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EXTENSIONS THROUGH CODIMENSION ONE TO SENSE PRESERVING MAPPINGS

by Charles J. TITUS

1. Introduction.

This paper is an exposition of the main problems in terms of motivation, results, approaches and conjectures.

All manifolds are oriented and of class C^∞; all mappings, unless specifically assumed otherwise, are of class C^∞. Let $\hat{M}$ be a connected manifold of dimension $n+1$ with the naturally oriented boundary $M = \partial\hat{M}$ which in turn is a compact manifold of dimension n; let N be a connected manifold (without boundary) of dimension $n+1$.

A mapping $F : \hat{M} \to N$ is *sense preserving*, $F \in SP$, provided that the derivative mapping

$$dF_x : T\hat{M}_x \to TN_y,\ y = f(x)$$

is for every $x \in \hat{M}$ either singular or bijective and sense preserving. If dF_x is never singular and $F \in SP$ then, of course, F is a sense preserving immersion of $\hat{M}$ in N.

A mapping $f: M \to N$ is SP *extendable to* $\hat{M}$ if there exists an extension of f to $F : \hat{M} \to N$ with $F \in SP$.

We formulate three general problems :

Problem 1. — Given M, N and an $\hat{M}$, characterize the mappings in $C^\infty(M, N)$ that are SP extendable to $\hat{M}$.

Problem 2. — Given M, N and an $\hat{M}$ characterize the immersions in $C^\infty(M, N)$ that are SP extendable to $\hat{M}$.

Problem 3. — Given M, N and an $\hat{M}$ characterize the immersions in $C^\infty(M, N)$ that are extendable to a sense preserving immersion of $\hat{M}$ in N.

For $n \geqslant 2$ Problems 1 and 2 have an essentially different character since the immersions in $C^\infty(M, N)$ are no longer dense in $C^\infty(M, N)$.

2. Motivation.

A problem which can be traced back at least as far as Picard (see [22], 310-314] was formulated by Lœwner and H. Hopf about 1948 in essentially the following form:

(A) Characterize the immersions of the circle in the complex line **C** which can be extended, modulo a sense preserving diffeomorphism on the circle, to a function complex analytic on the disk.

For generic immersions of the circle in **C** it is known (Titus [27]) that (A) is equivalent to.

(B) Characterize the generic immersions of the circle in **C** that are SP extendable (and so (A) is a special case of Problems 1 and 2).

Somewhat later several topologists became interested in the following (which is a special case of Problem 3):

(C) Characterize the generic immersions of the circle in $\mathbf{R}^2$ that are extendable to sense preserving immersions of the disk in $\mathbf{R}^2$.

Part of the charm of these problems, and occasionally some of the frustration, arises from the fact that no standard methods seem to apply. For example, the codimension of 1 as well as the « closed condition » for SP extendability make unlikely any direct application of the general theorems of Gromov, Haefliger, Phillips, et al.

Another motivation for the study of SP mappings lies in the following pair of easily proved Propositions. A mapping $F : \hat{M} \to N$ is called *interior* provided that all open sets in $\hat{M} - M$ are mapped into open sets in N and that $F^{-1}(y)$ is totally disconnected for all $y \in N$.

Proposition 1. — *If* $F : \hat{M} \to N$ *is* SP *and if* $F^{-1}(y)$ *is an isolated set in* $\hat{M} - M$ *for all* $y \in N$ *then* F *is interior.*

Proposition 2. — *If* $F : \hat{M} \to N$ *is interior (and* C^∞*) and a sense preserving homeomorphism on some open set then* $F \in SP$.

So the SP condition is closely related to the condition of interiority which for $n = 1$ characterizes the topology of complex analytic functions (see Stoilow [24], Whyburn [33]). For those of us interested in such generalizations of function theory the development of a generic theory inside the class of SP mappings seems natural. For example, let $\mathcal{S} \subset SP$ be the class of mappings $F : D \to \mathbf{R}^2$, D a closed disk, where dF_x has rank zero wherever dF_x is singular and where dF is non-singular on *bdy* D.

Conjecture 1. — If $F \in \mathcal{S}$ then the interior mappings in $\mathcal{S}$ are dense (e.g., in C^0 topology) in $\mathcal{S}$.

