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DAN BURGHELEA

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ON THE HOMOTOPY TYPE OF DIFF (M^n) AND CONNECTED PROBLEMS

by Dan BURGHELEA [1]

This paper is a report on my recent work on the homotopy type of Diff (M^n), M^n a compact differentiable manifold, as well as on the homotopy properties of Top_n, Top_n/O_n, (which can be derived as consequences of Theorem 1.3.).

1.

Given M^n a differentiable (throughout this paper « differentiable » means C^∞-differentiable) manifold with or without boundary, one defines Diff (M; K) (Homeo (M; K)) the topological group of all diffeomorphisms (homeomorphisms) which are the identity on K and are homotopic to identity among continous maps $h: (M, \partial M) \to (M, \partial M)$, $h|K = id$, endowed with $C^\infty(C^0)$-topology.

PROPOSITION 1.1. — *If* K *is a differentiable submanifold of* M, *with or without boundary,* Diff (M; K) *is a differentiable separable Frechet manifold and hence*

1) Diff (M; K) *has the homotopy type of a countable* CW-*complex.*

2) *As a topological space,* Diff (M; K) *is completely determined (up to a homeomorphism) by its homotopy type* (see [3], [8]).

In order to compare the groups Diff and Homeo, it is convenient to consider their semisimplicial analogues, namely the

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semisimplicial groups $S^d(\mathrm{Diff}\,(M^n; K))$, the singular differentiable complex of Diff $(M^n; K)$ and SHomeo $(M^n; K)$ the singular complex of Homeo $(M^n; K)$.

The semisimplicial groups S^d Diff, S Homeo, suggest that one should consider simultaneously the simplicial groups $\widetilde{\mathrm{Diff}}\,(M^n; K)$ and $\widetilde{\mathrm{Homeo}}\,(M^n; K)$ whose k-simplexes are diffeomorphisms, respectively homeomorphisms

$$h: \Delta[k] \times M \to \Delta[k] \times M$$

so that h restricts to the identity on $\Delta[k] \times K$

$$h(d_i \Delta[k] \times M) \subset d_i (\Delta[k] \times M),$$

and are homotopic to identity among the continous maps $f: (\Delta[K] \times M, \Delta[K] \times \partial M) \to (\Delta[K] \times M, \Delta[K] \times \partial M)$ with $f|\Delta[K] \times K = id$ and $f(d_i\Delta[K] \times M) \subset d_i\Delta[K] \times M$. (These simplicial groups have no s_i-operators hence are not semisimplicial, however we still have a good homotopy theory for them, see [11]). The relationship between these simplicial groups and S^d Diff, S Homeo is expressed by the following commutative diagram whose arrows are all inclusions

$$\begin{array}{ccc} S^d\,\mathrm{Diff}\,(M; K) & \to & S\,\mathrm{Homeo}\,(M; K) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Diff}}\,(M; K) & \to & \widetilde{\mathrm{Homeo}}\,(M; K). \end{array}$$

The surgery methods plus the smoothing theory permit (at least from a theoretic point of view) the description of the homotopy type of $\widetilde{\mathrm{Diff}}\,(M)$ (for homotopy groups see [2]).

For any closed differentiable manifold M^n we denote by $\mathscr{P}^d(M)$ respectively $\mathscr{P}^t(M)$ the topological groups Diff $(M \times I, M \times \{0\})$ respectively Homeo $(M \times I, M \times \{0\})$.

It is not difficult to check the commutativity of the following diagram of groups as well as the exacteness of its lines (the arrows are obviously defined)

$$\begin{array}{ccccccccccc} \pi_1(\mathrm{Diff}\,M) & \to & \pi_1(\widetilde{\mathrm{Diff}}\,M) & \to & \pi_0(\mathscr{L}^d(M)) & \to & \pi_0(\mathrm{Diff}\,M) & \to & \pi_0(\widetilde{\mathrm{Diff}}\,(M)) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \pi_1(\mathrm{Homeo}(M)) & \to & \pi_1(\widetilde{\mathrm{Homeo}}(M)) & \to & \pi_0(\mathscr{L}^t(M)) & \to & \pi_0(\mathrm{Homeo}(M)) & \to & \pi_0(\widetilde{\mathrm{Homeo}}(M)) & \to & 0 \end{array}$$

In [4] Cerf has proved that if $n \geqslant 5$, $\pi_1(M^n) = 0$ then $\pi_0(\mathscr{P}^d(M)) = 0$. Hatcher and Wagoner have computed $\pi_0(\mathscr{P}^d(M))$ when $\pi_1(M^n) \neq 0$ and $n \geqslant 7$ (see [7]).

As a consequence of Theorems 1.1 and 1.2 which will be stated below one has.

