ANNALES DE L'INSTITUT FOURIER

FLORIS TAKENS

Normal forms for certain singularities of vectorfields

Annales de l'institut Fourier, tome 23, nº 2 (1973), p. 163-195
<http://www.numdam.org/item?id=AIF_1973__23_2_163_0>

Ann. Inst. Fourier, Grenoble
23, 2 (1973), 163-195.

NORMAL FORMS FOR CERTAIN SINGULARITIES OF VECTORFIELDS

by Floris TAKENS

1. Introduction and statement of the results.

The main goal of this paper is to study singularities of vectorfields on $\mathbf{R}^1$ and singularities of vectorfields on $\mathbf{R}^2$ with a « rotation as 1-jet »; these are the simpelest non-hyperbolic singularities. For the first sort of singularities we obtain :

THEOREM 1. — *Let* X *be a* C^∞*-vectorfield on* $\mathbf{R}^1$ *of the form* $X = x^k F(x) \frac{\partial}{\partial x}$ *with* $F(0) \neq 0$ *and* $k \geqslant 2$. *Then there is a* C^∞ *orientation preserving diffeomorphism*

$$\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$$

such that, in some neighbourhood of the origin $0 \in \mathbf{R}^1$, $\varphi_*(X) = (\delta x^k + \alpha x^{2k-1}) \frac{\partial}{\partial x}$ *with* $\delta = \pm 1$ *and* $\alpha \in \mathbf{R}$; δ *and* α *are uniquely determined by the* $(2k-1)$ — *jet of* X *in* $0 \in \mathbf{R}^1$.

There is an analogue of this theorem for local diffeomorphisms :

THEOREM 2. — *Let* $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *be a* C^∞*-diffeomorphism such that* Ψ^2 *has the form* $\Psi^2(x) = x + x^k F(x)$ *with* $F(0) \neq 0$ *and* $k \geqslant 2$. *Then there is a* C^∞ *orientation* *preserving diffeomorphism* $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *such that, in some neighbourhood of* $0 \in \mathbf{R}^1$,

$$\varphi \Psi \varphi^{-1}(x) = \pm\, x + \delta x^k + \alpha x^{2k-1},$$

$\delta = \pm 1$ *and* $\alpha \in \mathbf{R}$; δ *and* α *are uniquely determined by the* $(2k-1)$ — *jet of* Ψ *in* $0 \in \mathbf{R}^1$; *if* Ψ *is orientation reversing, then* k *is odd.*

Remark. — The above two theorems, for $k = 1$, were proved by S. Sternberg [4]; in this case they should be formulated in a somewhat different way.

For vectorfields on $\mathbf{R}^2$, we obtain the following result.

THEOREM 3. — *Let* $X = X_1 \dfrac{\partial}{\partial x_1} + X_2 \dfrac{\partial}{\partial x_2}$ *be a* C^∞*-vectorfield on* $\mathbf{R}^2$ *such that the* 1*-jet of* X_1, *resp.* X_2, *in the origin equals the* 1*-jet of* $-2\pi x_2$, *resp.* $2\pi x_1$. *Then,*

either, there is a C^∞*-diffeomorphism* $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *such that*

$$\varphi_*(X) = f(x_1^2 + x_2^2)\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) + \tilde{X}_1 \frac{\partial}{\partial x_1} + \tilde{X}_2 \frac{\partial}{\partial x_2}$$

where f *is a* C^∞*-function,* $f(0,0) \neq 0$ *and* $\tilde{X}_1$, $\tilde{X}_2$ *are flat* C^∞*-functions* (i.e., *the* ∞*-jet of* X_i *is zero in* $(0,0)$),

or, there is a C^∞*-diffeomorphism* $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ *such that, in some neighbourhood of the origin,*

$$\varphi_*(X) = f_1(x_1, x_2)\left[2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} + (\delta(x_1^2 + x_2^2)^k + \alpha(x_1^2 + x_2^2)^{2k})\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)\right],$$

with f *a* C^∞*-function,* $f(0,0) = 1$, $\delta = \pm 1$, $k \in \mathbf{N}$, $k \geqslant 1$ *and* $\alpha \in \mathbf{R}$; δ, k *and* α *are uniquely determined by the* ∞*-jet of* X *in the origin.*

Remark. — It is clear that, using a coordinate change and/or multiplication with a constant, theorem 3 can be applied to any C^∞-vectorfield $X = X_1 \dfrac{\partial}{\partial x_1} + X_2 \dfrac{\partial}{\partial x_2}$ on $\mathbf{R}^2$

for which $X(0) = 0$ and $\left(\frac{\partial X_i}{\partial x_j}(0)\right)_{i,j}$ has 2 non-zero purely imaginary eigenvalues.