3. Simple Necessary Conditions. Examples and Conjectures.

The most important distinction between arbitrary and SP mappings is that there is no « folding ». More precisely one has directly from local differential degree theory (see e.g., Milnor [20]):

Condition 1. — If $F : \hat{M} \to N$, $F \in SP$, and if $y \notin F(M)$ then the local degree of F at y is positive if and only if $y \in F(\hat{M})$.

Also with $n = 2$ and with $\partial\hat{M}$ a simple oriented circle one can prove directly, by methods closely analogous to function theory, the following inequality on the tangent winding number of $f : \partial\hat{M} \to \mathbf{R}^2$, TWN$f$.

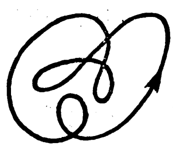
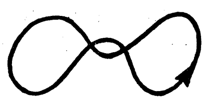
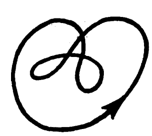
Condition 2. — If $f : \partial\hat{M} \to \mathbf{R}^2$ is an immersion and f is SP extendable to $\hat{M}$ then

$$\text{TWN}f \geqslant \chi(\hat{M})$$

if f is extendable to a sense preserving immersion of M then

$$\mathrm{TWN}f = \chi(\hat{\mathrm{M}}).$$

There is an interesting generalization of the second part of Condition 2 to higher dimensions, with N either a sphere or $\mathbf{R}^n$ due to Gramain [11]. However, these necessary conditions, Conditions 1 and 2, are nowhere near sufficient as the following Figures indicate. For with $\hat{\mathrm{M}}$ the disk: in figure 1, Condition 1 is satisfied but 2 is not; in figure 2, Condition 2 is satisfied but 1 is not; in figure 3, Conditions 1 and 2 are satisfied but there is no SP extension (a simple corollary of Titus [27]).

Now, Loewner and Hopf actually conjectured, with M a disk, that « f an immersion of ∂D with a non-negative winding number (= local degree) about every point in $\mathbf{R}^2 - f(\partial \mathrm{D})$ » should be sufficient for SP extendability to the disk. Although figure 1 represents a counterexample the following seems to be true:

Conjecture 2. — If $f \colon \mathrm{S}^1 \to \mathbf{R}^2$ is an immersion with a non-negative winding number about every point $y \in \mathbf{R}^2 - f(\mathrm{S}^1)$ then there exists a compact manifold M with $\partial \mathrm{M} = \mathrm{S}^1$ so that f is SP extendable to M.

Analogous conjectures in higher dimensions would also be interesting.

There is another « soft » result which is proved for $n = 2$ (unpublished) and which seems also to be provable by the same methods for $n \geqslant 2$.

Conjecture 3. — If $\mathrm{N} = \mathrm{S}^{n+1}$ and $f, g \colon \mathrm{M} \to \mathrm{N}$ are immersions in the same regular homotopy class then either both f and g are SP extendable or neither is SP extendable; thus, with $\mathrm{N} = \mathrm{S}^{n+1}$, SP extendability is a regular homotopy invariant.

A simple corollary of this result for $n = 1$ is that every immersion of S^1 in **C** is extendable, modulo a sense preserving diffeomorphism on S^1, to a meromorphic function on the disk (for a direct proof of this see Verhey [32]).

4. Combinatorial Approach with $n = 2$.

If $\partial \hat{M} = S^1$ and $f: S^1 \to \mathbf{R}^2$ is a transverse immersion then f may be described modulo diffeomorphic equivalence by an « intersection sequence ». The problem may then be transformed into a purely combinatorial one involving this diffeomorphic invariant and an algorithm in known (Titus [26]) which decides whether f is extendable to an interior mapping of the disk and, in particular, whether f is extendable to an immersion of the disk.

Also using different combinatorial data, not quite diffeomorphic invariants, Blank [2,3] gave a very elegant algorithm which decides whether f is extendable to an immersion of the disk and at the same time computes the number of distinct extension classes (two immersions $F, G: D \to \mathbf{R}^2$ are in the same extension class provided there exist sense preserving diffeomorphisms H and K on D and $\mathbf{R}^2$ respectively such that $G = K \circ F \circ H$).

In more recent work by Francis [8], Marx [15, 16, 17, 18], Verhey [19] these two algorithms have been combined and a complete combinatorial theory of the boundary behaviour of interior mappings from M to N (thus in particular of complex analytic and meromorphic functions on two dimensional Riemann Surfaces with boundary) now seems tractible.