THEOREM 1.4. — *If* $n \neq 4{,}5$ *then* $\pi_0(\mathcal{P}^d(M)) \simeq \pi_0(\mathcal{P}^t(M))$. (There exists also a p.l. version of $\mathcal{P}^d$, $\mathcal{P}^{Pl}$ in which case

$$\pi_0(\mathcal{P}^{Pl}(M)) = \pi_0(\mathcal{P}^d(M))).$$

Forgeting the differentiable structure on M^n we can consider the Top_n-principal bundle $\mathrm{Top}_n \to T(M^n) \to M^n$ associated to the tangent microbundle of M^n. Using the action of Top_n on Top_n/O_n we associate to $\mathrm{Top}_n \to T(M^n) \to M^n$ the bundle $\mathrm{Top}_n/O_n \to E \to M^n$. The differentiable structure of M^n defines a precised O_n-reduction of

$$\mathrm{Top}_n \to T(M^n) \to M^n,$$

hence a precised (up to a homotopy) crossection s of E. If M^n has nonempty boundary, the bundle

$$\mathrm{Top}_n/O_n \to E/\partial M \to \delta M$$

contains a well defined subbundle $\mathrm{Top}_{n-1}/O_{n-1} \to E^{\partial M} \to \partial M$ (associated with ∂M), and the section s restricts on ∂M to a section in $E^{\partial M}$. We denote by $\Gamma(E, s)$ the base point space of all continuous crossections of E with the compact open topology and section s as base-point, by $\Gamma(E, E^\partial, s)$ the subspace of all continuous crossections of E which restrict on ∂M to crossections in E^∂, by $\Gamma^K(E, s)$ the subspace of all continuous crossections in E which agree with s on K and by $\Gamma^K(E, E^\partial, s)$ the intersection $\Gamma^K(E, s) \cap \Gamma(E, E^\partial, s)$.

For a semisimplicial complex X we denote by $\{X\}$ its geometric realisation.

THEOREM 1.2. — *There exists the base-point preserving maps* i, *which are homotopy equivalences on any connected component and induce injective correspondance between connected components in any of the following cases*

1) M^n *closed*, $n \neq 4$,

$$i : \{S\ \mathrm{Homeo}\ (M^n)/S^d\ \mathrm{Diff}\ (M^n)\} \to \Gamma(E, s).$$

2) $\partial M^n \neq \emptyset$, $n \neq 4{,}5$ *or* $n = 5$, $\partial M^5 = S^4$,

$$i : \{S\ \mathrm{Homeo}\ (M^n; \partial M^n)/S^d\ \mathrm{Diff}\ (M^n; \partial M^n) \to \Gamma^{\partial M}(E, s).$$

3) $\partial M^n \neq \emptyset$, $n \neq 4,5$,

$$i : \{S\,\mathrm{Homeo}\,(M^n)/S^d\,\mathrm{Diff}\,(M^n)\} \to \Gamma(E, E^\partial, s).$$

4) $\partial M^n \neq \{\emptyset\}$ K *differentiable submanifold of* ∂M, $n \neq 4,5$,

$$i : \{S\,\mathrm{Homeo}\,(M^n, K)/S^d\,\mathrm{Diff}\,(M^n, K)\} \to \Gamma^K(E, E^\partial, s).$$

The base point of $\{S\,\mathrm{Homeo}\ldots/S^d\,\mathrm{Diff}\ldots\}$ is represented by the neutral element « O » of S Homeo...

Remark. — A similar theorem (without any dimensional restriction, comparing Diff (M) with $\mathrm{Homeo}^{Pl}(|M|)$ ($|M|$ being the p.l-manifold associated to M via the Whitehead triangulation) is true, but a precise statement will require some considerations about p.d topology which will make this paper much longer.

There exists a (strong H-space) map

$$\lambda_n : \Omega\,\mathrm{Diff}\,(D^n, \partial D^n) \to \mathrm{Diff}\,(D^{n+1}, \partial D^{n+1})$$

(see [1]) which associates to any differentiable path

$$\alpha : I \to \mathrm{Diff}\,(D^n, \partial D^n)$$

with $\alpha/\partial I = \{0\}$ the diffeomorphism

$$\tilde{\alpha} : D^n \times I \to D^n \times I\ (\tilde{\alpha}(x, t) = \alpha_t(x), t)$$

which restricts to the identity on $\partial(D^n \times I)$. $\tilde{\alpha}$ can be viewed as a diffeomorphism of D^{n+1} which restricts to the identity on ∂D^{n+1}.

THEOREM 1.3. — *For any* $n \neq 4$ *there exists a (strong H-space) homotopy equivalence*

$$h_n : \mathrm{Diff}\,(D^n, \partial D^n) \to \Omega^{n+1}(\mathrm{Top}_n/O_n)$$

so that the following diagram is homotopy commutative

$$\begin{array}{ccc} \Omega\,\mathrm{Diff}\,(D^n, \partial D^n) & \xrightarrow{\Omega h_n} & \Omega^{n+2}(\mathrm{Top}_n/O_n) \\ \downarrow{\scriptstyle \lambda_n} & & \downarrow{\scriptstyle \Omega^{n+2} i_n} \\ \mathrm{Diff}\,(D^{n+1}, \partial D^{n+1}) & \xrightarrow{h_{n+1}} & \Omega^{n+2}(\mathrm{Top}_{n+1}/O_{n+1}) \end{array}$$

with i_n *the natural map* $\mathrm{Top}_n/O_n \to \mathrm{Top}_{n+1}/O_{n+1}$.

