Dan Burghelea

On the homotopy type of $\text{Diff}(M^n)$ and connected problems

Annales de l’institut Fourier, tome 23, n°2 (1973), p. 3-17

<http://www.numdam.org/item?id=AIF_1973__23_2_3_0>
ON THE HOMOTOPY TYPE OF DIFF (M*)
AND CONNECTED PROBLEMS

by Dan BURGHELEA (1)

This paper is a report on my recent work on the homotopy type of Diff (M*), M* a compact differentiable manifold, as well as on the homotopy properties of Topn, Topn/O*, (which can be derived as consequences of Theorem 1.3.).

1.

Given M* a differentiable (throughout this paper « differentiable » means C*-differentiable) manifold with or without boundary, one defines Diff (M; K) (Homeo (M; K)) the topological group of all diffeomorphisms (homeomorphisms) which are the identity on K and are homotopic to identity among continuos maps h: (M, ∂M) → (M, ∂M), h|K = id, endowed with C*(C0)-topology.

**Proposition 1.1.** — If K is a differentiable submanifold of M, with or without boundary, Diff (M; K) is a differentiable separable Frechet manifold and hence

1) Diff (M; K) has the homotopy type of a countable CW-complex.

2) As a topological space, Diff (M; K) is completely determined (up to a homeomorphism) by its homotopy type (see [3], [8]).

In order to compare the groups Diff and Homeo, it is convenient to consider their semisimplicial analogues, namely the

(1) During the time this work has been done the author was supported by AARHUS, University (Math. Inst.), 1971 and by Sonderforschungsbereich Theoretische Mathematik an der Universität Bonn 1972.
semisimplicial groups $S^d(\text{Diff}(M^n; K))$, the singular differentiable complex of $\text{Diff}(M^n; K)$ and $\text{SHomeo}(M^n; K)$ the singular complex of $\text{Homeo}(M^n; K)$.

The semisimplicial groups $S^d\text{Diff}, \text{S Homeo}$, suggest that one should consider simultaneously the simplicial groups $\tilde{\text{Diff}}(M^n; K)$ and $\tilde{\text{Homeo}}(M^n; K)$ whose $k$-simplexes are diffeomorphisms respectively homeomorphisms

$$h : \Delta[k] \times M \to \Delta[k] \times M$$

so that $h$ restricts to the identity on $\Delta[k] \times K$

$$h(d_i \Delta[k] \times M) \leq d_i (\Delta[k] \times M),$$

and are homotopic to identity among the continuous maps $f: (\Delta[K] \times M, \Delta[K] \times \partial M) \to (\Delta[K] \times M, \Delta[K] \times \partial M)$ with $f|\Delta[K] \times K = id$ and $f(d_i \Delta[K] \times M) \leq d_i \Delta[K] \times M$. (These simplicial groups have no $s$-operators hence are not semi-simplicial, however we still have a good homotopy theory for them, see [11]). The relationship between these simplicial groups and $S^d\text{Diff}, \text{S Homeo}$ is expressed by the following commutative diagram whose arrows are all inclusions

$$S^d\text{Diff}(M; K) \to S\text{Homeo}(M; K) \quad \downarrow \quad \downarrow$$

$$\tilde{\text{Diff}}(M; K) \to \tilde{\text{Homeo}}(M; K).$$

The surgery methods plus the smoothing theory permit (at least from a theoretic point of view) the description of the homotopy type of $\tilde{\text{Diff}}(M)$ (for homotopy groups see [2]).

For any closed differentiable manifold $M^n$ we denote by $\mathcal{P}(M)$ respectively $\mathcal{P}(M)$ the topological groups $\text{Diff}(M \times I, M \times \{0\})$ respectively $\text{Homeo}(M \times I, M \times \{0\})$.

It is not difficult to check the commutativity of the following diagram of groups as well as the exactness of its lines (the arrows are obviously defined)

$$\pi_1(\text{Diff } M) \quad \pi_1(\tilde{\text{Diff }} M) \quad \pi_0(\mathcal{P}(M)) \quad \pi_0(\tilde{\text{Diff }} M) \quad \pi_0(\mathcal{P}(M)) \quad 0$$

$$\pi_1(\text{Homeo } M) \quad \pi_1(\tilde{\text{Homeo }} M) \quad \pi_0(\mathcal{P}(M)) \quad \pi_0(\tilde{\text{Homeo }} M) \quad \pi_0(\mathcal{P}(M)) \quad 0$$

In [4] Cerf has proved that if $n \geq 5$, $\pi_1(M^n) = 0$ then $\pi_0(\mathcal{P}(M^n)) = 0$. Hatcher and Wagoner have computed $\pi_0(\mathcal{P}(M^n))$ when $\pi_1(M^n) \neq 0$ and $n \geq 7$ (see [7]).
As a consequence of Theorems 1.1 and 1.2 which will be stated below one has.