5. An Approach via Restricted Homotopy.

It seems likely that these methods can produce interesting results with $n \geqslant 2$ for Problems 2 and 3; however we will restrict ourselves to the case $n = 1$ (where M is a finite union of circles).

With V a set of vectors and let $\mathscr{C}V$ be the linear convex closure; i.e., the set of all finite linear combinations of vectors from V with non-negative real coefficients. Let $K_E(x)$

be the curvature of $f: \mathrm{M} \to \mathrm{N}$ at $y = f(x)$ with respect to a choice of euclidean structure on TN_y (the following will not depend on this choice).

Next we construct a table (E) of « events » defined for an immersion f thought of as an intermediate immersion in a regular homotopy of M in N; each type of event is seen to be a sense preserving diffeomorphic invariant.

E^2 : $f^{-1}(y)$ consists of exactly 2 points x_1, x_2 and $f'(x_1), f'(x_2)$ are linearly dependent,

E^2_1 : $\mathscr{C}\{f'(x_1), f'(x_2)\}$ is a half-line,

E^2_2 : $\mathscr{C}\{f'(x_1), f'(x_2)\}$ is a line,

(E) E^2_{21} : $\mathrm{K_E}(x_1) + \mathrm{K_E}(x_2) > 0,$

E^2_{22} : $\mathrm{K_E}(x_1) + \mathrm{K_E}(x_2) < 0;$

E^3 : $f^{-1}(y)$ consists of exactly 3 points x_1, x_2, x_3 and $f'(x_1), f'(x_2), f'(x_3)$ are pairwise independent.

E^3_1 : $\mathscr{C}\{f'(x_1), f'(x_2), f'(x_3)\}$ is contained in a half-plane,

E^3_2 : $\mathscr{C}\{f'(x_1), f'(x_2), f'(x_3)\}$ is a plane.

A homotopy will be called GR (generic regular) if it is a regular homotopy during which only a finite number of events in (E) occur and at each such time there is exactly one such event; it is seen (Francis [10]) that the GR homotopies are dense and open in the C^2 topology in the space of regular homotopies of M in N.

A GR homotopy is called *restricted* if some of the events in (E) are disallowed; of course, in a restricted homotopy one can expect more invariants.

PROPOSITION 3. — (Francis [9]). If $f, g : \mathrm{M} \to \mathrm{N}(\dim \mathrm{M}=1)$ *are transverse immersions connected by an* E^2_{22} *restricted homotopy (in which the event* E^2_{22} *is disallowed) then for every* $\hat{\mathrm{M}}$, f *is extendable to* $\hat{\mathrm{M}}$ *by a sense preserving immersion if and only if* g *is so extendable*; *thus, extendability to a sense preserving immersion is an* E^2_{22} *restricted homotopy invariant.*

PROPOSITION 4. — (Francis [9]). If $f, g : \mathrm{M} \to \mathrm{N}(\dim \mathrm{M} = 1)$ *are transverse immersions connected by an* $\mathrm{E}^2_{22} \cup \mathrm{E}^3_2$

restricted homotopy invariant, then f is SP *extendable to an* $\hat{M}$ *if and only if g is so extendable*; i.e., SP *extendability is an* $E^2_{22} \cup E^3_2$ *restricted homotopy invariant.*

6. An Approach via Transformation Semigroups.

For more details in the general approach $(n \geq 1)$ see Titus [29]; for special cases $(n = 1)$ see Benson [1], Farias [4, 5, 6], Loewner [14], Titus [28, 30, 31].

Given $f, g: M \to N$ let us say f *grows to* g if there exists a homotopy $F: M \times [0, 1] \to N$ taking f to g where $F \in SP$. With proper technical conditions this relation gives a partial ordering on $C^\infty(M, N)$. Thus it is natural to search for a transformation semigroup $\mathcal{S}$ which acts on $C^\infty(M, N)$ so that the action produces this partial ordering of « growth ». Not only is this possible but there is a natural group $\mathcal{I} \supset \mathcal{S}$ which acts « nearly transitively » on $C^\infty(M, N)$.

Historically this approach has its roots primarily in the Heavyside Calculus in which various integral operators are approximated by operators with « degenerate » kernels which in turn can be thought of as differential operators (see especially [1, 5, 14, 31]).

The action of the semigroups $\mathcal{S}$ turns out to be a natural variation on the usual action of $\mathbf{R}$ on N as in flow theory.