Very important aknowledgement.

A weak form of Theorem 1.3, the isomorphism between the homotopy groups of $\mathrm{Diff}(D^n, \partial D^n)$ and of $\Omega^{n+1}(\mathrm{Top}_n/O_n)$ has been conjectured by Cerf [5], and proted for $n = 3$.

Theorem 1.3 *and Theorem* 1.2 *have been annonced by C. Morlet* [10] but since then no complete proofs have appeared. Partly this work is *my attempt to understand and prove Morlet's statements*. Actually it seems that the proof sketched in section 3 is different from Morlet's ideas. As it will be pointed out in § 3 the present proof owes very much to R. Lashof who also can prove Theorem 1.2 and 1.3.

Let us denote by G the topological semigroup $G = \varinjlim G_n$ where G_n is the topological semigroup of all homotopy equivalences of S^{n-1}, and $G_n \hookrightarrow G_{n+1}$ is induced by suspension $\pi_i(G_n) = \pi_{i+n}(S^n)$.

THEOREM 1.5. — *For any* (k, n, r) *positive integers* $r < n+k$ *there exists a bilinear pairing*

$$\psi_{k,n,r} : \pi_k(\mathrm{Diff}(D^n, \partial D^n)) \otimes \pi_r(G) \to \pi_{k+r}(\mathrm{Diff}(D^n, \partial D^n))$$

so that:

1) $\psi_{k-1,n+1,r}(\lambda_n^* \otimes id) = \lambda_n^* \circ \psi_{k,n,r}$ *with* λ_n^* *the homomorphism induced by* λ_n *between corresponding homotopy groups.*

2) $\lambda_1^* . \lambda_2^* . \ldots \lambda_n^* . \psi_{0,n,r}$ *is the Bredon pairing* (Annals of Math. 1967, Vol. 86).

Remarks. — 1) There exists an unstable version of this pairing.

2) The pairing $\psi_{k,n,r}$ is a strong generalization and extension of the Munkres-Milnor-Novikov pairing (see [1]) and it allows us to improve considerably the list of nontrivial homotopy groups $\pi_k(\mathrm{Diff}(D^n, \partial D^n))$.

In order to use this pairing for computational purposes one defines

$$\psi_{k,n;r_1,r_2,\ldots,r_s} : \pi_k(\mathrm{Diff}(D^n, \partial D^n)) \otimes \pi_{r_1}(G) \otimes \ldots \pi_{r_s}(G) \to \pi_{k+r_1+\ldots r_s}(\mathrm{Diff}(D^n, \partial D^n))$$

for $r_1 < n + k$, $r_2 < n + k + r_1, \ldots, r_s < n + k + \ldots r_{s-1}$, inductively by

$$\psi_{k,\, n;\, r_1, \ldots r_s} = \psi_{k + r_1 + \ldots r_{s-1},\, n;\, r_s}(\psi_{k,\, n;\, r_1, \ldots r_{s-1}} \otimes id)$$

and study its nontriviality using composition in $\pi_*(G)$.

I will mention only the following result.

THEOREM 1.6. — *Given an odd prime* p, *for any* $i \leqslant 2p(p-1)(p-2)+1$ *and*

$$n = 2p(p-1)^2 - i,\ \pi_i\,(\mathrm{Diff}\ D^n, \partial D^n)$$

contains a subgroup isomorphic to $\mathbf{Z}_p$ *and the composite*

$$\pi_i\,(\mathrm{Diff}\,(D^n, \partial D^n)) \xrightarrow{\lambda_n^*} \pi_i\,(\mathrm{Diff}(D^{n+1}, \partial D^{n+1})) \to \cdots \to \pi_0\,(\mathrm{Diff}\,(D^{n+i}, \partial D^{n+i})) = \Gamma_{n+i+1}$$

is nontrivial on this subgroup.

Remark. — This theorem gives first examples of nontrivial $\pi_i\,(\mathrm{Diff}\,(D^n, \partial D^n))$ surviving in Γ_{n+i+1} for $i > n$ (as also for $i > kn$ for any integer k).

Slightly better computations will be discussed in a forthcoming paper.

Few words about proofs. — The main theorem is Theorem 1.2 whose key step is point 2 which we will sketch in section 3. Points 1), 3), 4) follow from point 2), considering various natural fibrations.

Theorem 1.3 follows from Theorem 1.2.2 using the contractibility of Homeo $(D^n, \partial D^n)$.

In order to prove Theorem 1.4 we use Theorem 2.1. (see next section), a consequence of Theorem 1.2,2. and the mentioned result of Cerf [4] for the particular case $M^n = S^n$.