From conversations with R. Moussu and others I learned that theorem 2 has a consequence which has some importance in the theory of co-dimension one foliations:

THEOREM 4. — *Let* $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *be a* C^∞*-orientation preserving diffeomorphism of the form* $\Psi(x) = x + x^k F(x)$ *with* $F(0) \neq 0$ *and* $k \geqslant 2$. *Then there is a* C^∞*-vectorfield* X *on* $\mathbf{R}^1$ *such that, in a neighbourhood of the origin,* $\Psi = \mathscr{D}_{X,1}$, *where* $\mathscr{D}_{X,t} : \mathbf{R}^1 \to \mathbf{R}^1$ *is the time* t *integral of* X (*for* $k = 1$ *this result follows from Sternberg* [4]).

The rest of this paper is organized as follows. In § 2 we show that the above theorems 1, 2 and 3 are true « in the formal sense », i.e., that they are true modulo flat functions and vectorfields. In § 3 we prove theorem 1. In § 4 we prove the existence of solutions of a certain functional equation; this result is then used in § 5 to prove the theorems 2, 3 and 4.

We shall use the following notation: If X is a vectorfield on $\mathbf{R}^n$, then $\mathscr{D}_X : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$, the integral of X, is the map defined by $\mathscr{D}_X(p, 0) = p$ and $\frac{\partial}{\partial t}(\mathscr{D}_X(p, t)) = X(\mathscr{D}_X(p,t))$ (we shall always assume that $\mathscr{D}_X$ can be defined on all of $\mathbf{R}^n \times \mathbf{R}$). $\mathscr{D}_{X,t} : \mathbf{R}^n \to \mathbf{R}^n$ denotes the map defined by

$$\mathscr{D}_{X,t}(p) = \mathscr{D}_X(p, t).$$

2. Formal normal forms.

Before we start with the actual proofs, we have to state and prove two preliminary lemmas.

LEMMA (2,1). — *Let* $X = X_1 + X_2$ *and* Y *be* C^∞*-vectorfields on* $\mathbf{R}^n$ *such that* $[Y, X_1] = 0$. *If the* $(s_1 - 1)$ — *jet of* X_2 *and the* $(s_2 - 1)$ — *jet of* Y *are zero,* $s_2 \geqslant 2$, *then the* $(s_1 + s_2 - 1)$ — *jets of* $(\mathscr{D}_{Y,1})_* X$ *and* $(X_1 + X_2 - [Y, X_2]$ *are equal* (*jet means here always: jet in the origin of* $\mathbf{R}^n$).

Proof. — Because $s_2 \geqslant 2$, the 1-jet of $\mathscr{D}_{Y,t}$ is the 1-jet of the identity for all t. This implies that the s_1-jet of $(\mathscr{D}_{Y,t})_* X_2$ equals the s_1 — jet of X_2 for all t. One knows from differential geometry [2] that

$$\frac{d}{dt}((\mathscr{D}_{Y,t})_* X_2) = -[Y, (\mathscr{D}_{Y,t})_* X_2].$$

As the $(s_2 - 1)$ — jet of Y is zero, the $(s_1 + s_2 - 1)$ — jet of $[Y, (\mathscr{D}_{Y,t})_* X_2]$ is completely determined by the s_2 — jet of Y and the s_1 — jet of $(\mathscr{D}_{Y,t})_* X_2$; both are independent of t. Hence, for all t, the $(s_1 + s_2 - 1)$ — jet of $\frac{d}{dt}((\mathscr{D}_{Y,t})_* X_2)$ equals the $(s_1 + s_2 - 1)$ — jet of $-[Y, X_2]$. From this it follows that the $(s_1 + s_2 - 1)$ — jets of $(\mathscr{D}_{Y,1})_* X_2$ and $X_2 - [Y, X_2]$ are equal. The lemma now follows from the observation that $(\mathscr{D}_{Y,1})_* X_1 = X_1$ (because $[Y, X_1] = 0$).

LEMMA (2,2). — *Let Ψ be a diffeomorphism of $\mathbf{R}^n$ to itself such that the $(s_1 - 1)$-jet of Ψ in the origin equals the $(s_1 - 1)$-jet of the identity, $s_1 \geqslant 1$. Let Y be some vectorfield on $\mathbf{R}^n$ with zero $(s_2 - 1)$-jet, $s_2 \geqslant 2$. Then the $(s_1 + s_2 - 1)$-jet of $\mathscr{D}_{Y,1}\Psi(\mathscr{D}_{Y,1})^{-1}$ and $\mathscr{D}_{A_\Psi(Y),1}\Psi$ are equal,* $A_\Psi(Y) = Y - \Psi_*(Y)$.

