$ is divisible in $\hat{\mathcal{E}}'(\mathbf{R}^2)$ by $z_2 - iz_1$, while in the second case each element of $\mathcal{J}$ is divisible by $z_1^2 + z_2^2$ (cf. Lemma 3.2).

Suppose $\mathcal{J}$ has the Morera property and

$$[(z_2 - iz_1)^{-1}\,\hat{T}(z)]_{z=0} = 0$$

for all $T \in \mathcal{J}$, whence for all $T \in \mathcal{J}^*$. Then writing $\hat{T}(z) = (z_2 - iz_1)\,\hat{S}(z)$, $T \in \mathcal{J}^*$, we obtain

$$\frac{\partial}{\partial z_1}\,[\hat{T}(z)]_{z=0} = \frac{\partial}{\partial z_1}\,[(z_2 - iz_1)\,\hat{S}(z)]_{z=0} = 0, \qquad T \in \mathcal{J}^*\,.$$

But in view of (5) this says the function $p(x, y) = x$ is in V, contradicting the assumption that $\mathcal{J}$ has the Morera property. Similarly, assuming that $M_0 \subset Z$ and

$$[(z_1^2 + z_2^2)^{-1} \hat{T}(z)]_{z=0} = 0 , \qquad T \in \mathcal{J} .$$

We conclude that $p(x, y) = x^2$ is in V. Thus in iii), b) implies a).

Let H be the subspace of $\mathcal{E}(\mathbf{R}^2)$ of all entire holomorphic functions of ζ. By Theorem 3.4 H is spanned by the polynomial-exponential functions it contains. But if $pf_z \in H$ for some polynomial p and $z \in M_0^+$, then $p \in H$. Moreover, setting $z_2 = iz_1$ and differentiating f_z with respect to z_1, one obtains the function $\zeta \exp iz_1 \zeta$, and it is easy to see that this differentiation converges in $\mathcal{E}(\mathbf{R}^2)$. It follows that H is generated by the functions f_z with $z \in M_0^+$.

Suppose $M_0^+ \subset Z$ and $[(z_2 - iz_1)^{-1} \hat{T}(z)]_{z=0} \neq 0$ for some $T \in \mathcal{J}$. Then $Z = M_0^+$ and $V \supset \{f_z : z \in M_0^+\}$, so by the previous remarks $H \subset V$. On the other hand, if $pf_{z_0} \in V$ for some polynomial p and $z_0 \in M_0^+$, then by (5) $p(D) [\hat{T}(z)]_{z=z_0} = 0$, $T \in \mathcal{J}^*$. Applying the general Leibnitz formula (cf. (7)), we obtain

$$P(D) [\hat{T}(z)]_{z=z_0} = p(D) [(z_2 - iz_1) \hat{S}(z)]_{z=z_0} =$$

$$= i \left(\frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y}\right) (D) [\hat{S}(z)]_{z=z_0} = 0 .$$

Since by i)

$$\{S \in \mathcal{E}'(\mathbf{R}^2) : \hat{T}(z) = (z_2 - iz_1) \hat{S}(z) \quad \text{for some} \quad T \in \mathcal{J}\}$$

has the Pompeiu property, we conclude that p satisfies the Cauchy-Riemann equations

$$\frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y} = 0 .$$

Hence $V \subset H$, so $\mathcal{J}$ has the Morera property.

Finally, assume that b) of iii) holds. As shown above the space H is generated by $\{f_z : z \in M_0^+\}$, hence also the space of conjugates of functions in H is generated by $\{f_z : z \in M_0^-\}$. Thus the space of harmonic functions on $\mathbf{R}^2$ is generated by $\{f_z : z \in M_0\}$, so that assumption that $M_0 \subset Z$ implies V contains all harmonic functions. For $T \in \mathcal{J}^*$, let R, $U \in \mathcal{E}'(\mathbf{R}^2)$ such that

$$\hat{T}(z) = (z_2 + iz_1) \hat{R}(z) = (z_1^2 + z_2^2) \hat{U}(z) , \qquad z \in \mathbf{C}^2 .$$

Then by i) and ii) $\{R\}$ has the Morera property and $\{U\}$ has the Pompeiu property. If $pf_{z_0} \in V$ for some polynomial p and $z \in M_0$, then the Leibnitz formula gives

$$p(\mathrm{D})\,[\hat{\mathrm{T}}(z)]_{z=z_0} = i\left(\frac{\partial p}{\partial x} - i\frac{\partial p}{\partial y}\right)(\mathrm{D})\,[\hat{\mathrm{R}}(z)]_{z=z_0} = 0\ .$$

Thus $\dfrac{\partial p}{\partial x} - i\dfrac{\partial p}{\partial y}$ is analytic, so the Cauchy-Riemann equations imply $\nabla^2 p = 0$. If $z_0 \neq 0$, then

$$p(\mathrm{D})\,[\hat{\mathrm{T}}(z)]_{z=z_0} = \left(2z_1\frac{\partial p}{\partial x} + 2z_2\frac{\partial p}{\partial y} + \nabla^2 p\right)(\mathrm{D})\,[\hat{\mathrm{U}}(z)]_{z=z_0} = 0\ ,$$

so since p is harmonic

$$z_1\frac{\partial p}{\partial x} + z_2\frac{\partial p}{\partial y} = 0\ .$$

Thus p, whence pf_{z_0}, is analytic or conjugate-analytic, according as $z_0 \in \mathrm{M}_0^+$ or M_0^-. Hence V consists entirely of harmonic functions, so b) implies a). This completes the proof.