**Theorem 1.4.** — If \( n \neq 4,5 \) then \( \pi_0(\mathcal{P}(M)) \simeq \pi_0(\mathcal{P}(\mathcal{M})) \).
(There exists also a p.l. version of \( \mathcal{P}, \mathcal{P}\pi \) in which case
\[ \pi_0(\mathcal{P}\pi(M)) = \pi_0(\mathcal{P}(\mathcal{M})). \]

Forgetting the differentiable structure on \( M^n \) we can consider the Top\(_n\)-principal bundle Top\(_n\) \( \rightarrow \mathcal{T}(M^n) \rightarrow M^n \) associated to the tangent microbundle of \( M^n \). Using the action of Top\(_n\) on Top\(_n\)/On we associate to Top\(_n\) \( \rightarrow \mathcal{T}(M^n) \rightarrow M^n \) the bundle Top\(_n\)/On \( \rightarrow E \rightarrow M^n \). The differentiable structure of \( M^n \) defines a precise O\(_n\)-reduction of
\[ \text{Top}_n \rightarrow \mathcal{T}(M^n) \rightarrow M^n, \]

hence a precise (up to a homotopy) crosssection \( s \) of \( E \).
If \( M^n \) has nonempty boundary, the bundle
\[ \text{Top}_n/\text{On} \rightarrow E/\partial M \rightarrow \partial M \]
contains a well defined subbundle \( \text{Top}_{n-1}/\text{On}_{n-1} \rightarrow E^{\partial M} \rightarrow \partial M \)
(associated with \( \partial M \)), and the section \( s \) restricts on \( \partial M \) to a section in \( E^{\partial M} \). We denote by \( \Gamma(E, s) \) the base point space of all continuous crosssections of \( E \) with the compact open topology and section \( s \) as base-point, by \( \Gamma(E, E^0, s) \) the subspace of all continuous crosssections of \( E \) which restrict on \( \partial M \) to crosssections in \( E^0 \), by \( \Gamma^K(E, s) \) the subspace of all continuous crosssections in \( E \) which agree with \( s \) on \( K \) and by \( \Gamma^K(E, E^0, s) \) the intersection \( \Gamma^K(E, s) \cap \Gamma(E, E^0, s) \).

For a semisimplicial complex \( X \) we denote by \( \{X\} \) its geometric realisation.

**Theorem 1.2.** — There exists the base-point preserving maps \( i \), which are homotopy equivalences on any connected component and induce injective correspondence between connected components in any of the following cases

1) \( M^n \) closed, \( n \neq 4, \)
\[ i : \{S \text{ Homeo}(M^n)/S^4 \text{ Diff}(M^n)\} \rightarrow \Gamma(E, s). \]

2) \( \partial M^n \neq \emptyset, n \neq 4,5 \) or \( n = 5, \partial M^n = S^4, \)
\[ i : \{S \text{ Homeo}(M^n; \partial M^n)/S^4 \text{ Diff}(M^n; \partial M^n)\} \rightarrow \Gamma^{\partial M}(E, s). \]
3) $\partial M^n \neq \emptyset$, $n \neq 4,5$,
\[ i: \{S \text{ Homeo} (M^n)/S^d \text{ Diff} (M^n)\} \to \Gamma(E, E^n, s). \]

4) $\partial M^n \neq \emptyset$ K differentiable submanifold of $\partial M$, $n \neq 4,5$,
\[ i: \{S \text{ Homeo} (M^n, K)/S^d \text{ Diff} (M^n, K)\} \to \Gamma^k(E, E^n, s). \]

The base point of $\{S \text{ Homeo}.../S^d \text{ Diff}...\}$ is represented by the neutral element « 0 » of S Homeo...

Remark. — A similar theorem (without any dimensional restriction, comparing Diff (M) with Homeo$^p(|M|)$ ($|M|$ being the p.l-manifold associated to M via the Whitehead triangulation) is true, but a precise statement will require some considerations about p.d topology which will make this paper much longer.