Proof. — Let Ψ_t, for $t \in \mathbf{R}$, be the diffeomorphism defined by $\Psi_t = \mathscr{D}_{Y,t}\Psi\mathscr{D}_{Y,-t}$. From the fact that the 1-jet of $\mathscr{D}_{Y,t}$ is the 1-jet of the identity for all t, it follows that the s_1-jet of Ψ_t is independent of t. Now we define the vectorfield Z_t on $\mathbf{R}^n$, depending on $t \in \mathbf{R}$, to be the vectorfield such that each $p \in \mathbf{R}^n$ and $t \in \mathbf{R}$, $Z_t(p)$ is the tangent vector of the curve $u \longmapsto \Psi_{t+u}(\Psi_t)^{-1}(p)$. This definition of Z_t is equivalent with $Z_t = Y - (\Psi_t)_* Y = A_{\Psi_t}(Y)$; it is clear that, for each $p \in \mathbf{R}^n$ and $t \in \mathbf{R}$, $\frac{\partial}{\partial t}(\Psi_t(p)) = Z_t(\Psi_t(p))$. Hence the lemma is proved once we know that the (s_1+s_2-1)-jet of Z_t is independent of t. This last fact follows from the fact that the s_1-jet of Ψ_t is independent of t and that the $(s_2 - 1)$-jet of Y is zero: this namely implies that the $(s_1 + s_2 - 1)$-jet of $(\Psi_t)_* Y$, and hence of Z_t, is independent of t.

PROPOSITION (2,3). — *Let X be a C^∞-vectorfield on $\mathbf{R}^1$ of the form $X = x^k F(x) \frac{\partial}{\partial x}$ with $F(0) \neq 0$ and $k \geqslant 2$.*

Then there is a C^∞*-orientation preserving diffeomorphism* $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *such that the* ∞*-jet of* $\varphi_*(X)$ *in the origin equals the* ∞*-jet of* $(\delta x^k + \alpha x^{2k-1}) \frac{\partial}{\partial x}$ *for some* $\delta = \pm 1$ *and* $\alpha \in \mathbf{R}$; δ *and* α *are uniquely determined by the* $(2k-1)$*-jet of* X *in the origin.*

Proof. — First we notice that for fixed X, the $(1 + k - 1)$-jet of $\varphi_*(X)$ only depends on the 1-jet of φ. Next we construct, by induction, a sequence of diffeomorphisms $\varphi_i : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0), i = 1, 2, \ldots,$ with the following properties:

(i) the $(i-1)$-jet of φ_i is the $(i-1)$-jet of the identity;

(ii) φ_i is the time 1 integral $\mathscr{D}_{Y_i, 1}$ of the vectorfield $Y_i = a_i x^i \frac{\partial}{\partial x}$ for some $a_i \in \mathbf{R}$;

(iii) for each i, the $(i + k - 1)$-jet of

$$(\varphi_i)_*(\varphi_{i-1})_* \ldots (\varphi_1)_* X = (\varphi_i \varphi_{i-1} \ldots \varphi_1)_*(X) = X^i$$

is as in the conclusion of proposition (2,3).

Construction of φ_i, $i = 1, 2, \ldots$.

(*a*) $i = 1$. For $Y_1 = a_1 x \frac{\partial}{\partial x}$ we have $\varphi_1(x) = e^{a_1} x$ and hence the k-jet of

$$(\varphi_1)_*(X) = (e^{-a_1}x)^k . F(e^{-a_1}x) \cdot \frac{\partial}{\partial(e^{-a_1}x)} = e^{(1-k)a_1} . x^k . F(e^{-a_1}x) \frac{\partial}{\partial x}.$$

As $k \geqslant 2$, there is exactly one $a_1 \in \mathbf{R}$ such that $e^{(1-k)a_1}.\ F(0) = \pm 1$; hence there is exactly one a_1 such that $\mathscr{D}_{Y_1, 1}$ transforms the k-jet of X in the required form.

(*b*) $i = 2, \ldots, k-1$. We have now $Y_i = a_i x^i \frac{\partial}{\partial x}$; the k-jet of $(\varphi_{i-1} \ldots \varphi_1)_* X = X^{i-1}$ has the form $\delta x^k \frac{\partial}{\partial x}$ with $\delta = \pm 1$. From lemma (2,1) we now obtain that the $(k + i - 1)$-jet of $(\varphi_i)_* X^{i-1}$ $(\varphi_i = \mathscr{D}_{Y_i, 1})$ equals the $(k + i - 1)$-jet of $X^{i-1} - [Y_i, X^{i-1}]$. The $(k + i - 1)$-jet of this last vectorfield is given by $X^{i-1} - (k - i) a_i x^{k+i-1} \frac{\partial}{\partial x}$. As $(k - i) \neq 0$, there is exactly one a_i such that the

$(k+i-1)$-jet of $(\mathcal{D}_{Y_i,1})_* X^{i-1}$ has the required form.