Let τ be a top order differential form on N so that $\tau_y > 0$ defines the positively oriented bases in TN_y.

Let $\mathcal{V}$ be the real vector space of all tangent vector fields Y on N such that:

(*a*) the solution, $\Phi^Y: N \times \mathbf{R} \to N$, to the differential equations defined by Y exists for all time t,

(*b*) Y is divergence free with respect to τ.

Let $\mathcal{Y}$ be a subspace of $\mathcal{V}$ (generally in most applications $\mathcal{Y}$ is finite dimensional).

Let $\mathcal{A}$ be a subring of $C^\infty(M, \mathbf{R})$ which contains all the constant functions.

It is in terms of M, N, τ, $\mathcal{Y}$, $\mathcal{A}$ that the group $\mathfrak{J}$ and the semigroup $\mathcal{S}$ are defined.

On $\mathcal{A} \times \mathcal{Y}$ defined the equivalence relation $(\alpha, Y) = (\beta, Z)$ if and only if there exist $a, b \in \mathbf{R}$, not both zero, so that

$aY = bZ$ and $a^2\beta = b^2\alpha$. One has

$$\{(\beta, Z) | (\beta, Z) = (0, O)\} = \{(\beta, Z) | \beta = 0 \quad \text{or} \quad Z = O\}$$

and that

$$\{(\beta, Z) | (\beta, Z) = (\alpha, Y)\} = \left\{ a^2\alpha, \frac{1}{a} Y | a \in \mathbf{R}, \ a \neq 0 \right\}.$$

Let $\mathfrak{J}(Y)$ be the set $\{(\alpha, Y) | \alpha \in \mathcal{A}\}$ together with the binary operation (which makes sense for the equivalence classes), $(\beta, Y)(\alpha, Y) = (\alpha + \beta, Y)$; thus each $\mathfrak{J}(Y)$ is an abelian group in which inverses are representable by $(\alpha, Y)^{-1} = (-\alpha, Y)$ and the common identity by

$$(o, Y) = (o, O).$$

The group $\mathfrak{J}$ is defined as the (finite) free product of the abelian groups $\mathfrak{J}(Y)$, $Y \in \mathcal{Y}$; thus every $G \in \mathfrak{J}$, except the identity, is represented as a (reduced) product

$$G = (\alpha_m, Y_m) \ldots (\alpha_1, Y_1), \ \alpha_k \in \mathcal{A}, \ Y_k \in \mathcal{Y},$$

where $\mathfrak{J}(Y_k) \neq \mathfrak{J}(Y_{k+1})$ and where each element (α_k, Y_k) is uniquely determined (modulo the equivalence relation).

The semigroup $\mathcal{S}$ is defined to be the sub-semigroup of $\mathfrak{J}$ given by

$$\mathcal{S} = \{(\alpha_m, Y_m) \ldots (\alpha_1, Y_1) | m \geqslant 1, \alpha_k \geqslant 0\},$$

(where again this makes sense for equivalence classes since $\alpha \geqslant 0$ and $(\alpha, Y) = (\beta, Z)$ imply $\beta \geqslant 0$).

Next we define the action of $\mathfrak{J}$ (and therefore of $\mathcal{S}$) on $C^\infty(M, N)$. First we need a top order nowhere degenerate form σ on M so that $\sigma_x > 0$ defines the positive bases in TM_x.

Now, construct the mapping,

$$A : \mathcal{Y} \times C^\infty(M, N) \to C^\infty(M, \mathbf{R}), (Y, f) \longmapsto \lambda_f^Y,$$

where, for each $x \in M$ and basis $\varepsilon_1, \ldots, \varepsilon_n$ of TM_x, λ_f^Y is defined, with $y = f(x)$, by

$$\lambda_f^Y(x) = \frac{\tau_y(Y(y), df_x e_1, \ldots, df_x e_n)}{\sigma_x(e_1, \ldots, e_n)};$$

it is seen that this assigned value is independent of the choice of basis and also that $\lambda_f^Y \in C^\infty(M, \mathbf{R})$.