As a first application of Theorem 4.1 we arrive at an interesting equivalence of the Pompeiu problem for domains in **C** whose boundaries are rectifiable closed curves and the Morera problem for such curves.

Notation. – Let D be a domain in the complex plane and let Γ be a rectifiable curve. We denote by μ_{D} area measure on D and by ν_Γ and σ_Γ the measure $d\zeta$ and normalized linear measure on Γ, respectively. If D is the disk and Γ the circle of radius r about the origin, we denote μ_{D}, ν_Γ and σ_Γ by μ_r, ν_r, and σ_r, respectively.

THEOREM 4.2. – *Let $\mathcal{D}$ be a collection of domains in the complex plane whose boundaries are rectifiable closed curves. Then $\{\mu_{\mathrm{D}} : \mathrm{D} \in \mathcal{D}\}$ has the Pompeiu property if and only if $\{\nu_\Gamma : \Gamma = \partial\mathrm{D},\ \mathrm{D} \in \mathcal{D}\}$ has the Morera property.*

Proof. – By Green's Theorem we have for $\mathrm{D} \in \mathcal{D}$ with boundary Γ,

$$\nu_\Gamma = \left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial x}\right)\mu_{\mathrm{D}}\ ,$$

the derivatives being taken in the sense of distributions. Thus

$$\hat{\nu}_\Gamma(z_1, z_2) = i(iz_1 - z_2)\,\hat{\mu}_D(z_1, z_2), \qquad (z_1, z_2) \in \mathbf{C}^2 . \quad (11)$$

One completes the proof by applying i), ii) and the last assertion of Theorem 4.1, since $\mu_D \geqslant 0$.

Remark. – This theorem also follows from Green's Theorem and the fact that given $f \in C(\mathbf{R}^2)$ there exists $u \in C(\mathbf{R}^2)$ such that $\dfrac{\partial u}{\partial x} + i\,\dfrac{\partial u}{\partial y} = f$.

Note that area measure on a disk does not have the Pompeiu property, and thus $d\zeta$ on a circle cannot have the Morera property. Indeed, the reason for this, as we shall now show, is that since the disk is rotation invariant all rigid motions of the disk are obtained by translations.

THEOREM 4.3. – *For every* $T \in \mathcal{E}'(\mathbf{R}^2)$ *such that the support of* T *contains at least two points there exists a nonharmonic exponential function* f *such that* $T * f = 0$.

Proof. – The assertion is that $\hat{T}(z) = 0$ for some $z \in \mathbf{C}^2 \setminus M_0$. Since every $T \in \mathcal{E}'(\mathbf{R}^2)$ is of finite order it suffices to consider the case when T is given by a measure μ. By the Hadamard Factorization Theorem and the growth conditions (3) on $\hat{\mu}$, for any real m

$$\hat{\mu}(\zeta, m\zeta) = \zeta^k\, e^{ia\zeta}\, P(\zeta), \qquad \zeta \in \mathbf{C} ,$$

where P is of the form

$$P(\zeta) = \prod_{n=1}^{\infty} \left(1 - \frac{\zeta}{\alpha_n}\right) \exp p\left(\frac{\zeta}{\alpha_n}\right) \cdot$$

If $P(\zeta) \not\equiv 1$, then $\hat{\mu}(\alpha_1, m\alpha_1) = 0$ and $\alpha_1 \neq 0$. Thus $z = (\alpha_1, m\alpha_1)$ is the desired point not in M_0. We can deal similarly with the function $\hat{\mu}(0, \zeta)$. On the other hand, if $P(\zeta) \equiv 1$, then since $\hat{\mu}$ is bounded for real z_1 and z_2, we must have $k = 0$ and a real. Thus suppose that for each $m \in \mathbf{R} \cup \{\infty\}$ there is a real number a_m such that

$$\begin{cases} \hat{\mu}(\zeta, m\zeta) = e^{ia_m \zeta}, & \zeta \in \mathbf{C},\ m \in \mathbf{R} , \\ \hat{\mu}(0, \zeta) = e^{ia_\infty \zeta}, & \zeta \in \mathbf{C} . \end{cases} \qquad (12)$$

Differentiating (12) with respect to ζ, we find that $a_m = a_0 + m a_\infty$, $m \in \mathbf{R}$. If we now fix ζ and consider $\hat{\mu}(\zeta, m\zeta)$ as a function of m we have by analytic continuation that (12) holds for all $m \in \mathbf{C}$, whence

$$\hat{\mu}(z_1, z_2) = e^{i(a_0 z_1 + a_\infty z_2)}, \qquad (z_1, z_2) \in \mathbf{C}^2,$$

contradicting the assumption concerning the support of μ.