There exists a (strong H-space) map
\[ \lambda_a : \Omega \text{ Diff} (D^n, \partial D^n) \to \text{ Diff} (D^{n+1}, \partial D^{n+1}) \]
(see [1]) which associates to any differentiable path
\[ \alpha : I \to \text{ Diff} (D^n, \partial D^n) \]
with $\alpha/\partial I = \{0\}$ the diffeomorphism
\[ \tilde{\alpha} : D^n \times I \to D^n \times I (\tilde{\alpha}(x, t) = \alpha(x), t) \]
which restricts to the identity on $\partial(D^n \times I)$. $\tilde{\alpha}$ can be viewed as a diffeomorphism of $D^{n+1}$ which restricts to the identity on $\partial D^{n+1}$.

Theorem 1.3. — For any $n \neq 4$ there exists a (strong H-space) homotopy equivalence
\[ h_n : \text{ Diff} (D^n, \partial D^n) \to \Omega^{n+2}(\text{Top}_n/O_n) \]
so that the following diagram is homotopy commutative
\[
\begin{array}{ccc}
\Omega \text{ Diff} (D^n, \partial D^n) & \xrightarrow{\Omega h_n} & \Omega^{n+2}(\text{Top}_n/O_n) \\
\downarrow \lambda_n & & \downarrow \Omega^{n+2} i_n \\
\text{ Diff} (D^{n+1}, \partial D^{n+1}) & \xrightarrow{h_{n+1}} & \Omega^{n+2}(\text{Top}_{n+1}/O_{n+1})
\end{array}
\]
with $i_n$ the natural map $\text{Top}_n/O_n \to \text{Top}_{n+1}/O_{n+1}$.
Very important acknowledgement.

A weak form of Theorem 1.3, the isomorphism between the homotopy groups of $\text{Diff}(D^n, \partial D^n)$ and of $\Omega^{n+1}(\text{Top}_n/O_n)$ has been conjectured by Cerf [5], and proven for $n = 3$.

**Theorem 1.3 and Theorem 1.2 have been announced by C. Morlet [10] but since then no complete proofs have appeared. Partly this work is my attempt to understand and prove Morlet's statements. Actually it seems that the proof sketched in section 3 is different from Morlet's ideas. As it will be pointed out in § 3 the present proof owes very much to R. Lashof who also can prove Theorem 1.2 and 1.3.**

Let us denote by $G$ the topological semigroup $G = \lim \to G_n$ where $G_n$ is the topological semigroup of all homotopy equivalences of $S^{n-1}$, and $G_n \to G_{n+1}$ is induced by suspension $\pi_i(G_n) = \pi_{i+n}(S^n)$.

**Theorem 1.5.** — For any $(k, n, r)$ positive integers $r < n+k$ there exists a bilinear pairing

$$
\psi_{k,n,r} : \pi_k(\text{Diff}(D^n, \partial D^n)) \otimes \pi_r(G) \to \pi_{k+r}(\text{Diff}(D^n, \partial D^n))
$$

so that:

1) $\psi_{k-1,n+1,r}(\lambda_n^* \otimes \text{id}) = \lambda_n^* \circ \psi_{k,n,r}$ with $\lambda_n^*$ the homomorphism induced by $\lambda_n$ between corresponding homotopy groups.

2) $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*, \psi_{0,n,r}$ is the Bredon pairing (Annals of Math. 1967, Vol. 86).

**Remarks.** — 1) There exists an unstable version of this pairing.

2) The pairing $\psi_{k,n,r}$ is a strong generalization and extension of the Munkres-Milnor-Novikov pairing (see [1]) and it allows us to improve considerably the list of nontrivial homotopy groups $\pi_n(\text{Diff}(D^n, \partial D^n))$.

In order to use this pairing for computational purposes one defines

$$
\psi_{k,n,r_1,r_2,\ldots,r_s} : \pi_k(\text{Diff}(D^n, \partial D^n)) \otimes \pi_{r_1}(G) \otimes \ldots \otimes \pi_{r_s}(G) \to \pi_{k+r_1+\ldots+r_s}(\text{Diff}(D^n, \partial D^n))
$$
for \( r_1 < n + k, r_2 < n + k + r_1, \ldots, r_s < n + k + \ldots r_{s-1}, \)
inductively by
\[
\psi_{k,n;r_1,\ldots,r_s} = \psi_{k+n;\ldots;\psi_{k,n;r_1,\ldots,r_{s-1}}(\otimes \text{id})}
\]
and study its nontriviality using composition in \( \pi_*(G) \).