Using Theorem 1.3 we can define $\psi_{k,n,r}$ as a bilinear pairing:

$$\psi_{k,n,r} : \pi_{n+k+1}(\mathrm{Top}_n/\mathrm{O}_n) \otimes \pi_{r+n+k+1}(S^{n+k+1}) \to \pi_{k+n+r+1}(\mathrm{Top}_n/\mathrm{O}_n)$$

taking $\psi_{k,n,r}(\alpha, \beta) = \alpha\,.\,\beta$. To prove Theorem 1.6 one needs Toda's result about p.-primary components of $\pi_*(G)$ [12].

2.

It is known from Milnor that the fibration $O \to \mathrm{Top} \to \mathrm{Top}/O$ induces for homotopy an exact sequence

$$0 \to \pi_i(O) \to \pi_i(\mathrm{Top}) \to \pi_i(\mathrm{Top}/O) \to 0.$$

(Because according to Kirby-Siebennman

$$\mathrm{Top}_n/\mathrm{PL}_n \simeq \mathrm{Top}/\mathrm{PL} \simeq K(Z_2, z) n \geqslant 5$$

we will discuss here only Top_n, Top_n/O_n, Top/O, similar results are true for PL_n, PL_n/O_n, PL/O).

From now on we will assume $n \geqslant 5$. (The results remain correct without dimensional restriction for PL... instead of Top...).

It is known from Baratt-Mohawald that $\pi_i(O_n) \to \pi_i(O)$ is a split surjection for $i \leqslant 2n-2$.

Haefliger and Wall have proved in [6] that $\pi_i(\mathrm{Top}/\mathrm{Top}_n)=0$ for $i \leqslant n$.

It is therefore natural to conjecture

(C_1) *Conjecture* 1 : $0 \to \pi_i(O_n) \to \pi_i(\mathrm{Top}_n) \to \pi_i(\mathrm{Top}_n/O_n) \to 0$ is an exact sequence for $i \leqslant 2n-2$.

(C_2) *Conjecture* 2 : $\pi_i(\mathrm{Top}_n) \to \pi_i(\mathrm{Top})$ is surjective for $i \leqslant 2n-2$. Although we cannot prove C_1 and C_2 at the moment, we will state some results which supply some evidence for the truth of these conjectures.

THEOREM 2.1. — 1) $\pi_i(\mathrm{Top}_n/O_n) \to \pi_i(\mathrm{Top}_{n+k}/O_{n+k})$, $k \geqslant 1$, *are bijective for all* $i \leqslant n+1$ *and surjective for all* $i \leqslant n+2$;

2) $\pi_i(\mathrm{Top}_n/O_n) \to \pi_{i-1}(O_n)$ *is zero for* $i \leqslant n+1$.

THEOREM 2.2. — 1) $\pi_i(\mathrm{Top}_n) \to \pi_i(\mathrm{Top})$ *are isomorphisms for* $i \leqslant n-2$ (Haefliger-Wall [6]).

2) $0 \to \pi_n(O, O_n) \to \pi_{n-1}(\mathrm{Top}_n) \to \pi_{n-1}(\mathrm{Top}) \to 0$ *and*

$$0 \to \pi_{n+1}(O, O_n) \to \pi_n(\mathrm{Top}_n) \to \pi_n(\mathrm{Top}) \to 0$$

are exact sequences;

3) $\pi_{n+1}(\mathrm{Top}_n) \to \pi_{n+1}(\mathrm{Top})$ *is surjective.*

The proof of Theorem 2.1 uses Theorem 1.3 for 1 and 2 plus the E.H.P.-sequence for spheres for 3. Theorem 2.2 uses

Theorem 2.1 plus well known facts about some homotopy of Stiefel manifolds.

Detecting exotic homotopy of Top_n.

It is easy to see that all the elements $\psi_{0,n;r_1,\dots r_s}(\alpha)$,

$$\alpha \in \pi_0(\mathrm{Diff}(D^n, \partial D^n)) = \pi_{n+1}(\mathrm{Top}_n/O_n)$$

satisfy $\delta . \psi_{0,n;r_1,\dots r_s}(\alpha) = 0$, $\delta: \pi_*(\mathrm{Top}_n/O_n) \to \pi_{*-1}(O_n)$. Therefore we can find $\beta \in \pi_{n+1+r_1+\dots r_s}(\mathrm{Top}_n/O_n)$ with $j_*(\beta) = \psi_{0,n;r_1,\dots r_s}(\alpha)$, j being the canonical map

$$\mathrm{Top}_n \to \mathrm{Top}_n/O_n.$$

In fact all the elements in $\pi_*(\mathrm{Diff}(D^n, \partial D^n)) = \pi_{*+n+1}(\mathrm{Top}_n/O_n)$ constructed in [1] lift in $\pi_{*+n+1}(\mathrm{Top}_n)$.

These elements are «exotic» because they represent topological bundles over spheres with n-euclidean space as fibre which does not reduces to differentiable bundles (notice we construct such topological bundles over S^k with $k \gg n$. There are a lot of such examples for $k < n$).

3.