Althouth, as noted above, area measure on a disk does not have the Pompeiu property, we have the following "two circle theorem", which was refferred to in § 1.

Notation. – For $n = 0, 1, \ldots$ let $J_n(\zeta)$ be the n-th Bessel function of the first kind (see [4, Chapter 8]), and for each n and each $\zeta_0 \in \mathbf{C}$ let

$$Q_n(\zeta_0) = \{\zeta_1/\zeta_2 : J_n(\zeta_1) = J_n(\zeta_2) = \zeta_0\}.$$

Remark. – Note that the set $Q_1(0)$ of exceptional values which appears in the following theorem is a countable dense subset of $\mathbf{R}$. This follows from the well-known fact that the zeroes of J_1 are all real and are distributed in a regular fashion over $\mathbf{R}$.

THEOREM 4.4. – *Given positive numbers r_1 and r_2, the following are equivalent.*

i) $\{\mu_{r_1}, \mu_{r_2}\}$ *has the Pompeiu property.*

ii) $\{\nu_{r_1}, \nu_{r_2}\}$ *has the Morera property.*

iii) $r_1/r_2 \notin Q_1(0)$.

Proof. – For $\xi_1, \xi_2 \in \mathbf{R}$ and $r > 0$,

$$\hat{\mu}_r(\xi_1, \xi_2) = \int_{\mathbf{R}^2} e^{i(\xi_1 x + \xi_2 y)} d\mu(x, y) =$$

$$= \int_0^r \int_0^{2\pi} e^{i\rho(\xi_1 \cos\theta + \xi_2 \sin\theta)} \rho d\theta\, d\rho .$$

Setting $\xi_1 = R\cos\varphi$ and $\xi_2 = R\sin\varphi$, the inner integral becomes

$$\int_0^{2\pi} e^{i\rho R\cos(\theta - \varphi)} d\theta = 2\pi J_0(\rho R) . \qquad (13)$$

Thus

$$\hat{\mu}_r(\xi_1, \xi_2) = 2\pi \int_0^r \rho J_0(\rho R)\, d\rho = \frac{2\pi r}{R} J_1(rR) =$$

$$= \frac{2\pi r\, J_1(r\sqrt{\xi_1^2 + \xi_2^2})}{\sqrt{\xi_1^2 + \xi_2^2}} .$$

Since J_1 is an odd function, the last expression above is a real entire function of ξ_1 and ξ_2. Thus upon complexification we have

$$\hat{\mu}_r(z_1, z_2) = \frac{2\pi r\, J_1(r\sqrt{z_1^2 + z_2^2})}{\sqrt{z_1^2 + z_2^2}}, \qquad (z_1, z_2) \in \mathbf{C}^2 . \qquad (14)$$

By Theorem 4.1, i) holds if and only if $\hat{\mu}_{r_1}$ and $\hat{\mu}_{r_2}$ have no common zeros, which by (14) means $r_1/r_2 \notin Q_1(0)$. Since i) and ii) are equivalent by Theorem 4.2, this completes the proof.

Similarly we can obtain a recent result of S.P. Ponomarev [15] on the Morera problem.

THEOREM 4.5. – *If $\{r_n\}$ is a convergent sequence of distinct positive numbers, then $\{\mu_{r_n} : n \geqslant 1\}$ has the Pompeiu property and $\{\nu_{r_n} : n \geqslant 1\}$ has the Morera property.*

Proof. – If $\hat{\mu}_r$ vanishes on M_α, then by (14) $J_1(r\sqrt{\alpha}) = 0$. Since the r_n are all distinct and the zeros of J_1 have no finite limit point, the result follows from Theorems 4.1 and 4.2.

In [5] J. Delsarte characterized harmonic functions by the following strengthened converse of the mean-value property.

THEOREM 4.6. – *Given positive real numbers r_1 and r_2, the following are equivalent.*

i) *A function $f \in C(\mathbf{R}^2)$ is harmonic if and only if*

$$f(x, y) = f * \sigma_{r_1}(x, y) = f * \sigma_{r_2}(x, y), \qquad (x, y) \in \mathbf{R}^2 .$$

ii) $r_1/r_2 \notin Q_0(1)$.

Proof. – If δ_0 denotes the unit point mass at the origin, then we have from equation (13)

$$(\sigma_r - \delta_0)\hat{}(z_1, z_2) = J_0(r\sqrt{z_1^2 + z_2^2}) - 1 .$$

The proof is completed as before by applying Theorem 4.1.