I will mention only the following result.

**Theorem 1.6.** — Given an odd prime \( p \), for any
\( i \leq 2p(p-1)(p-2) + 1 \)
and
\( n = 2p(p-1)^2 - i, \pi_i(\text{Diff} D^n, \partial D^n) \)
contains a subgroup isomorphic to \( Z_p \) and the composite
\[
\pi_i(\text{Diff}(D^n, \partial D^n)) \xrightarrow{\lambda_0^n} \pi_i(\text{Diff}(D^{n+1}, \partial D^{n+1}))
\]
\( \rightarrow \cdots \rightarrow \pi_0(\text{Diff}(D^{n+i}, \partial D^{n+i})) = \Gamma_{n+i+1} \)
is nontrivial on this subgroup.

**Remark.** — This theorem gives first examples of nontrivial
\( \pi_i(\text{Diff}(D^n, \partial D^n)) \) surviving in \( \Gamma_{n+i+1} \) for \( i > n \) (as also for \( i > kn \) for any integer \( k \)).
Slightly better computations will be discussed in a forthcoming paper.

**Few words about proofs.** — The main theorem is Theorem 1.2
whose key step is point 2 which we will sketch in section 3.
Points 1), 3), 4) follow from point 2), considering various
natural fibrations.

Theorem 1.3 follows from Theorem 1.2.2 using the contractibility of \( \text{Homeo}(D^n, \partial D^n) \).

In order to prove Theorem 1.4 we use Theorem 2.1. (see next
section), a consequence of Theorem 1.2.2, and the mentioned
result of Cerf [4] for the particular case \( M^n = S^n \).

Using Theorem 1.3 we can define \( \psi_{k,n,r} \) as a bilinear
pairing:
\[
\psi_{k,n,r}: \pi_{n+k+1}(\text{Top}_n/O_n) \otimes \pi_{r+n+k+1}(S^{n+k+1}) \rightarrow \pi_{k+n+r+1}(\text{Top}_n/O_n)
\]
taking \( \psi_{k,n,r}(\alpha, \beta) = \alpha \cdot \beta \). To prove Theorem 1.6 one
needs Toda's result about \( p \)-primary components of
\( \pi_*(G) \) [12].
It is known from Milnor that the fibration \( 0 \rightarrow \text{Top} \rightarrow \text{Top/O} \)
induces for homotopy an exact sequence
\[
0 \rightarrow \pi_i(\text{O}) \rightarrow \pi_i(\text{Top}) \rightarrow \pi_i(\text{Top/O}) \rightarrow 0.
\]
(Because according to Kirby-Siebenmann
\[
\text{Top}_n/\text{PL}_n \cong \text{Top}/\text{PL} \cong K(\mathbb{Z}_2, 2n) n \geq 5
\]
we will discuss here only \( \text{Top}_n, \text{Top}_n/\text{O}_n, \text{Top}/\text{O}, \) similar
results are true for \( \text{PL}_n, \text{PL}_n/\text{O}_n, \text{PL}/\text{O} \).)

From now on we will assume \( n \geq 5 \). (The results remain
correct without dimensional restriction for \( \text{PL} \) instead of
\( \text{Top} \).)

It is known from Baratt-Mohawald that \( \pi_i(\text{O}) \rightarrow \pi_i(\text{O}) \) is
a split surjection for \( i \leq 2n - 2 \).

Haefliger and Wall have proved in [6] that \( \pi_i(\text{Top}/\text{Top}_n) = 0 \)
for \( i \leq n \).

It is therefore natural to conjecture

\((C_1)\) Conjecture 1: \( 0 \rightarrow \pi_i(\text{O}) \rightarrow \pi_i(\text{Top}_n) \rightarrow \pi_i(\text{Top}_n/\text{O}_n) \rightarrow 0 \)
is an exact sequence for \( i \leq 2n - 2 \).

\((C_2)\) Conjecture 2: \( \pi_i(\text{Top}_n) \rightarrow \pi_i(\text{Top}) \) is surjective for
\( i \leq 2n - 2 \). Although we cannot prove \( C_1 \) and \( C_2 \) at the
moment, we will state some results which supply some evi-
dence for the truth of these conjectures.