In this section I will sketch the proof of Theorem 1.2,2). Let M^n be a compact connected differentiable manifold with nonempty boundary,

$$\tilde{M} = M^n \cup \partial M \times [0, \infty] \quad M^n_\alpha = M^n \cup \partial M \times [0, \alpha],$$

P an interior collar of ∂M in M, and N^n an open manifold without boundary. We define the semisimplicial Kan complexes $\mathbf{Diff}(M^n; P)$, $\mathbf{Homeo}\ (M^n; P)$, $\mathbf{E}^d(M^n; N)$, $\mathbf{E}^t(M^n, N)$, $\mathbf{I}^d m(M^n; N)$, $\mathbf{I}^t m(M^n; N)$, $\mathbf{R}^d(M^n; N)$, $\mathbf{R}^t(M^n; N)$, $\mathbf{R}^d(M^n, P^n; \tilde{M})$ and $\mathbf{R}^t(M^n, P^n; \tilde{M})$ as follows:

Diff (M^n, P^n) and **Homeo** (M^n, P^n):

— A k-simplex is a diffeomorphism (homeomorphism)

$$\begin{array}{ccc} \sigma: \Delta[k] \times M & \to & \Delta[k] \times M \ (^2) \\ & \searrow \quad \swarrow & \\ & \Delta[k] & \end{array}$$

which agrees with the identity on a neighborhood of $\Delta[k] \times P^n$

(2) When we write $S \times X \to S \times Y$ we mean a commutative diagram with $S \times X \to S$ and $S \times Y \to S$ the first component projection.

in $\Delta[k] \times M^n$. (d_i and s_i are obviously defined). These two semisimplicial complexes are semi-simplicial groups.

$\mathbf{E}^d(M^n, N)$, $\mathbf{E}^t(M^n, N)(\mathbf{I}^d m(M^n, N^n)$, $\mathbf{I}^t m(M^n, N^n))$:

A k-simplex of $\mathbf{E}^d$... or $\mathbf{E}^t$... is an equivalence class of differentiable (topological) embeddings

$$\begin{tikzcd}[column sep=small] \sigma: \Delta[k] \times Q \arrow[rr] \arrow[dr] & & \Delta[k] \times N \arrow[dl] \\ & \Delta[k] & \end{tikzcd}$$

with Q an open neighborhood of M in $\tilde{M}$. Two such embeddings (σ_1, Q_1) (σ_2, Q_2) are equivalent if they agree on $\Delta[K] \times Q$, Q a neighborhood of M contained in Q_i, $i = 1, 2$. Replacing embeddings by immersions we define similarly the k-simplexes of $\mathbf{I}^d m$, ..., $\mathbf{I}^t m$ . . .

$\mathbf{R}^d(M^n; N^n)$, $\mathbf{R}^t(M^n; N^n)(\mathbf{R}^d(M^n, P^n; \tilde{M}^n)$, $\mathbf{R}^t(M^n, P^n; \tilde{M}^n))$:

The k-simplexes are continuous linear representations respectively (germs of topological representations) of the tangent bundle $T(M^n)$ in $T(N^n)$, i.e.

$$\begin{tikzcd}[column sep=small] f: \Delta[k] \times T(M) \arrow[rr] \arrow[dr] & & \Delta[k] \times T(N) \arrow[dl] \\ & \Delta[k] & \end{tikzcd}$$

where $T(M)$ and $T(N)$ are the tangent vector bundles of M and N. $\mathbf{R}^d(M^n, P; \tilde{M}^n)$ and $\mathbf{R}^t(M^n, P; \tilde{M})$ are the semisimplicial subcomplexes of $\mathbf{R}^d(M; \tilde{M}^n)$ and $\mathbf{R}^t(M^n; \tilde{M}^n)$ consisting of those simplexes f which restrict to the identity on $\Delta[k] \times P$.

One has the following commutative diagram :

$$\begin{tikzcd}[column sep=tiny]
 & \mathrm{Diff}(M^n, P) \arrow[rr] \arrow[dl, "i"'] \arrow[dddd, "d^d"] & & E^d(M^n; \tilde{M}) \arrow[rr] \arrow[dl, "i_1"'] \arrow[dd] & & E^d(P^n; \tilde{M}^n) \arrow[dl, "i_2"'] \arrow[dd, hook] \\
\mathrm{Homeo}(M^n, P) \arrow[rr] \arrow[dddd, "d^t"'] & & E^t(M^n, \tilde{M}) \arrow[rr] \arrow[dd, hook] & & E^t(P^n, \tilde{M}^n) \arrow[dd, hook] & \\
 & & & \mathbf{I}^d m(M^n; \tilde{M}^n) \arrow[rr] \arrow[dl, "j_1"'] \arrow[dd, "d^d"] & & \mathbf{I}^d m(P^n, \tilde{M}^n) \arrow[dl, "j_2"'] \arrow[dd, "d^d"] \\
 & & \mathbf{I}^t m(M^n; \tilde{M}) \arrow[rr] \arrow[dd, "d^t"] & & \mathbf{I}^t m(P^n, \tilde{M}^n) \arrow[dd, "d^t"] & \\
 & R^d(M^n, P, \tilde{M}) \arrow[rr] \arrow[dl, "S"'] & & R^d(M^n, \tilde{M}^n) \arrow[rr] \arrow[dl, "S_1"'] & & R^d(P^n, \tilde{M}^n) \arrow[dl, "S_2"'] \\
R^t(M^n, P, \tilde{M}) \arrow[rr] & & R^t(M, \tilde{M}) \arrow[rr] & & R^t(P, \tilde{M}) &
\end{tikzcd}$$

where (1), (2), (3), (4) are Kan fibrations (in fact principal fibrations because **Diff**... and **Homeo**..., are semisimplicial groups and $\mathbf{R}^d(M^n, P; M^n)$ and $\mathbf{R}^t(M^n, P; M^n)$ semisimplicial semigroups).