Remark. – Using asymptotic properties of J_0 Delsarte has shown that the exceptional set in Theorem 4.6 is finite.

The following version of Delsarte's theorem using area measure appears in the paper [19] of Zalcman.

THEOREM 4.7. – *Let* $Q = \left\{ \zeta_1/\zeta_2 \ : \ J_1(\zeta_1)/\zeta_1 = J_1(\zeta_2)/\zeta_2 = \frac{1}{2} \right\}$. *Then given positive numbers* r_1 *and* r_2, *the following are equivalent.*

i) *A function* $f \in C(\mathbf{R}^2)$ *is harmonic if and only if*

$$\pi r_i^2 f(x, y) = f * \mu_{r_i}(x, y), \qquad (x, y) \in \mathbf{R}^2, \ i = 1, 2 .$$

ii) $r_1/r_2 \notin Q$.

Proof. – Equation (14) gives

$$\left(\frac{1}{\pi r^2} \mu_r - \delta_0\right)\hat{}(z_1, z_2) = \frac{2 J_1(r\sqrt{z_1^2 + z_2^2})}{r\sqrt{z_1^2 + z_2^2}} - 1, \quad (z_1, z_2) \in \mathbf{C}^2 .$$

5. Some examples.

In this section we shall make some specific calculations which, together with Theorems 4.1 and 4.2, show that various measures have the Pompeiu or Morera property. As a first example we consider an elliptical region. We preserve the notation established in § 4.

THEOREM 5.1. – *Let* D *denote the elliptical region*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1, \qquad 0 < b < a$$

and Γ *its boundary. Then* μ_D *has the Pompeiu property and* ν_Γ *has the Morera property.*

Proof. – By a linear change of variables, equation (13) yields

$$\hat{\mu}_{D}(z_1, z_2) = \frac{2\pi ab J_1(\sqrt{a^2 z_1^2 + b^2 z_2^2})}{\sqrt{a^2 z_1^2 + b^2 z_2^2}}, \qquad (z_1, z_2) \in \mathbf{C}^2 .$$

It is then easy to see that $\hat{\mu}(0) \neq 0$ and $\hat{\mu}$ cannot vanish identically on any M_α, $\alpha \neq 0$, so the theorem follows.

The remainder of this section is devoted to exhibiting a class of measures which have the Pompeiu property and which includes the measure μ_D if D is, for example, any polygonal region. The proof will be based on some estimates of the growth of Fourier transforms on certain curves in the manifolds M_α. Namely, we let $z = (z_1, z_2)$ vary on the manifold M_α in such a way that

$$\begin{cases} z_1 = t \\ z_2 = z_2(t) = -it\left(1 - \dfrac{\alpha}{t^2}\right)^{\frac{1}{2}} \qquad t > 0 \\ \qquad = -it + 0\left(\dfrac{1}{t}\right), \quad t \to +\infty . \end{cases} \tag{15}$$

Observe that the square root is chosen so that the last equation holds and that for all t, $(z_1, z_2) \in M_\alpha$ if $\alpha \neq 0$ and $(z_1, z_2) \in M_0^-$ for $\alpha = 0$.

LEMMA 5.2. – *If μ and ν are measures on $\mathbf{R}^2$ with compact support, and if*

$$|\hat{\mu}(t, z_2(t))| = o(|\hat{\nu}(t, z_2(t))|), \qquad t \to +\infty ,$$

then $\hat{\mu} + \hat{\nu}$ does not vanish identically on M_α if $\alpha \neq 0$ or on M_0^- if $\alpha = 0$. The same result holds on M_0^+ if we set $z_2 = it$.

LEMMA 5.3. – *Let μ be a measure with compact support contained in $\{(x, y) : y \leqslant a\}$ and ν a measure with compact support such that*

$$e^{at} = o(|\hat{\nu}(t, z_2(t))|), \qquad t \to +\infty .$$

Then $\hat{\mu} + \hat{\nu}$ does not vanish identically on M_α if $\alpha \neq 0$ or on M_0^- if $\alpha = 0$.

Proof. – Under the assumption on the support of μ, a direct estimation shows that

$$|\hat{\mu}(t, z_2(t))| = 0(e^{at}), \qquad t \to +\infty, \tag{16}$$

so Lemma 5.2 applies.

LEMMA 5.4. – *Let $a < b$ and let Γ denote the sum of the (oriented) line segments from the point (x_0, a) to $(0, b)$ and from $(0, b)$ to the point (x_1, a). Then for all $\alpha \in \mathbf{C}$ and for (z_1, z_2) as in* (15) *we have*

$$|\hat{\nu}_\Gamma(t, z_2(t))| \geqslant \text{const.} \frac{e^{bt}}{t}, \qquad t \to +\infty. \tag{17}$$

The same result holds if ν_Γ is replaced by σ_Γ. Moreover, in the case of σ_Γ the same estimate holds when $z_2 = it$, $t > 0$.