**Theorem 2.1.** 1) \( \pi_i(\text{Top}_n/\text{O}_n) \rightarrow \pi_i(\text{Top}_{n+k}/\text{O}_{n+k}), \ k \geq 1 \),
are bijective for all \( i \leq n + 1 \) and surjective for all \( i \leq n + 2 \);

2) \( \pi_i(\text{Top}_n/\text{O}_n) \rightarrow \pi_{i-1}(\text{O}_n) \) is zero for \( i \leq n + 1 \).

**Theorem 2.2.** 1) \( \pi_i(\text{Top}_n) \rightarrow \pi_i(\text{Top}) \) are isomorphisms
for \( i \leq n - 2 \) (Haefliger-Wall [6]).

2) \( 0 \rightarrow \pi_n(\text{O}, \text{O}_n) \rightarrow \pi_{n-1}(\text{Top}_n) \rightarrow \pi_{n-1}(\text{Top}) \rightarrow 0 \) and
\( 0 \rightarrow \pi_{n+1}(\text{O}, \text{O}_n) \rightarrow \pi_n(\text{Top}_n) \rightarrow \pi_n(\text{Top}) \rightarrow 0 \)
are exact sequences;

3) \( \pi_{n+1}(\text{Top}_n) \rightarrow \pi_{n+1}(\text{Top}) \) is surjective.

The proof of Theorem 2.1 uses Theorem 1.3 for 1 and 2 plus
the E.H.P.-sequence for spheres for 3. Theorem 2.2 uses
Theorem 2.1 plus well known facts about some homotopy of Stiefel manifolds.

**Detecting exotic homotopy of \( \text{Top}_n \).**

It is easy to see that all the elements \( \psi_{0,n::r_0,...,r_s}(\alpha) \),

\[
\alpha \in \pi_0 (\text{Diff} (D^n, \partial D^n)) = \pi_{n+1}(\text{Top}_n/O_n)
\]
satisfy \( \delta . \psi_{0,n::r_0,...,r_s}(\alpha) = 0 \), \( \delta : \pi_* (\text{Top}_n/O_n) \to \pi_{*-1}(O_n) \).

Therefore we can find \( \beta \in \pi_{n+1 + r_1 + ... + r_s}(\text{Top}_n/O_n) \) with \( j_*(\beta) = \psi_{0,n::r_0,...,r_s}(\alpha) \), \( j \) being the canonical map \( \text{Top}_n \to \text{Top}_n/O_n \).

In fact all the elements in \( \pi_* (\text{Diff} (D^n, \partial D^n)) = \pi_{*+n+1}(\text{Top}_n/O_n) \)
constructed in [1] lift in \( \pi_{*+n+1}(\text{Top}_n) \).

These elements are «exotic» because they represent topological bundles over spheres with \( n \)-euclidean space as fibre which does not reduces to differentiable bundles (notice we construct such topological bundles over \( S^k \) with \( k \gg n \). There are a lot of such examples for \( k < n \).

3.

In this section I will sketch the proof of Theorem 1.2,2). Let \( M^n \) be a compact connected differentiable manifold with nonempty boundary,

\[
\tilde{M} = M^n \cup \partial M \times [0, \infty] \quad M^n_2 = M^n \cup \partial M \times [0, \alpha],
\]
P an interior collar of \( \partial M \) in \( M \), and \( N^n \) an open manifold without boundary. We define the semisimplicial Kan complexes \( \text{Diff} (M^n; P) \), \( \text{Homeo} (M^n; P) \), \( E^d(M^n; N) \), \( E'(M^n, N) \), \( \text{Im}(M^n; N) \), \( \text{R}^d(M^n; N) \), \( \text{R}'(M^n; N) \), \( \text{R}^d(M^n, P^n; \tilde{M}) \) and \( \text{R}'(M^n, P^n; \tilde{M}) \) as follows:

\( \text{Diff} (M^n, P^n) \) and \( \text{Homeo} (M^n, P^n) \):

- A \( k \)-simplex is a diffeomorphism (homeomorphism)

\[
\sigma : \Delta[k] \times M \to \Delta[k] \times M (2)
\]

which agrees with the identity on a neighborhood of \( \Delta[k] \times P^n \)

(2) When we write \( S \times X \to S \times Y \) we mean a commutative diagram with \( S \times X \to S \) and \( S \times Y \to S \) the first component projection.
in $\Delta[k] \times M^n$. ($d_i$ and $s_i$ are obviously defined). These two semisimplicial complexes are semi-simplicial groups.