By imersion theory (Smale-Hirsch in differentiable case and J. Lees [9] in topological case d^d and d^t are homotopy equivalences. Therefore, as soon as we prove that the inclusion $(E^t, E^d) \hookrightarrow (\mathbf{I}^t m..., \mathbf{I}^d m...)$ induces an isomorphism for all homotopy groups (and π_1) with respect to any base point, we conclude that

$$d: (\mathbf{Homeo}\,(M^n; P^n), \mathbf{Diff}\,(M^n, P^n)) \to (\mathbf{R}^t(M^n, P^n, \tilde{M}^n), \mathbf{R}^d(M^n, P^n, \tilde{M}^n))$$

induces an isomorphism for all homotopy groups (and π_1) with respect to any base point. Because the inclusions $\mathbf{Diff}\,(M^n, P^n) \subset S^d\,\mathrm{Diff}\,(M^n, \partial M^n)$ and

$$\mathbf{Homeo}\,(M^n, P^n) \subset S\,\mathrm{Homeo}\,(M_n, \partial M^n)$$

are homotopy equivalences by standard homotopy theoretic arguments one obtains, Theorem 1.2,2) except injectivity on connected components (for this we have to use « Smoothing theory » « à la » Lashof [9]).

To check that the inclusion $(\mathbf{E}^t, \mathbf{E}^d) \hookrightarrow (\mathbf{I}^t m..., \mathbf{I}^d m...)$ induces (for any base point) an isomorphism between homotopy groups (and π_1) it suffices to check for $(\mathbf{I}^t m..., \mathbf{E}^d...) \subset (\mathbf{I}^t m..., \mathbf{E}^t...)$. This fact will be obtained as a consequence of « Fibrewise Smoothing Theorem » (Theorem 3.1. (k)), and the topological Ambient Isotopy Theorem.

We define a k-fibrewise smoothing as a differentiable structure θ on $I^k \times \tilde{M}$, so that:

1) the first factor projection $(I^k \times \tilde{M})_\theta \to I^k$ is a submersion;

2) $(\{0\} \times \tilde{M})_\theta$ is concordant to the initial differentiable structure on $\tilde{M}$.

THEOREM 3.1. (k). — *Assume* $n \neq 4,5$ *or* $n = 4$ *and* $\partial M^5 = S^4$; *given any* α, β, γ, $0 \leqslant \alpha < \beta < \gamma$, *and* θ *a k-fibrewise* *smoothing there exists a differentiable embedding*

$$\begin{tikzcd} I^k \times M_\beta \arrow[rr, "\varphi"] \arrow[dr] & & (I^k \times \tilde{M})_\theta \arrow[dl] \\ & I^k & \end{tikzcd}$$

so that:

1) $\varphi(I^k \times M_\beta) \subset \mathrm{Int}\,(I^k \times M_\gamma)$, $I^k \times M_\alpha \subset \varphi(I^k \times \mathrm{Int}\, M_\beta)$.

2) $\varphi(\{0\}) \times M_\beta)\setminus\{0\} \times \mathrm{Int}\, M_\alpha$ *is homeomorphic to*

$$\partial M \times [0,1].$$

3) if

$$\begin{tikzcd} I^{k'} \times \{0\} \times M_\beta \arrow[rr, "\psi"] \arrow[dr] & & (I^{k'} \times \{0\} \times \tilde{M})_\theta \arrow[dl] \\ & I^{k'} \times \{0\} & \end{tikzcd}$$

is a differentiable embedding compatible with 1) *and* 2) *then one can choose* φ *so that* $\varphi/I^{k'} \times \{0\} \times M_\beta = \psi$.

Proof of the isomorphism

$$\pi_i(\mathbf{I}^{d'}m\ldots, \mathbf{E}^{d'}\ldots) \simeq \pi_i(\mathbf{I}^{t}m\ldots, \mathbf{E}^{t}\ldots).$$