Proof. – We shall give the proof for ν_Γ ; the other case may be carried out analogously. A direct calculation shows that

$$\hat{\nu}_\Gamma(z_1, z_2) = \alpha(z_1, z_2)\, e^{ibz_2} + \beta(z_1, z_2)\, e^{i(x_1 z_1 + a z_2)} - \\ - \gamma(z_1, z_2)\, e^{i(x_0 z_1 + a z_2)}, \tag{18}$$

where

$$\left\{\begin{aligned} \alpha(z_1, z_2) &= \frac{-x_0 + i(b-a)}{i(-x_0 z_1 + (b-a) z_2)} - \frac{x_1 + i(a-b)}{i(x_1 z_1 + (a-b) z_2)}, \\ \beta(z_1, z_2) &= \frac{x_1 + i(a-b)}{i(x_1 z_1 + (a-b) z_2)}, \\ \gamma(z_1, z_2) &= \frac{-x_0 + i(b-a)}{i(-x_0 z_1 + (b-a) z_2)}. \end{aligned}\right. \tag{19}$$

Setting $z_1 = t$, $z_2 = -it + 0\left(\frac{1}{t}\right)$ in (18) and (19), we obtain

$$|\alpha(t, z_2(t))| \geqslant \text{const.} \frac{1}{t},$$

and hence

$$|\hat{\nu}_\Gamma(t, z_2(t))| \geqslant \text{const.} \frac{e^{bt}}{t} - 0\left(\frac{e^{at}}{t}\right),$$

which implies (17) since $a < b$.

COROLLARY 5.5. – *Let μ be a measure with compact support contained in $\{(x, y) : y \leqslant a\}$. Let $a < b$ and let Γ be as in Lemma 5.4. Then $\mu + \sigma_\Gamma$ has the Pompeiu property if and only if $(\hat{\mu} + \hat{\sigma}_\Gamma)(0,0) \neq 0$.*

Proof. – By Lemmas 5.3 and 5.4 $\hat{\mu} + \hat{\sigma}_\Gamma$ does not vanish identically on M_α for all $\alpha \neq 0$ and on M_0^-. And when $z_1 = t$, $z_2 = it$, $t > 0$, then $(z_1, z_2) \in M_0^+$, and the estimate (16) also holds here. Thus, by Lemmas 5.2 and 5.4 $\hat{\mu} + \hat{\sigma}$ does not vanish identically on M_0^+ also. By Theorem 4.1 $\mu + \sigma$ has the Pompeiu property.

COROLLARY 5.6. – *Let μ be as in Corollary 5.5 with $\mu \geqslant 0$. Let $a < b$, and denote by Δ the triangle determined by the points (x_0, a), (x_1, a) and $(0, b)$. Then $\mu + \mu_\Delta$ has the Pompeiu property.*

Proof. – Since $\mu + \mu_\Delta \geqslant 0$, we have $(\hat{\mu} + \hat{\mu}_\Delta)(0,0) \neq 0$. Thus we need only consider the manifolds M_α for $\alpha \neq 0$. Since the estimate (16) holds for the measure $d\zeta$ on the line segment from (x_1, a) to (x_0, a), we may apply Lemma 5.4 to conclude that (17) holds for $\Gamma = \partial\Delta$. But then (11) gives

$$|\hat{\mu}_\Delta(t, z_2(t))| \geqslant \text{const.} \frac{e^{bt}}{t^2}, \qquad t \to +\infty .$$

Our corollary now follows from Lemma 5.2.

COROLLARY 5.7. – *Let μ be as in Corollary 5.5 and let Γ denote the (oriented) line segment from the point (x_0, a) to $(0, b)$. Then $\mu + \nu_\Gamma$ and $\mu + \sigma_\Gamma$ have the Pompeiu property if and only if*

$$(\hat{\mu} + \hat{\nu}_\Gamma)(0,0) \neq 0 \quad \textit{and} \quad (\hat{\mu} + \hat{\sigma}_\Gamma)(0,0) \neq 0 ,$$

respectively.

Proof. – A calculation yields estimates like (17) for ν_Γ and σ_Γ. One then proceeds as in Corollary 5.5.

COROLLARY 5.8. – *Let μ be as in Corollary 5.5 and ν_Γ as in Lemma 5.4. Then $\hat{\mu} + \hat{\nu}_\Gamma$ does not vanish identically on any M_α for $\alpha \neq 0$ or on M_0^-.*

As a consequence of these corollaries we can settle the Pompeiu and Morera problems for polygonal regions and their boundaries.