$E^d(M^n, N), E^t(M^n, N)(I^d m(M^n, N^n), I^t m(M^n, N^n))$:

A $k$-simplex of $E^d...$ or $E^t...$ is an equivalence class of differentiable (topological) embeddings

$$\sigma: \Delta[k] \times Q \to \Delta[k] \times N$$

with $Q$ an open neighborhood of $M$ in $\tilde{M}$. Two such embeddings $(\sigma_1, Q_1)$ ($\sigma_2, Q_2$) are equivalent if they agree on $\Delta[K] \times Q$, $Q$ a neighborhood of $M$ contained in $Q_i$, $i = 1, 2$.

Replacing embeddings by immersions we define similarly the $k$-simplexes of $I^d m, ..., I^t m ...$

$R^d(M^n; N^n), R^t(M^n; N^n)(R^d(M^n, P^n; \tilde{M}^n), R^t(M^n, P^n; \tilde{M}^n))$:

The $k$-simplexes are continuous linear representations respectively (germs of topological representations) of the tangent bundle $T(M^n)$ in $T(N^n)$, i.e.

$$f: \Delta[k] \times T(M) \to \Delta[k] \times T(N)$$

where $T(M)$ and $T(N)$ are the tangent vector bundles of $M$ and $N$. $R^d(M^n, P; \tilde{M}^n)$ and $R^t(M^n, P; \tilde{M})$ are the semisimplicial subcomplexes of $R^d(M; \tilde{M}^n)$ and $R^t(M; \tilde{M}^n)$ consisting of those simplexes $f$ which restrict to the identity on $\Delta[k] \times P$.

One has the following commutative diagram:
where (1), (2), (3), (4) are Kan fibrations (in fact principal fibrations because Diff... and Homeo..., are semisimplicial groups and \( R^d(M^n, P; M^n) \) and \( R^d(M^n, P; M^n) \) semisimplicial semigroups).

By immersion theory (Smale-Hirsch in differentiable case and J. Lees [9] in topological case \( d^d \) and \( d^l \) are homotopy equivalences. Therefore, as soon as we prove that the inclusion \((E^l, E^i) \hookrightarrow (I^m..., I^i m...)\) induces an isomorphism for all homotopy groups (and \( \pi_1 \)) with respect to any base point, we conclude that

\[
d : (\text{Homeo} (M^n; P^n), \text{Diff} (M^n, P^n)) \rightarrow (R^d(M^n, P^n, M^n), R^d(M^n, P^n, \tilde{M}^n))
\]

induces an isomorphism for all homotopy groups (and \( \pi_1 \)) with respect to any base point. Because the inclusions \( \text{Diff} (M^n, P^n) \subset \text{S}^d \text{Diff} (M^n, \partial M^n) \) and

\[
\text{Homeo} (M^n, P^n) \subset S \text{Homeo} (M^n, \partial M^n)
\]

are homotopy equivalences by standard homotopy theoretic arguments one obtains, Theorem 1.2,2) except injectivity on connected components (for this we have to use « Smoothing theory » « à la » Lashof [9]).

To check that the inclusion \((E^l, E^i) \hookrightarrow (I^m..., I^i m...)\) induces (for any base point) an isomorphism between homotopy groups (and \( \pi_1 \)) it suffices to check for \((I^m..., E^i...) \subset (I^m..., E^i...)\). This fact will be obtained as a consequence of « Fibrewise Smoothing Theorem » (Theorem 3.1. (k)), and the topological Ambient Isotopy Theorem.

We define a \( k \)-fibrewise smoothing as a differentiable structure \( \theta \) on \( I^k \times \tilde{M} \), so that:

1) the first factor projection \((I^k \times \tilde{M})_0 \rightarrow I^k\) is a submersion;
2) \((\{0\} \times \tilde{M})_0\) is concordant to the initial differentiable structure on \( \tilde{M} \).