Surjectivity: Start with a topological immersion

$$\begin{tikzcd} I^k \times \tilde{M} \arrow[rr, "\sigma"] \arrow[dr] & & I^k \times N \arrow[dl] \\ & I^k & \end{tikzcd}$$

so that σ restricts on $\partial I^k \times \tilde{M}$ to an embedding and on $\{0\} \times \tilde{M}$ to a differentiable embedding which represents the base point. σ induces on $I^k \times \tilde{M}$ a fibrewise smoothing, the pull back of the differentiable structure of $I^k \times N$ by σ. We have then

$$\begin{tikzcd} I^k \times \tilde{M} \arrow[rr, bend left, "\sigma"] \arrow[dr, "id"] \arrow[ddr] & & I^k \times N \arrow[ddl] \\ & (I^k \times \tilde{M})_\theta \arrow[ur, "\bar{\sigma}"] \arrow[d] & \\ & I^k & \end{tikzcd}$$

with $\bar{\sigma}$ and $id/\{0\} \times \tilde{M}$ differentiable immersions. Applying

Theorem 3.1 (k) with $\psi/\{0\} \times M_\beta = id$ we find a differentiable embedding $\varphi^1 : I^k \times M_\beta \to (I^k \times \tilde{M})_\theta$ so that

$$\begin{tikzcd} I^k \times M_\beta \arrow[rr] \arrow[dr] & & (I^k \times \tilde{M})_\theta \arrow[dl] \\ & I^k & \end{tikzcd}$$

$$\varphi^1/\{0\} \times M_\beta = id$$

and applying the topological ambient isotopy theorem ([9] pp 145) we can find $\varphi^t : I^t \times M_{\beta_1} \to (I^k \times \tilde{M})_\theta$ topological

$$\begin{tikzcd} I^t \times M_{\beta_1} \arrow[rr] \arrow[dr] & & (I^k \times \tilde{M})_\theta \arrow[dl] \\ & I^k & \end{tikzcd}$$

embedding so that $\varphi^0 = id$; $\varphi^t/\{0\} \times M_{\beta_1} = id$ and φ^t depends continously of t. We take $\sigma^t = \bar{\sigma}.\varphi^t$ (by the definition of $I^t m$, E^t, $(\sigma, \tilde{M})$ is equivalent to $(\sigma_0, \text{Int } M_{\beta_1})).\sigma_1$ represents an element in $\pi_k(I^d m..., E^d...)$ which is homotopic to $(\sigma = \sigma_0, \tilde{M})$.

Injectivity. — We use the same kind of arguments although the proof is easier. The proof of Theorem 3.1 follows (in a slightly different form) a beautiful inductive argument due to R. Lashof. (The proof which I indicated in my talk (Juny 1972) was more technical and more complicated and was essentially based on the Witehead theory of transverse fields).

In order to make clear this inductive argument we need Lemma 3.2. Proposition 3.3 and Corollary 3.4.

LEMMA 3.2. — *Given* $\varphi_i : I^k \times M^n_\alpha \to (I^k \times \tilde{M})_\theta$ $i = 1,2$ *differentiable imbeddings, so that* $\varphi_1(I^k \times M_\alpha) \subset \text{Int } \varphi_2(I^k \times \text{Int } M_\alpha)$ *and* $\varphi_2(\{0\} \times M_\alpha) \setminus \varphi_1(\{0\} \times \text{Int } M_\alpha)$ *is diffeomorphic to* $\partial M \times [0,1]$ *then for any* $\alpha' > \alpha$ *there exists*

$$\begin{tikzcd} \psi : I^k \times M_{\alpha'} \arrow[rr] \arrow[dr] & & (I^k \times \tilde{M})_\theta \arrow[dl] \\ & I^k & \end{tikzcd}$$

with $\psi/I^k \times M_\alpha = \varphi_1$ *and* $\psi(I^k \times M_\alpha) \equiv \varphi_2(I^k \times M_\alpha)$ (*as sets*).

The proof is an immediate consequence of the differentiable ambient isotopy theorem.

We denote by I^k_1, I^k_2, I^k_0, the products $I^{k-1} \times [0,3/4]$, $I^{k-1} \times [1/4, 1]$ and $I^{k-1} \times [1/4, 3/4]$.

PROPOSITION 3.3. — *Given two differentiable embeddings*

$$\begin{tikzcd} \varphi_i : \mathrm{I}^k \times \mathrm{M}_\alpha \arrow[rr] \arrow[dr] & & (\mathrm{I}^k \times \tilde{\mathrm{M}})_\theta \arrow[dl] \\ & \mathrm{I}^k & \end{tikzcd}$$

so that φ_i/I^k_0 *satisfy the conditions of Lemma* 3.2, *there exists a differentiable embedding*

$$\begin{tikzcd} \psi : \mathrm{I}^k \times \mathrm{M}_\alpha \arrow[rr] \arrow[dr] & & (\mathrm{I}^k \times \tilde{\mathrm{M}})_\theta \arrow[dl] \\ & \mathrm{I}^k & \end{tikzcd}$$

with $\psi/\mathrm{I}^{k-1} \times [0, 1/4] \times \mathrm{M}_\alpha = \varphi_1$, $\psi(\mathrm{I}^k_0 \times \mathrm{Int}\,\mathrm{M}_\alpha) \supset \varphi_1(\mathrm{I}^k_0 \times \mathrm{M}_\alpha)$ *and* $\psi(\mathrm{I}^{k-1} \times [3/4,1] \times \mathrm{M}_\alpha) \equiv \varphi_2(\mathrm{I}^{k-1} \times [3/4, 1] \times \mathrm{M}_\alpha)$ *(as sets).*

Using Lemma 3.2 the proof is immediate.