(*c*) $i = k$. We take again $Y_k = a_k \frac{\partial}{\partial x}$, $\varphi_k = \mathcal{D}_{Y_k,1}$. By the same calculations as above, we find that the $(2k-1)$-jet of $(\varphi_k)_* X^{k-1}$ is given by

$$X^{k-1} - [Y_k, X^{k-1}] = X^{k-1} - (k-k) a_k x^{k+i-1} \frac{\partial}{\partial x} = X^{k-1}.$$

In other words, we are not able to change the $(2k-1)$-jet of X. As α in proposition (2,3) is allowed to be any real number, the $(2k-1)$-jet of X^{k-1} was already in the required form. From the above arguments it follows that this time we may chose for a_k any real number.

(*d*) $i > k$. By the same argument as under (*b*), we obtain exactly one a_i for each i such that the $(k+i-1)$-jet of $(\mathcal{D}_{Y_i,1})_* X^{i-1}$ has the required form if $Y_i = a_i x^i \frac{\partial}{\partial x}$.

To prove proposition (2,3) we first construct φ, using the above sequence $\varphi_1, \varphi_2, \ldots$. Let $\tilde{\varphi}_l$, $l = 1, 2, \ldots$, be the map $\tilde{\varphi}_l = \varphi_l \varphi_{l-1} \ldots \varphi_1$. Then, for any $i \leqslant l, l'$ the i-jets of $\tilde{\varphi}_l$ and $\tilde{\varphi}_{l'}$, are equal. Hence, by Borel's theorem [3], there is a diffeomorphism $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ such that for all $i \leqslant l$, the i-jets of φ and $\tilde{\varphi}_l$ are the same. From the construction of the sequence $\varphi_1, \varphi_2, \ldots$ above it is clear that the ∞-jet of $\varphi_*(X)$ is given by $(\delta x^k + \alpha x^{2k-1}) \frac{\partial}{\partial x}$ for some $\delta = \pm 1$ and $\alpha \in \mathbf{R}$.

Finally we have to show that δ and α are uniquely determined by the $(2k-1)$-jet of X. Notice that the ∞-jet of φ is not uniquely determined by the requirements in proposition (2,3) (see under (*c*) above), but the $(k-1)$-jet of φ is uniquely determined by these requirements (for given $(2k-2)$-jet of X). The $(2k-2)$-jet, and by (*c*) above even the $(2k-1)$-jet of $\varphi_*(X)$ is then uniquely determined by the $(k-1)$-jet of φ and the $(2k-1)$-jet of X. Hence δ and α, are uniquely determined by the $(2k-1)$-jet of X and the requirements imposed on φ.

Proposition (2,4). — *Let* $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *be a* C^∞-*orientation preserving diffeomorphism of the form* $\Psi(x) = x + x^k F(x)$ *with* $F(0) \neq 0$ *and* $k \geqslant 2$. *Then there* *is a* C^∞*-orientation preserving diffeomorphism* φ :

$$(\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$$

such that $\varphi\Psi\varphi^{-1}(x) = x + \delta x^k + \alpha x^{2k-1} + g(x)$ *for some flat function* g, $\delta = \pm 1$ and $\alpha \in \mathbf{R}$; δ *and* α *are uniquely determined by the* $(2k-1)$*-jet of* Ψ.

Remark. — The same proposition holds if we require $\delta = \pm a$ for some $a \neq 0$.

Proof. — The proof of this proposition is completely analogous to the proof of proposition (2,3); we only have to use lemma (2,2) instead of lemma (2,1). Hence, instead of $[Y_i, X]$, we have to calculate here $A_\Psi(Y_i)$ for $Y_i = a_i x^i \frac{\partial}{\partial x}$. This goes as follows:

$$\begin{aligned}\Psi_*(Y_i)(x) &= a_i(\Psi^{-1}(x))^{+i}.\Psi'(\Psi^{-1}(x))\cdot\frac{\partial}{\partial x}\\ &= a_i(x^i + (k-i)x^{k+i-1}.F(0) + 0(|x^{k+i}|))\frac{\partial}{\partial x}.\end{aligned}$$

Hence the $(k+i-1)$-jet of $A_\Psi(Y) = Y - \Psi_*(Y)$ is given by $a_i(i-k)x^{k+i-1}F(0)\frac{\partial}{\partial x}$.

One can now construct a sequence $\varphi_1, \varphi_2, \ldots$ as in the proof of proposition (2,3). This time one also has to use the following (rather trivial) fact: if $\Psi_1, \Psi_2 : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ have the same $(i-1)$-jet, then there is a vectorfield Z, with zero $(i-1)$-jet, such that the i-jets of $\mathcal{D}_{Z,1}\Psi_1$ and Ψ_2 are the same; the i-jet of Z is completely determined by the i-jets of Ψ_1 and Ψ_2. If $i = 1$, one has to assume that both Ψ_1 and Ψ_2 are orientation preserving.