THEOREM 5.9. – *Let* D *be a polygonal region in* $\mathbf{R}^2$ *with boundary* Γ. *Then* μ_D *and* σ_Γ *have the Pompeiu property and* ν_Γ *has the Morera property.*

Proof. – After possibly rotating and translating D we can bring it to a position such that for some real number a $\nu_\Gamma = \mu + \nu$, where μ has support in $\{(x , y) : y \leqslant a\}$ and ν denotes the measure $d\zeta$ on the sum of two line segments as in Lemma 5.4. Cauchy's theorem implies that $\hat{\nu}_\Gamma$ vanishes identically on M_0^+, so applying Corollary 5.8 and Theorem 4.1 we see that ν_Γ has the Morera property. By Theorem 4.2 μ_D has the Pompeiu property. Similarly, Corollary 5.5 gives the Pompeiu property for σ_Γ.

We shall now give a "perturbation" of Lemma 5.4, replacing the line segments by suitably smooth curves. This will allow us to deal with a much larger class of regions.

LEMMA 5.10. – *Let* Γ_1 *denote the line segment from* (x_0 , a) *to* $(0 , b)$, *where* $a < b$, *and let* Γ_2 *be a curve terminating at* $(0 , b)$ *and tangent to* Γ_1 *at that point. Suppose* Γ_2 *is given by*

$$\Gamma_2 : x = f(y) , \qquad y_0 \leqslant y \leqslant b ,$$

where

i) f *is of Lipschitz type, and*

ii) $\lim_{y \to b} f'(y) = - \dfrac{x_0}{b - a}$. *(Recall that* $f'(y)$ *exists a.e.)*

Then for all $\alpha \in \mathbf{C}$ *and* (z_1 , z_2) *as in* (15) *we have*

$$| \hat{\nu}_{\Gamma_1}(t , z_2(t)) - \hat{\nu}_{\Gamma_2}(t , z_2(t)) | = o\left(\frac{e^{bt}}{t}\right), \qquad t \to +\infty . \quad (20)$$

The same conclusion holds with ν *replaced by* σ.

Proof. – To simplify the notation, let us assume that $b = 1$ and $a = y_0 = 0$. Then Γ_1 is given by $x = \lambda(y - 1)$, $0 \leqslant y \leqslant 1$, where $\lambda^{-1} = - x_0^{-1}$ is the slope of Γ_1, and ii) becomes $\lim_{y \to b} f'(y) = \lambda$.

Since f' is bounded near $y = b$ by ii), let us assume it is bounded on [0,1]. Write

$$\hat{\nu}_{\Gamma_1}(z_1, z_2) - \hat{\nu}_{\Gamma_2}(z_1, z_2) = \int_0^1 e^{i[\lambda(y-1)z_1 + yz_2]} (\lambda + i)\, dy -$$

$$- \int_0^1 e^{i[f(y)z_1 + yz_2]} (f'(y) + i)\; dy$$

$$= \int_0^1 e^{iyz_2} \left(e^{i\lambda(y-1)z_1} - e^{if(y)z_1} \right) (\lambda + i)\, dy +$$

$$+ \int_0^1 e^{i[f(y)z_1 + yz_2]} (\lambda - f'(y))\, dy$$

$$= \mathrm{I}_1 + \mathrm{I}_2 .$$

(The second equality follows from the first by adding and subtracting

$$\int_0^1 e^{i[f(y)z_1 + yz_2]} \lambda dy$$

and collecting terms.) We shall estimate each of the integrals I_1 and I_2 on the points (z_1, z_2) as in (15).

We begin with I_2. Define

$$\varepsilon(\delta) = \sup_{\delta \leqslant y \leqslant 1} |\lambda - f'(y)| , \qquad 0 \leqslant \delta < 1 ,$$

so that $\varepsilon(\delta) \to 0$ as $\delta \to 1^-$. Set

$$\mathrm{M} = \varepsilon(0) = \sup_{0 \leqslant y \leqslant 1} |\lambda - f'(y)| .$$

Note that when $z_2(t) = -it + 0(t^{-1})$ we have

$$|e^{iyz_2}| \leqslant \mathrm{C}\, e^{yt} , \qquad 0 \leqslant y \leqslant 1,\ 1 < t < \infty .$$

Thus for (z_1, z_2) as in (15) we have for $t > 1$,

$$|\mathrm{I}_2| \leqslant \mathrm{C} \int_0^1 e^{yt} |\lambda - f'(y)|\, dy = \mathrm{C} \left(\int_0^\delta + \int_\delta^1 \right)$$

$$\leqslant \mathrm{C} \left[\mathrm{M} \frac{e^{\delta t}}{t} + \varepsilon(\delta) \frac{e^t}{t} \right] = \mathrm{C} \frac{e^t}{t} \left[\mathrm{M} e^{-(1-\delta)t} + \varepsilon(\delta) \right] .$$

It follows that $|\mathrm{I}_2| = o\left(\frac{e^t}{t}\right)$ as $t \to +\infty$.