COROLLARY 3.4 (k). — *Given* $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3$ *and* θ *a k-fibrewise smoothing so that* $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3$ *there exists a differentiable embedding* $\varphi : \mathrm{I}^k \times \mathrm{M}_{\beta_2} \to (\mathrm{I}^k \times \tilde{\mathrm{M}})_\theta$ *such that*

$$\begin{tikzcd} \mathrm{I}^k \times \mathrm{M}_{\beta_2} \arrow[rr] \arrow[dr] & & (\mathrm{I}^k \times \tilde{\mathrm{M}})_\theta \arrow[dl] \\ & \mathrm{I}^k & \end{tikzcd}$$

1) $\varphi(\mathrm{I}^k \times \mathrm{M}_{\beta_i}) \subset \mathrm{I}^k \times \mathrm{Int}\ \mathrm{M}_{\alpha_{i+1}}$, $\mathrm{I}^k \times \mathrm{M}_{\alpha_i} \subset \varphi(\mathrm{I}^k \times \mathrm{Int}\ \mathrm{M}_{\beta_i})$;

2) $\varphi(\{0\} \times \mathrm{M}_{\beta_i}) \setminus \varphi(\{0\} \times \mathrm{Int}\ \mathrm{M}_{\alpha_i})$ *is homeomorphic to* $\partial \mathrm{M} \times [0,1]$.

Theorem 3.1 (k) plus the s-cobordism theorem plus Proposition 3.3 implies Corollary 3.4 (k), and Corollary 3.4 (k) implies Theorem 3.1 ($k+1$) as follows. One take $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_2$ with $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $\alpha_2 = \gamma$ and for any $t \in [0,1|$ one construct the differentiable embedding

$$\begin{tikzcd} \varphi_t : \mathrm{I}^k \times \{t\} \times \mathrm{M}_{\beta_2} \arrow[rr] \arrow[dr] & & (\mathrm{I}^k \times \{t\} \times \mathrm{M})_\theta \arrow[dl] \\ & \mathrm{I}^k \times \{t\} & \end{tikzcd}$$

applying Corollary 3.4 (k). Because the first factor projection $(\mathrm{I}^{k+1} \times \mathrm{M})_\theta \to \mathrm{I}^{k+1}$ is a submersion one can extend φ to a differentiable embedding φ_t,

$$\begin{tikzcd} \varphi_t : \mathrm{I}^k \times [t-\varepsilon, t+\varepsilon] \times \mathrm{M}_{\beta_2} \arrow[rr] \arrow[dr] & & (\mathrm{I}^k \times [t-\varepsilon, t+\varepsilon] \times \tilde{\mathrm{M}})_\theta. \arrow[dl] \\ & \mathrm{I}^k \times [t-\varepsilon, t+\varepsilon] & \end{tikzcd}$$

Using the compactness of $[0,1]$ one can find

$$t_0 = 0 < t_1 < \cdots < t_N = 1$$

so that $\varphi_i = \varphi_{t_i}$ are defined on $I^k \times [t_{i-1}, t_{i+1}] \times M_{\beta_2}$ and satisfy the requirements of Corollary 3.4. Applying successively Proposition 3.3. to $\varphi_1 = \varphi_i/I^k \times [t_i - \varepsilon, t_i + \varepsilon] \times M_{\beta_1}$ and to $\varphi_2 = \varphi_{i+1}/I^k \times [t_i - \varepsilon, t_i + \varepsilon] \times M_{\beta_2}$ we construct φ « step by step » with the required properties. Clearly we can start with φ_0 any given differentiable embedding compatible with the conditions 1) and 2) in which case we choose conveniently α_2, β_2.

Added in proofs:

1) The results prensented in this report will appear in « detailed and more complete version » as a joint paper with R. Lashof « On the homotopy type of the groups of automorphisms » I and II (who also obtained mostly of them).

2) It turns out that in Theorem 1.2 one can drop the restrictions $n \neq 5$ (point 2).

3) The results stated in Theorem 1.6 are much better (see the forthcoming paper Burghelea-Lashof ... II).

4) Recently I. A. Volodin, who also proved (independently) the results about pseudoisotopy announced by Hatcher-Wagoner, has succeeded in the computation of π_1 (Diff D^n). His computation implies

$$\begin{aligned}\pi_1 (\mathrm{Diff}\,(D^n, D^{n-1})) &= \pi_{n+2}\,(\mathrm{Top}_n/O_n, \mathrm{Top}_{n-1}/O_{n-1}) \\ &= Z_2 \oplus \mathrm{Wh}^3(O)\end{aligned}$$

if $n \geqslant 8$. The existence of an element of order 2 (and I think the generator of Z_2) has been conjectured by Chenciner who found an explicit candidat for.

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Dan BURGHELEA,

Mathematics Institute,
Academy of Science,

Calea Grivitei 21
Bucuresti 12 (Roumanie)