The rest of the proof of proposition (2,3) carries over to the present case without any difficulty.

LEMMA (2,5). — *Let* $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *be an orientation reversing diffeomorphism of the form* $\Psi(x) = -x + x^2F(x)$ (*we do not assume here that* $F(0) \neq 0$). *Then there is a* C^∞*-orientation preserving diffeomorphism* $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *such that* $\varphi\Psi\varphi^{-1}(x) = -x + x^3G(x^2) + g(x)$, *where* G *is a* C^∞*-function and* g *is a flat* C^∞*-function.*

Proof. — We use again the same procedure as in the proofs of the propositions (2,3) and (2,4). Now we want a sequence of diffeomorphisms φ_i such that

(i) the $(i-1)$-jet of φ_i is the $(i-1)$-jet of the identity;

(ii) φ_i is the time 1 integral $\mathscr{D}_{Y_i, 1}$ of $Y_i = a_i x^i \frac{\partial}{\partial x}$ for some $a_i \in \mathbf{R}$;

(iii) for each i, the i-jet of $\varphi_i \varphi_{i-1} \dots \varphi_1 \Psi \varphi_1^{-1} \dots \varphi_i^{-1}$ is as in the conclusion of lemma (2,5).

To show that such a sequence exists, we have to calculate the i-jet of $A_\Psi\left(a_i x^i \frac{\partial}{\partial x}\right)$ for each i. Clearly the i-jet of $\Psi_*\left(a_i x^i \frac{\partial}{\partial x}\right)$ is the i-jet of $(-1)^{i+1} a_i x^i \frac{\partial}{\partial x}$. Hence the i-jet of $A_\Psi\left(a_i x^i \frac{\partial}{\partial x}\right) = a_i x^i \frac{\partial}{\partial x} - \Psi_*\left(a_i x^i \frac{\partial}{\partial x}\right)$ is given by:

$$\begin{array}{ll} 2a_i x^i \dfrac{\partial}{\partial x} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even.} \end{array}$$

From this the existence of the sequence $\varphi_1, \varphi_2, \dots$ follows; the rest of the proof is as in the proof of proposition (2,3).

Proposition (2,6). — *Let* $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *be a* C^∞-*diffeomorphism such that* Ψ^2 *has the form* $\Psi^2(x) = x + x^k F(x)$ *with* $F(0) \neq 0$ *and* $k \geqslant 2$. *Then there is a* C^∞-*orientation preserving diffeomorphism* $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *such that* $\varphi \Psi \varphi^{-1}(x) = \pm x + \delta x^k + \alpha x^{2k-1} + g(x)$ *with* $\delta = \pm 1, \alpha \in \mathbf{R}$ *and* g *a flat function;* δ *and* α *are uniquely determined by the* $(2k-1)$-*jet of* Ψ *in* $0 \in \mathbf{R}^1$; *if* Ψ *is orientation reversing, then* k *is odd.*

Proof. — If Ψ is orientation preserving, Ψ is of the form $\Psi(x) = x + x^k \tilde{F}(x)$, where $\tilde{F}(0) \neq 0$. In this case we can apply simply proposition (2,4). If Ψ is orientation reversing, we may assume (because of lemma (2,5)) that Ψ is of the form $\Psi(x) = -x + x^3 G(x^2) + g(x)$ where g is a flat function. If G is flat, $\Psi^2(x) = x + \tilde{g}(x)$ for some flat function $\tilde{g}$; this contradicts the assumption on Ψ^2, so we have to assume that G is not flat. We can now write Ψ in the form $\Psi(x)=-x+x^{2l+1}.\tilde{G}(x^2)+g(x)$ with $\tilde{G}(0)\neq 0$. If we now calculate Ψ^2, we obtain $\Psi^2(x)=x-2.x^{2l+1}.\tilde{G}(0)+$ terms of order $\geqslant 2l$ in x; hence we have $k=2l+1$ which is odd.

Let $\bar{\delta}=\pm 2$ and $\bar{\alpha}\in\mathbf{R}$ be such that there is a diffeomorphism $\bar{\varphi}$, such that $\bar{\varphi}\Psi^2\bar{\varphi}^{-1}(x)=\bar{\delta}x^k+\bar{\alpha}x^{2k-1}+$ some flat function of x. $\bar{\delta}$ and $\bar{\alpha}$ exist and are uniquely determined by the $(2k-1)$-jet of Ψ (see proposition (2,4)). Next we define $\Psi_{\delta_0,\alpha_0}\colon(\mathbf{R}^1,0)\to(\mathbf{R}^1,0)$, for

$\delta_0=\pm 1$, $\alpha_0\in\mathbf{R}$ by $\Psi_{\delta_0,\alpha_0}(x)=-x+\delta_0x^{2l+1}+\alpha_0x^{4l+1}$.