To estimate I_1, let A be an arbitrary positive number, and set $\delta = 1 - A/t$. Then as in the estimate above, for $t > 1$ we have

$$|I_1| \leqslant C\left(\int_0^\delta + \int_\delta^1\right) e^{yt}\,|e^{itg(y)} - 1|\,|\lambda + i|\,dy\,, \qquad (21)$$

where $g(y) = \lambda(y-1) - f(y)$, $0 \leqslant y \leqslant 1$. When $\delta \leqslant y < 1$ we have

$$|g(y)| = |g(y) - g(1)| \leqslant (1-\delta)\,\varepsilon(\delta)$$

since g is absolutely continuous, whence

$$|e^{itg(y)} - 1| \leqslant t\,|g(y)| \leqslant t(1-\delta)\,\varepsilon(\delta) = A\varepsilon\left(1 - \frac{A}{t}\right)\cdot$$

Substituting this in (21), we obtain

$$\begin{aligned} |I_1| &\leqslant 2C\,\frac{e^{\delta t}}{t} + CA\,\frac{e^t}{t}\;\varepsilon\left(1 - \frac{A}{t}\right) \\ &= \frac{e^t}{t}\left[2C\,e^{-A} + CA\;\varepsilon\left(1 - \frac{A}{t}\right)\right], \qquad t > 1\,. \end{aligned}$$

Consequently, since $\varepsilon\left(1 - \frac{A}{t}\right) \to 0$ as $t \to \infty$, we have

$$\limsup_{t \to +\infty} te^{-t}\,|I_1| \leqslant 2Ce^{-A}.$$

Since A was arbitrarily chosen, we have $\limsup te^{-t}\,|I_1| = 0$, or $|I_1| = o\left(\frac{e^t}{t}\right)$, and the proof for ν is complete. The σ case is handled similarly.

Remark. – Lemmas 5.4 and 5.10 may be applied to yield results which generalize Corollaries 5.5-5.8. We leave the formulation of these results to the reader and proceed directly to the main result of this section.

DEFINITION. – *Let* $\Gamma = \Gamma(t)$, $-1 \leqslant t \leqslant 1$, *be a Lipschitz curve with well-defined (a.e.) unit tangent vectors* $T(t) = \Gamma'(t)/|\Gamma'(t)|$. *The point* $p = \Gamma(0)$ *is a* corner *of* Γ *if the right-hand and left-hand limits of* $T(t)$ *as* $t \to 0$ *exist and are not collinear.*

THEOREM 5.11. – *Let* D *be a compact region in* $\mathbf{R}^2$. *Suppose that there is a half-plane* H *in* $\mathbf{R}^2$ *and a unique point* $p \in \mathrm{D} \cap \mathrm{H}$ *of maximal distance from* $\partial\mathrm{H}$ *such that near* p *the boundary of* D *is given by a Lipschitz curve with* p *as a corner. Then* μ_{D} *has the Pompeiu property. If* $\partial\mathrm{D}$ *is a rectifiable curve* Γ, *then* ν_Γ *has the Morera property and* σ_Γ *has the Pompeiu property.*

Proof. – After rotating and translating D, we may assume that $p = (0, b)$ and $\mathrm{H} = \{(x, y) : y \geqslant a\}$ for some $a < b$. Then, by a minor modification in the choice of H it may be assumed that $(\partial\mathrm{D}) \cap \mathrm{H}$ consists of a Lipschitz curve $\Gamma = \Gamma(t)$, $-1 \leqslant t \leqslant 1$, with unit tangent vectors $\mathrm{T}(t) = (x(t), y(t))$, such that Γ has a corner at $p = \Gamma(0)$ and

$$\lim_{t \to 0^-} y(t) \neq 0 \neq \lim_{t \to 0^+} y(t) .$$

This means that Γ is given by two curves $x = f(y)$ and $x = g(y)$, $a \leqslant y \leqslant b$, each of Lipschitz type, such that $f(b) = g(b) = 0$ and the limits

$$\lim_{y \to b} f'(y) \quad \text{and} \quad \lim_{y \to b} g'(y)$$

exist and are unequal.

Let Γ_2 denote the curve $x = f(y)$, $a \leqslant y \leqslant b$, and let Γ_1 be the line segment from $(0, b)$ to a point (x_0, a) such that

$$\lim_{y \to b} f'(y) = -\frac{x_0}{b-a},$$

so that Γ_2 is tangent to Γ_1 at $(0, b)$. By Lemma 5.10, (20) holds for all $\alpha \in \mathbf{C}$ and (z_1, z_2) as in (15). We may proceed similarly to obtain (20) for the curve $x = g(y)$, $a \leqslant y \leqslant b$. Combining these results with Lemma 5.4, we obtain (17) for the curve Γ above. The proof now proceeds like those of Corollaries 5.5 and 5.6 and Theorem 5.9.

