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EQUIVARIANT ALGEBRAIC TOPOLOGY

by Sören ILLMAN

This talk covered the main parts of my thesis [5], see also [4] and [6]. Our main objective is to provide the equivariant analogue of ordinary singular homology and cohomology, and the equivariant analogue of the theory of CW complexes. For actions of **discrete** groups we have the work by Bredon [1], [2], and more recently also the article by Bröcker [3].

1. Equivariant singular homology and cohomology.

Let G be a topological group. By a G-space X we mean a topological space X together with a left action of G on X. A G-pair (X, A) consists of a G-space X and a G-subspace A of X. The notions G-map, G-homotopy, G-homotopy equivalence, etc... have the usual meaning.

DEFINITION 1.1. — *An orbit type family* $\mathcal{F}$ *for* G *is a family of subgroups of* G *which is closed under conjugation.*

Thus the family of all subgroups and the family of all finite subgroups of G are examples of orbit type families for G. A more special example is the following. Let $G = O(n)$ and let $\mathcal{F}$ be the family of all subgroups conjugate to $O(r)$ (standard imbedding) for some r, where $0 \leqslant r \leqslant n$.

Let R be a ring with unit. By an R-module we mean a unitary left R-module.

DEFINITION 1.2. — *Let* $\mathcal{F}$ *be an orbit type family for* G. *A covariant coefficient system* k *for* $\mathcal{F}$, *over the ring* R, *is a covariant functor from the category of G-spaces of the form* G/H, *where* $H \in \mathcal{F}$, *and G-homotopy classes of G-maps, to the category of R-modules.*

A contravariant coefficient system m is defined by the contravariant version of the above definition.

THEOREM 1.1. — *Let* G *be a topological group. Let* $\mathcal{F}$ *be an orbit type family for* G, *and* k *a covariant coefficient system for* $\mathcal{F}$, *over the ring* R. *Then there exists an equivariant homology theory* $H_*^G(\ ;k)$ *defined on the category of all G-pairs and* G-*maps (and with values in the category of* R-*modules), which satisfies all seven equivariant Eilenberg-Steenrod axioms, and which has the given coefficient system* k *as coefficients.*

By the statement that $H_*^G(\ ;k)$ satisfies the equivariant dimension axiom and has k as coefficients we mean the following. If $H \in \mathcal{F}$ then

$$H_n^G(G/H;k) = 0 \qquad \text{for} \qquad n \neq 0,$$

and there exists a natural isomorphism

$$\gamma : H_0^G(G/H;k) \xrightarrow{\cong} k(G/H).$$

The meaning of the rest of Theorem 1.1. is clear.

THEOREM 1.2. — *Let* G *and* $\mathcal{F}$ *be as above, and let* m *be a contravariant coefficient system for* $\mathcal{F}$, *over the ring* R. *Then there exists an equivariant cohomology theory* $H_G^*(\ ;m)$ *defined on the category of all G-pairs and G-maps (and with values in the category of* R-*modules), which satisfies all seven equivariant Eilenberg-Steenrod axioms, and which has the given coefficient system* m *as coefficients.*

For the details we refer to [5] and [6]. We call $H_*^G(\ ;k)$ for equivariant singular homology with coefficients in k, and $H_G^*(\ ;m)$ for equivariant singular cohomology with coefficients in m. Further properties, like functoriality in the transformation group G, transfer homomorphisms, Kronecker index, and a cup-product for equivariant singular cohomology are also given in [5] and [6].

2. Equivariant CW complexes.

In this section G denotes a **compact Lie group.**

DEFINITION 2.1. — *Let* X *be a Hausdorff G-space and* A *a closed* G*-subset of* X, *and* n *a non-negative integer. We say that* X *is obtainable from* A *by adjoining equivariant n-cells if there exists a collection* $\{c_j^n\}_{j\in J}$ *of closed* G*-subsets of* X *such that.*

1) $X = A \cup \left(\bigcup_{j\in J} c_j^n\right)$, *and* X *has the topology coherent with* $\{A, c_j^n\}_{j\in J}$.