Then $\Psi^2_{\delta_0,\alpha_0}(x)=x-2\delta_0x^{2l+1}+(\delta_0^2.(2l+1)-2\alpha_0)x^{4l+1}+$ terms of order $\geqslant 4l+2$ in x. Using proposition (2,4) and the fact that $k=2l+1$ it is clear that there is a diffeomorphism $\tilde{\varphi}$ such that the ∞-jet of $\tilde{\varphi}\bar{\varphi}\Psi^2\bar{\varphi}^{-1}\tilde{\varphi}^{-1}$ and $\Psi^2_{\delta_0,\alpha_0}$ are the same if and only if $-2\delta_0=\bar{\delta}$ and $(\delta_0^2.(2l+1)-2\alpha_0)=\bar{\alpha}$. This last pair of equations has a unique solution for fixed $\bar{\delta}$ and $\bar{\alpha}$; call it δ, α. From the proof of proposition (2,4) it is clear that the ∞-jets of $\bar{\varphi}$ and $\tilde{\varphi}$ are not uniquely determined. For $\bar{\varphi}$ we simply make a choice and then keep it fixed. Now we make the ∞-jet of $\tilde{\varphi}$ unique by requiring that the k-jet ($=$ the $(2l+1)$-jet) of $\tilde{\varphi}$ is the k-jet of the identity.

Finally we have to show that the ∞-jets of $\tilde{\varphi}\bar{\varphi}\Psi\bar{\varphi}^{-1}\tilde{\varphi}^{-1}$ and $\Psi_{\delta,\alpha}$ are the same. This follows from the following calculation in which $=$ means that the ∞-jets on both sides are the same. We know that $\tilde{\varphi}$ is uniquely determined by

(*a*) the k-jet of $\tilde{\varphi}$ is the k-jet of the identity;

(*b*) $\Psi^{-2}_{\delta,\alpha}\tilde{\varphi}(\bar{\varphi}\Psi\bar{\varphi}^{-1})^2=\tilde{\varphi}$.

Now we define $\tilde{\tilde{\varphi}}$ by $\tilde{\tilde{\varphi}}=\Psi^{-1}_{\delta,\alpha}\tilde{\varphi}(\bar{\varphi}\Psi\bar{\varphi}^{-1})$. As the k-jets of $\Psi_{\delta,\alpha}$ and $\bar{\varphi}\Psi\bar{\varphi}^{-1}$ are the same, the k-jet of $\tilde{\tilde{\varphi}}$ equals the k-jet of the identity. $\tilde{\tilde{\varphi}}$ also satisfies (*b*) above so $\tilde{\varphi}=\tilde{\tilde{\varphi}}$. This means that $\tilde{\varphi}=\Psi^{-1}_{\delta,\alpha}\tilde{\varphi}(\bar{\varphi}\Psi\bar{\varphi}^{-1})$ or $\Psi_{\delta,\alpha}=\tilde{\varphi}\bar{\varphi}\Psi\bar{\varphi}^{-1}\tilde{\varphi}^{-1}$.

Hence, for $\varphi=\tilde{\varphi}\bar{\varphi}$, we have

$$\varphi\Psi\varphi^{-1}(x)=-x+\delta x^k+\alpha x^{2k+1}+g(x)$$

for some flat function g and $\delta=\pm 1$, $\alpha\in\mathbf{R}$. The fact that

δ and α are uniquely determined follows from the way in which they were constructed.

LEMMA (2,7). — *Let* X *be a vectorfield on* $\mathbf{R}^1$ *of the form* $X = x^k F(x) \frac{\partial}{\partial x}$ *with* $k \geqslant 2$ *and* $F(0) \neq 0$. $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *denotes the map* $\mathscr{D}_{X,1}$. *Let* $\delta_X = \pm 1$, $\alpha_X \in \mathbf{R}$, $\delta_\Psi = \pm 1$ *and* $\alpha_\Psi \in \mathbf{R}$ *be such that there are orientation preserving diffeomorphisms* φ_1, φ_2 : $(\mathbf{R}^1, 0) \to (\mathbf{R}^2, 0)$ *such that the* ∞*-jet of* $(\varphi_1)_*(X)$ *is given by* $(\delta_X x^k + \alpha_X x^{2k-1}) \frac{\partial}{\partial x}$ *and the* ∞*-jet of* $\varphi_2 \Psi \varphi_2^{-1}$ *by* $x \longmapsto x + \delta_\Psi x^k + \alpha_\Psi x^{2k-1}$. *Then* $\delta_X = \delta_\Psi$ *and* $\left(\alpha_X + \frac{1}{2} k\right) = \alpha_\Psi$.