**Theorem 3.1. (k).** — Assume \( n \neq 4,5 \) or \( n = 4 \) and \( \partial M^6 = S^4 \); given any \( \alpha, \beta, \gamma, 0 \leq \alpha < \beta < \gamma \), and \( \theta \) a \( k \)-fibrewise
smoothing there exists a differentiable embedding
\[ \varphi : I^k \times M_\beta \to (I^k \times \bar{M})_\theta \]
so that:
1) \( \varphi(I^k \times M_\beta) \subset \text{Int}(I^k \times M_\gamma), I^k \times M_\Delta \subset \varphi(I^k \times \text{Int} M_\beta). \)
2) \( \varphi(\{0\} \times M_\beta) \setminus \{0\} \times \text{Int} M_\Delta \) is homeomorphic to
\( \partial M \times [0,1]. \)
3) if \( \psi : I^{k'} \times \{0\} \times M_\beta \to (I^{k'} \times \{0\} \times \bar{M})_\theta \)
\[ \leftarrow I^{k'} \times \{0\} \]
is a differentiable embedding compatible with 1) and 2) then one can choose \( \varphi \) so that \( \varphi/I^{k'} \times \{0\} \times M_\beta = \psi. \)

Proof of the isomorphism
\[ \pi_i(I^{d'}m..., E^{d'}...) \cong \pi_i(I^m..., E^t...). \]

Surjectivity: Start with a topological immersion
\[ \sigma : I^k \times \bar{M} \to I^k \times N \]
\[ \leftarrow I^k \]
so that \( \sigma \) restricts on \( \partial I^k \times \bar{M} \) to an embedding and on \( \{0\} \times \bar{M} \) to a differentiable embedding which represents the base point. \( \sigma \) induces on \( I^k \times \bar{M} \) a fibrewise smoothing, the pull back of the differentiable structure of \( I^k \times N \) by \( \sigma. \)
We have then

\[ \sigma \]
\[ \text{id} \]
\[ \varphi \]
\[ \sigma \]
\[ \text{id} \]
\[ \varphi \]

with \( \sigma \) and \( \text{id}/\{0\} \times \bar{M} \) differentiable immersions. Applying
Theorem 3.1 (k) with $\psi/\{0\} \times M_\beta = id$ we find a differentiable embedding $\varphi^1: I^k \times M_\beta \to (I^k \times \hat{M})_0$ so that

$$\varphi^1/\{0\} \times M_\beta = id$$

and applying the topological ambient isotopy theorem ([9] pp 145) we can find $\varphi^t: I^t \times M_\beta \to (I^t \times \hat{M})$ topological embedding so that $\varphi^0 = id$; $\varphi^t/\{0\} \times M_\beta = id$ and $\varphi^t$ depends continously of $t$. We take $\sigma^t = \sigma. \varphi^t$ (by the definition of $I^t m$, $E^t$, $(\sigma, \hat{M})$ is equivalent to $(\sigma_0, \text{Int } M_\beta)$). $\sigma_1$ represents an element in $\pi_k(I^t m..., E^d...)$ which is homotopic to $(\sigma = \sigma_0, \hat{M})$.

**Injectivity.** — We use the same kind of arguments although the proof is easier. The proof of Theorem 3.1 follows (in a slightly different form) a beautiful inductive argument due to R. Lashof. (The proof which I indicated in my talk (Juny 1972) was more technical and more complicated and was essentially based on the Witehead theory of transverse fields).

In order to make clear this inductive argument we need Lemma 3.2. Proposition 3.3 and Corollary 3.4.

**Lemma 3.2.** — Given $\varphi_i: I^k \times M_\alpha \to (I^k \times \hat{M})_0$ $i = 1, 2$ differentiable imbeddings, so that $\varphi_1(I^k \times M_\alpha) \subset \text{Int } \varphi_2(I^k \times \text{Int } M_\alpha)$ and $\varphi_2(\{0\} \times M_\alpha) \setminus \varphi_1(\{0\} \times \text{Int } M_\alpha)$ is diffeomorphic to $\partial M \times [0, 1]$ then for any $\alpha' > \alpha$ there exists

$$\psi: I^k \times M_\alpha \to (I^k \times \hat{M})_0$$

with $\psi/I^k \times M_\alpha = \varphi_1$ and $\psi(I^k \times M_\alpha) = \varphi_2(I^k \times M_\alpha)$ (as sets).

The proof is an immediate consequence of the differentiable ambient isotopy theorem.

We denote by $I^k_1$, $I^k_2$, $I^k_3$ the products $I^{k-1} \times [0, 3/4]$, $I^{k-1} \times [1/4, 1]$ and $I^{k-1} \times [1/4, 3/4]$. 
ON THE HOMOTOPY TYPE OF DIFF (M^k)

PROPOSITION 3.3. — Given two differentiable embeddings

\[ \varphi_i : I^k \times M_\alpha \to (I^k \times \tilde{M})_0 \]

so that \( \varphi_i/I^k_0 \) satisfy the conditions of Lemma 3.2, there exists a differentiable embedding

\[ \psi : I^k \times M_\alpha \to (I^k \times \tilde{M})_0 \]

with \( \psi/I^{k-1} \times [0, 1/4] \times M_\alpha = \varphi_1, \psi(I^k_0 \times \text{Int } M_\alpha) \supseteq \varphi_1(I^k_0 \times M_\alpha) \) and \( \psi(I^{k-1} \times [3/4, 1] \times M_\alpha) = \varphi_2(I^{k-1} \times [3/4, 1] \times M_\alpha) \) (as sets).