For a given $Y \in \mathscr{Y}$ let the mapping

$$\Phi^Y : N \times \mathbf{R} \to N, (y, t) \longmapsto \Phi^Y_t(y),$$

be the solution to the differential equation defined by Y. We have then for all $Y \in \mathscr{Y}$, by convention or hypothesis, that

(a) Φ^Y_0 is the identity,

(b) $\frac{\partial}{\partial t} \Phi^Y = Y \circ \Phi^Y$,

(c) $\Phi^Y_s \circ \Phi^Y_t = \Phi^Y_{s+t}$ for all $s, t \in \mathbf{R}$,

(d) $\det [(d\Phi^Y_t)y] = 1$ for all $(y, t) \in N \times \mathbf{R}$, (since Y is divergence free).

Finally, we can define a mapping of

$$\mathfrak{J} \times C^\infty(M, N) \to C^\infty(M, N)$$

which can be shown to be an action of $\mathfrak{J}$. First, for $(\alpha, Y) \in \mathfrak{J}(Y)$ define

$$(\alpha, Y) : C^\infty(M, N) \to C^\infty(M, N),$$

by $$f \longmapsto (\alpha, Y)f = (\Phi^Y_{\alpha\lambda_f^Y}) \circ f.$$

It follows directly that $(\alpha, Y) f = (\beta, Z) f$ for all f if and only if $(\alpha, Y) = (\beta, Z)$ and thus, for every $G \in \mathfrak{J}$, that $Gf = (\alpha_m, Y_m) \ldots (\alpha_1, Y_1) f$ is well defined. This gives the desired action of $\mathfrak{J}$ as well as, of course, the action of $\mathscr{S}$.

PROPOSITION 5 (Titus [29]). *Given* $\mathbf{M}$, N, σ, τ, $\mathscr{Y}$, $\mathfrak{A}$ *(which define* $\mathscr{S}$ *and its action), and given any* $S \in \mathscr{S}$, *and any* $\hat{M}$ *then* $f : M \to N$, SP *extendable to* $\hat{M}$, *implies that* $g = Sf$ *is also* SP *extendable to* $\hat{M}$; *thus,* SP *extendability is invariant under the action of the semigroup* $\mathscr{S}$.

The idea of this proof is to construct, using the action of $\mathscr{S}$, a homotopy $F : Mx[0,1] \to N$ from f to g where $F \in SP$ (and so represents a growth). This property of the action of $\mathscr{S}$ allows for the generation of SP extendable mappings as the union of orbits of $\mathscr{S}$ on any collection of mappings $\mathscr{D}$ which are known *a priori*, by virtue of simplicity, degeneracy or whatever, to be SP extendable. In special cases much more information is available as, for example, in Proposition 6.

In $\mathbf{R}^2$ let φ be an imbedding representing the positively oriented unit circle in $\mathbf{R}^2$ (with respect to some euclidean structure on $\mathbf{R}^2$); and let $\mathcal{D}_\varphi$ be the class of imbeddings of circle S^1 in $\mathbf{R}^2$ which are diffeomorphically (sense preserving) equivalent to φ; thus each $f \in \mathcal{D}_\varphi$ is trivially SP extendable.

Proposition 6 (Farias [5]) *Given* $M = S^1$, σ *and* τ *usual forms on* S^1 *and* $\mathbf{R}^2$, $\mathcal{Y}$ *the constant vector field on* $\mathbf{R}^2$ *and* $\mathcal{A} = C^\infty(S^1, \mathbf{R})$ *then every* SP *extendable mapping* $f\colon S^1 \to \mathbf{R}^2$ *is contained in* $\mathcal{S}(\mathcal{D}_\varphi)$.

The following is not difficult.

Proposition 7 (Titus, to appear). *Given* $\dim M = 1$ (M *a union of circles*), $N, \sigma, \tau, \mathcal{A} = C^\infty(M, N)$ *and* $\mathcal{Y}$ *such that, for every* $y \in N$, *the vectors* $\{Y(y) \mid Y \in \mathcal{Y}\}$ *span* TN_y *then the group* $\mathfrak{J}$ *acts transitively on every homotopy class of immersions in* $C^\infty(M, N)$.

Conjecture 4. — Given M with $\dim M = 2$,

$$N\ (\dim N = 3),\ \sigma,\ \tau,\ \mathcal{A} = C^\infty(M, \mathbf{R})$$

and $\mathcal{Y}$ such that, for every $y \in N$, the vectors $\{Y(y) \mid Y \in \mathcal{Y}\}$ span TN_y then the group $\mathfrak{J}$ acts transitively on every homotopy class of generic mappings; note here $(n = 2)$ the immersions are not dense in $C^\infty(M, N)$.