Proof. — Without loss of generality we may assume that X is already in the form $X = (\delta_X x^k + \alpha_X x^{2k-1} +$ terms of order $\geqslant 2k$ in $x) \frac{\partial}{\partial x}$. We want to compute

$$\mathscr{D}_X(x, t) = \left[\sum_{i=1}^{2k-1} a_i(t) . x^i + \text{terms of order} \geqslant 2k \text{ in } x\right].$$

It is clear that $a_1(t) = 1$, $a_2(t) = \cdots = a_{k-1}(t) = 0$. To compute $a_k(t), \ldots, a_{2k-1}(t)$ we use $\frac{\partial}{\partial t}(\mathscr{D}_X(x, t)) = X(\mathscr{D}_X(x, t))$; this gives, modulo terms of order $\geqslant 2k$ in x,

$$\sum_{i=1}^{2k-1} a_i'(t) x^i = \delta_X \left(\sum_{i=1}^{2k-1} a_i(t) x^i\right)^{kl} + \alpha_X \left(\sum_{i=1}^{2k-1} a_i(t) x^i k\right)^{2k-1}.$$

Using the fact that $a_1 = 1$ and $a_2 = \cdots =: a_{k-1} = 0$ we get (again modulo x^{2k}) :

$$\sum_{i=1}^{2k-1} a_i'(t) x^i = \delta_X x^k + k \delta_X a_k(t) x^{2k-1} + \alpha_X x^{2k-1}.$$

From this we obtain, using that $\mathscr{D}_X(x, 0) = x$ for all $x \in \mathbf{R}^1$, or $a_i(0) = 0$ for $i > 1$:

$a_k'(t) = \delta_X$, which implies $a_k(t) = \delta_X . t$;

$a_{k+1}(t) = \cdots = a_{2k-2}(t) = 0$ and $a_{2k-1}'(t) = k . \delta_X . \delta_X . t + \alpha_X$, which implies $a_{2k-1}(t) = \frac{1}{2} kt^2 + \alpha_X t$.

Hence the $(2k-1)$-jet of Ψ is given by

$$x \longmapsto x + \delta_X x^k + \left(\alpha_X + \frac{1}{2}k\right)x^{2k-1}.$$

From this and proposition (2,4) the lemma follows.

PROPOSITION (2,8). — *Let* $X = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2}$ *be a* C^∞-*vectorfield on* $\mathbf{R}^2$ *such that the* 1-*jet of* X_1, *resp.* X_2, *in the origin equals the* 1-*jet of* $-2\pi x_2$, *resp.* $2\pi x_1$. *Then*

either there is a C^∞-diffeomorphism $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that

$$\varphi_*(X) = f(x_1^2 + x_2^2)\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) + \tilde{X}_1 \frac{\partial}{\partial x_1} + \tilde{X}_2 \frac{\partial}{\partial x_2},$$

where f is a C^∞-function, $f(0) \neq 0$ and $\tilde{X}_1$, $\tilde{X}_2$ are flat C^∞-functions (i.e., the ∞-jet of both X_1 and X_2 is zero in the origin),

or there is a C^∞-diffeomorphism $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that

$$\varphi_*(X) = f(x_1, x_2)\Big[2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} + (\delta(x_1^2 + x_2^2)^k + \alpha(x_1^2 + x_2^2)^{2k})\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)\Big] + \tilde{X}_1 \frac{\partial}{\partial x_1} + \tilde{X}_2 \frac{\partial}{\partial x_2},$$

with f a C^∞-function, $f(0, 0) = 1$, $\tilde{X}_1$ and $\tilde{X}_2$ flat C^∞-functions and $\delta = \pm 1$, $\alpha \in \mathbf{R}$, $k \geqslant 1$. δ, α and k are uniquely determined by the ∞-jet of X in the origin.

Proof. — According to [5] one may assume that X has the following form

$$X = h_1(x_1^2 + x_2^2)\left(2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1}\right) + h_2(x_1^2 + x_2^2)\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) + \tilde{X}_1 \frac{\partial}{\partial x_1} + \tilde{X}_2 \frac{\partial}{\partial x_2}$$

with h_1, h_2, $\tilde{X}_1$, $\tilde{X}_2$ C^∞-functions, $h_1(0) = 1$, $h_2(0) = 0$ and $\tilde{X}_1$, $\tilde{X}_2$ flat. If h_2 is a flat function, we are done: we then have the first of the two alternatives in the conclusion of proposition (2,8).