Using Lemma 3.2 the proof is immediate.

COROLLARY 3.4 (k). — Given \( \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3 \) and \( \theta \) a \( k \)-fibrewise smoothing so that \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 \) there exists a differentiable embedding \( \varphi : I^k \times M_{\beta_i} \to (I^k \times \tilde{M})_0 \) such that

1) \( \varphi(I^k \times M_{\beta_i}) \subseteq I^k \times \text{Int } M_{\alpha_i+\epsilon}, \ I^k \times M_{\alpha_i} \subseteq \varphi(I^k \times \text{Int } M_{\beta_i}) \);
2) \( \varphi(\{0\} \times M_{\beta_i}) \setminus \varphi(\{0\} \times \text{Int } M_{\alpha_i}) \) is homeomorphic to \( \partial M \times [0,1] \).

Theorem 3.1 (k) plus the s-cobordism theorem plus Proposition 3.3 implies Corollary 3.4 (k), and Corollary 3.4 (k) implies Theorem 3.1 (k + 1) as follows. One take \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 \) with \( \alpha_1 = \alpha, \beta_1 = \beta \) and \( \alpha_2 = \gamma \) and for any \( t \in [0,1] \) one construct the differentiable embedding

\[ \varphi_t : I^k \times \{t\} \times M_{\beta_i} \to (I^k \times \{t\} \times \tilde{M})_0 \]

applying Corollary 3.4 (k). Because the first factor projection \( (I^{k+1} \times M)_0 \to I^{k+1} \) is a submersion one can extend \( \varphi \) to a differentiable embedding \( \varphi_i \),

\[ \varphi_i : I^k \times [t - \epsilon, t + \epsilon] \times M_{\beta_i} \to (I^k \times [t - \epsilon, t + \epsilon] \times \tilde{M})_0. \]
Using the compactness of $[0,1]$ one can find

$$t_0 = 0 < t_1 < \cdots < t_n = 1$$

so that $\varphi_i = \varphi_{i+1}$ are defined on $I^k \times [t_{i-1}, t_{i+1}] \times M_\beta$ and satisfy the requirements of Corollary 3.4. Applying successively Proposition 3.3. to $\varphi_1 = \varphi_i/I^k \times [t_i - \varepsilon, t_i + \varepsilon] \times M_\beta$ and to $\varphi_2 = \varphi_{i+1}/I^k \times [t_i - \varepsilon, t_i + \varepsilon] \times M_\beta$ we construct $\varphi$ « step by step » with the required properties. Clearly we can start with $\varphi_0$ any given differentiable embedding compatible with the conditions 1) and 2) in which case we choose conveniently $\alpha_2, \beta_2$.

*Added in proofs:*

1) The results presented in this report will appear in «detailed and more complete version» as a joint paper with R. Lashof «On the homotopy type of the groups of automorphisms» I and II (who also obtained mostly of them).

2) It turns out that in Theorem 1.2 one can drop the restrictions $n \neq 5$ (point 2).

3) The results stated in Theorem 1.6 are much better (see the forthcoming paper Burghelea-Lashof ... II).

4) Recently I. A. Volodin, who also proved (independently) the results about pseudoisotopy announced by Hatcher-Wagoner, has succeeded in the computation of $\pi_1 (\text{Diff } D^n)$. His computation implies

$$\pi_1 (\text{Diff } (D^n, D^{n-1})) = \pi_{n+2} (\text{Top}_n/O_n, \text{Top}_{n-1}/O_{n-1}) = Z_2 \oplus \text{Wh}^n(O)$$

if $n \geq 8$. The existence of an element of order 2 (and I think the generator of $Z_2$) has been conjectured by Chenciner who found an explicit candidate for.

**BIBLIOGRAPHY**


Dan Burghelea,
Mathematics Institute,
Academy of Science,
Calea Grivitei 21
Bucuresti 12 (Roumanie)