COROLLARY 5.12. – *Let* D *be a compact convex set in* $\mathbf{R}^2$ *with non-empty interior such that for some point* $p \in \Gamma = \partial\mathrm{D}$ *there is no unique line of support for* D *through* p. *Then* μ_{D} *and* σ_Γ *have the Pompeiu property and* ν_Γ *has the Morera property.*

Proof. – It is well known that Γ is a Lipschitz curve and that p is a corner of Γ in the sense defined above if there is no unique line of support for D through p. Thus the corollary follows easily from Theorem 5.11.

6. Bounded functions.

We shall conclude by pointing out how much simpler the Pompeiu problem becomes if the functions f under consideration in the formulation of the problem are assumed to be bounded. In particular, one needs only to consider the Fourier-Stieltjes transforms of measures in the usual sense, and the rotations enter in only a superficial way. Moreover, the measures need not have compact support, and the problem may be considered in a much more general context, as follows.

Let G be a locally compact abelian group with character group Γ. The Fourier-Stieltjes transform $\hat{\mu}$ of a measure μ in the measure algebra M(G) of G is the bounded, uniformly-continuous function on Γ given by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma(x)}\, d\mu(x), \qquad \gamma \in \Gamma .$$

For a full discussion of the concepts discussed here we refer the reader to [16]. If $f \in L^\infty(G)$ the *spectrum* $\sigma(f)$ of f is the set of all characters γ on G contained in the weak*-closed, translation-invariant subspace of $L^\infty(G)$ generated by f, and we say f *admits spectral synthesis* if f is in the weak*-closed subspace of $L^\infty(G)$ generated by $\sigma(f)$. It follows immediately from the Wiener Tauberian Theorem that every nonzero function in $L^\infty(G)$ has a nonvoid spectrum (see [16, Chapter 7]).

If I is a closed ideal in $L^1(G)$ and E a closed subset of Γ, define

$$Z(I) = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0 \quad \text{for all} \quad f \in I\}$$

and

$$I(E) = \{f \in L^1(G) : \hat{f}(\gamma) = 0 \quad \text{for all} \quad \gamma \in E\} .$$

The following easy proposition has as an immediate consequence the solution of the Pompeiu problem for bounded functions. Though the result is well known, we shall include the proof in the interest of completeness.

PROPOSITION 6.1. – *Let f be a bounded continuous function on* G *and* $\mu \in M(G)$. *If* $f * \mu = 0$ *then* $\sigma(f) \subset \hat{\mu}^{-1}(0)$. *Conversely, if f admits spectral synthesis and* $\sigma(f) \subset \hat{\mu}^{-1}(0)$, *then* $f * \mu = 0$.

Proof. – If $f * \mu = 0$, then $(g * \mu) * f = 0$ for all $g \in L^1(G)$, so

$$L^1(G) * \mu \subset I[f] = \{g \in L^1(G) : g * f = 0\} .$$

If $\gamma \in \sigma(f)$, then γ annihilates $I[f]$, so

$$g * \mu * \gamma = \hat{g}(\gamma)\, \hat{\mu}(\gamma)\, \gamma = 0$$

for all $g \in L^1(G)$, giving $\hat{\mu}(\gamma) = 0$. Thus $\sigma(f) \subset \hat{\mu}^{-1}(0)$.

Suppose $\sigma(f) \subset \hat{\mu}^{-1}(0)$. Let I be the closure in $L^1(G)$ of $L^1(G) * \mu$. Then I is a closed ideal in $L^1(G)$ with $Z(I) = \hat{\mu}^{-1}(0)$. Since $\sigma(f) \subset Z(I)$, we have

$$I(\sigma(f)) \supset I(Z(I)) \supset I .$$

If f admits spectral synthesis, then $I(\sigma(f)) = I[f]$. Hence $I[f] \supset I$, so $\mu * f$ annihilates all of $L^1(G)$, giving $\mu * f = 0$.

COROLLARY 6.2. – *Let $\mathfrak{M}$ be a subset of* M(G) *such that*

$$\cap \{\hat{\mu}^{-1}(0) : \mu \in \mathfrak{M}\} = \emptyset$$

If f is a bounded continuous function on G *such that* $f * \mu = 0$ *for all* $\mu \in \mathfrak{M}$, *then* $f = 0$.

Proof. – If f satisfies the hypotheses, then by the Proposition $\sigma(f) = \emptyset$, which as mentioned above implies $f = 0$.

COROLLARY 6.3. – *For each* $\alpha \geqslant 0$, *let* C_α *be the circle* $x^2 + y^2 = \alpha$. *Given* $\mu \in M(\mathbf{R}^2)$, *the following are equivalent.*

i) *If f is a bounded continuous function on* $\mathbf{R}^2$ *such that* $\int_{\mathbf{R}^2} f \circ \sigma \, d\mu = 0$ *for all* $\sigma \in \Sigma$, *then* $f = 0$.

ii) $C_\alpha \not\subset \hat{\mu}^{-1}(0)$ *for all* $\alpha \geqslant 0$.

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Manuscrit reçu le 17 mai 1972
accepté par B. Malgrange

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