2) *Denote* $\dot{c}_j^n = c_j^n \cap A$, *then*

$$(c_j^n - \dot{c}_j^n) \cap (c_i^n - \dot{c}_i^n) = \emptyset \quad \text{if} \quad i \neq j.$$

3) *For each* $j \in J$ *there exists a closed subgroup* H_j *of* G *and a* G*-map*

$$f_j\colon (E^n \times G/H_j, S^{n-1} \times G/H_j) \to (c_j^n, \dot{c}_j^n)$$

such that $f_j(E^n \times G/H_j) = c_j^n$, *and* f_j *maps* $(E^n - S^{n-1}) \times G/H_j$ *homeomorphically onto* $c_j^n - \dot{c}_j^n$.

DEFINITION 2.2. — *An equivariant relative* CW *complex* (X, A) *consists of a Hausdorff* G*-space* X, *a closed* G*-subset* A *of* X, *and an increasing filtration of* X *by closed* G*-subsets* $(X, A)^k$ $k = 0, 1, \ldots$, *such that.*

1) $(X, A)^0$ *is obtainable from* A *by adjoining equivariant 0-cells, and for* $k \geqslant 1$ $(X, A)^k$ *is obtainable from* $(X, A)^{k-1}$ *by adjoining equivariant k-cells.*

2) $X = \bigcup_{k=0} (X, A)^k$, *and* X *has the topology coherent with* $\{(X, A)^k\}_{k\geqslant 0}$.

The closed G-subset $(X, A)^k$ is called the k-skeleton of (X, A). If $A = \emptyset$ we call X an equivariant CW complex and denote the k-skeleton by X^k.

It is not difficult to prove the equivariant analogues of the standard elementary properties of CW complexes, that is, the equivariant homotopy extension property for an equivariant relative CW complex (X. A), the equivariant skeletal approximation theorem, and an equivariant Whitehead theorem. These results are stated as Propositions 2.3-2.5 in [4], and will not be repeated here. I understand that both C. Vaseekaran and S. Willson have also proved these three results.

Our main result about equivariant CW complexes is the following theorem, see [4] and [5].

THEOREM 2.1. — *Every differentiable G-manifold* M *is an equivariant* CW *complex.*

This result has also been proved by T. Matsumoto, see [7] Proposition 4.4. In fact a stronger result is true. We proved in [5] that every differentiable G-manifold can be equivariantly triangulated. The proof of this uses the theorem of C.T. Yang [9] that the orbit space G/M can be triangulated, the existence of slices, and the « covering homotopy theorem » of Palais [8] Theorem 2.4.1. It should be observed that Proposition 4.4. in Matsumoto [7] also proves this stronger result.

3. Equivariant singular homology and cohomology of finite dimensional equivariant CW complexes.

We are still assuming that G is a compact Lie group. An equivariant CW complex X is called finite dimensional if $X = X^m$ for some m. Let us for simplicity assume that the orbit type family $\mathcal{F}$ under consideration is the family of all closed subgroups of G. We say that a covariant coefficient system k is finitely generated if $k(G/H)$ is a finitely generated R-module for every closed subgroup H of G.

THEOREM 3.1. — *Let* X *be a finite dimensional equivariant* CW *complex. Then the n-th homology of the chain complex*

$$\dots \overset{\partial}{\leftarrow} H^G_{n-1}(X^{n-1}, X^{n-2}; k) \overset{\partial}{\leftarrow} H^G_n(X^n, X^{n-1}; k) \overset{\partial}{\leftarrow} \dots$$

is isomorphic to $H^G_n(X; k)$.

Together with Theorem 2.1 this gives us.

COROLLARY 3.2. — *Let* M^m *be an* m-*dimensional differentiable* G-*manifold. Then*

$$H_r^G(M^m; k) = 0 \qquad for \qquad r > m.$$

If M^m moreover is compact, and k is a finitely generated coefficient system over a noetherian ring R, then $H_r^G(M^m; k)$ is a finitely generated R-module for every r.

The corresponding results for cohomology are true.

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