However, when $n \geqslant 3$, a result as in Conjecture 4 is no longer possible because of the structure of the singularities of generic mappings for $n \geqslant 3$ and because of

Proposition 5. — (Titus [29]). *Given any* $M, N, \sigma, \tau, \mathcal{A}, \mathcal{Y}$ *with* $n \geqslant 2$, *and given an* $x \in M$ *with rank* $df_x \leqslant n - 3$ *then, for all* $G \in \mathfrak{J}$, *the rank* $(dGf)_x \leqslant n - 3$ *and*

$$(Gf)(x) = f(x);$$

thus all such points are invariant under the action of $\mathfrak{J}$.

Since $\mathfrak{J}$ is a group the above also leads to another invariant of the action of $\mathfrak{J}$ when $n \geqslant 2$; namely, if f is such that rank $df_x \geqslant n - 2$ for all $x \in M$ then, for all $G \in \mathfrak{J}$, Gf has the same property. It is still possible that $\mathfrak{J}$ acts transitively on each generic homotopy class with a common singularity structure of some sort.

7. Other Applications of the Group $\mathfrak{J}$.

With $M = S^1$, $N = \mathbf{R}^2$, σ and τ as usual, $\mathcal{A} = \mathbf{R}$ and $\mathcal{Y}$ the constant vector fields on $\mathbf{R}^2$, it is a detailed geometrical study of the action of $\mathcal{S}$ on $C^\infty(S^1, \mathbf{R})$ that is the main idea in a proof of the Carathéodory Conjecture on umbilic points as well as a proof of the related « higher order » Loewner Conjecture, see Titus [30].

There is an algebraic action of $\mathfrak{J}$ and $\mathcal{S}$, rather than a differential action, on the space of polynomial mappings of $\mathbf{R}$ to $\mathbf{R}^2$ which gives respectively, for example, a geometric form of the Euclidean Algorithm and of Sturm Sequences; this idea has been developed by Norton [21], there is a related approach in Cohn [3]. This algebraic action is, in a natural sense, a dual theory to the differential action via a Laplace Transform.

There is also a close relation between the differential action of $\mathfrak{J}$ and the classical theory of integral operators. For example, suppose $M = S^1$, $N = \mathbf{R}^2$, σ ant τ as usual, $\mathcal{Y}$ the constant vector fields in $\mathbf{R}^2$ and $\mathcal{A} = \mathbf{R}$. Let $\nu \in \mathbf{R}^2$, $f \in C^\infty(S^1, \mathbf{R})$, $S \in \mathcal{S}$ then it follows that $S(f\nu)$ is represented by a curve in $\mathbf{R}^2$ of the form $y_1 = Af$, $y_2 = Bf$ where A and B are real polynomial differential operators with B separating A positively (i.e., $\deg B = 1 + \deg A$, A and B have all real roots, the roots of B interlace the roots of A, the product of the highest coefficients is positive). For x, y, $f \in C^\infty(S^1, \mathbf{R})$ one sees that $x = Af$ has a unique solution and thus one can form the linear operator on $C^\infty(S^1, \mathbf{R})$ given by $y = (BA^{-1})x$. Such operators are used classically to effectively approximate, for example, the Hilbert Operator given by

$$y(s) = \mathcal{P}\int_{-\pi}^{\pi} \cot \tau x(t - \tau)\, d\tau, \quad y = Hx.$$

But in this context the result of Farias [5] shows essentially that if one stays in the « parametric » form $x = Af$, $y = Bf$, and allows periodic coefficients (let $\mathcal{A} = C^\infty(S^1, \mathbf{R})$ instead of $\mathcal{A} = \mathbf{R}$) then, essentially, all curves (x, Hx) are of the form $S(f\nu)$, $S \in \mathcal{S}$, where S has, of course, *finite* order. This approach has three advantages over the classical theory.

First, the minimum order of S required to give a given curve has many of the properties of a polynomial degree. Second, with the operators in parametric form, no elimination theory (as leads to $y = BA^{-1} x$ from $x = Af$, $y = Bf$) is required; this allows then the consideration of non-linear targets as in the general theory $(n \geqslant 1)$ of the actions of $\mathfrak{J}$ and $\mathscr{S}$. Third, the differential operators, $G \in \mathfrak{J}$, have differential operators as inverses so that integration is never necessary,

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