Next we assume that h_2 is not flat.

$$\text{Let } \overline{X} = (h_1(x_1^2 + x_2^2))^{-1} . X;$$

this is, at least in a neighbourhood of the origin, well defined. The ∞-jet of $\overline{X}$ equals

$$\overline{h}_2(x_1^2 + x_2^2) \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) + 2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1},$$

where $\overline{h}_2(x_1^2+x_2^2) = (h_1(x_1^2+x_2^2))^{-1} . (h_2(x_1^2+x_2^2))$; $\overline{h}_2$ is not flat. We now want to show that there is a C^∞-diffeomorphism φ : $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that the ∞-jet of $\varphi_*(\overline{X})$ has the form

$$2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} + (\delta(x_1^2 + x_2^2)^k + \alpha(x_1^2 + x_2^2)^{2k}) \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right)$$

for some $k \geqslant 1$, $\delta = \pm 1$ and $\alpha \in \mathbf{R}$.

The k, occuring above, is the integer for which we have $\overline{h}_2(x_1^2 + x_2^2) = (x_1^2 + x_2^2)^k . H(x_1^2 + x_2^2)$ for some C^∞-function H with $H(0) \neq 0$. To construct φ, we make again a sequence of diffeomorphisms φ_i: $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$, $i = 0, 1, 2, \ldots$, such that

(i) for each i, the $2i$-jet of φ_i equals the $2i$-jet of the identity;

(ii) φ_i is the time 1 integral $\mathscr{D}_{Y_i, 1}$ of a vectorfield

$$Y_i = a_i(x_1^2 + x_2^2)^i \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \quad \text{for some} \quad a_i \in \mathbf{R};$$

(iii) the $(2k + 2i + 1)$-jet of $(\varphi_i)_* \ldots (\varphi_0)_* \overline{X}$ has the required form. The construction of this sequence $\varphi_0, \varphi_1, \ldots$ and the proof that the required diffeomorphism φ exists goes exactly in the same way as in the proof of proposition (2,3).

Finally we have to show that k, δ and α are uniquely determined by the ∞-jet of X. To do so, we first have to define the Poincaré map for a vectorfield X as above. Let l : $(\mathbf{R}, 0) \to (\mathbf{R}^2, (0, 0))$ be some embedding; then the Poincaré map $P_{l, X}$, associated with X and l, is a map from a neighbourhood U of $0 \in \mathbf{R}$ to $\mathbf{R}$ such that, for $x \in U$, $l(P_{l,X}(x))$ is the first intersection of the positive integral curve of X, starting in $l(x)$, with $l(0, \infty)$ if $x > 0$ and with $l(-\infty, 0)$ if $x < 0$, $P_{l,X}(0) = 0$. Using the blow up construction in [5], it is easy to see (*a*) that $P_{l,X}$ is a local diffeomorphism, (*b*) that, if $l' : (\mathbf{R}, 0) \to (\mathbf{R}^2, (0, 0))$ is another embedding, there is a diffeomorphism $\lambda_{l'l}$: $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that (in a neighbourhood of $0 \in \mathbf{R}$) $\lambda_{l'l} P_{l,X} (\lambda_{l'l})^{-1} = P_{l',X}$ and (*c*) that the ∞-jet of $P_{l,X}$ is determined by the ∞-jet of X. From these properties of $P_{l,X}$ and proposition (2,4) it follows that for some λ, the ∞-jet of $\lambda P_{l,X} \lambda^{-1}$ equals the ∞-jet of $t \longmapsto t + \bar{\delta} t^{\bar{k}} + \bar{\alpha} t^{2\bar{k}+1}$ for some $\bar{k} \geqslant 2$, $\bar{\delta} = \pm 1$ and $\bar{\alpha} \in \mathbf{R}$; $\bar{k}$, $\bar{\delta}$ and $\bar{\alpha}$ are uniquely determined by the ∞-jet of X (the case that the ∞-jet of $P_{l,X}$ equals the ∞-jet of the identity is excluded because we assumed that h_2 is not flat). From the definitions it is clear that $\bar{k}$, $\bar{\delta}$, $\bar{\alpha}$ do not change if we replace X by $\varphi_*(X)$ for any diffeomorphism $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$. We proved the existence of a map φ such that

$$\varphi_*(X) = f(x_1, x_2) \left[2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} + (\delta(x_1^2 + x_2^2)^k + \alpha(x_1^2 + x_2^2)^{2k-1}) \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \right] + \tilde{X}_1 \frac{\partial}{\partial x_1} + \tilde{X}_2 \frac{\partial}{\partial x_2}$$

with $f(0, 0) = 1$, $\delta = \pm 1$, $k \geqslant 1$, $\alpha \in \mathbf{R}$ and $\tilde{X}_1$, $\tilde{X}_2$ flat functions. Using lemma (2,7) it follows that $\bar{k}$, $\bar{\delta}$, $\bar{\alpha}$ and k, δ, α are related in the following way:

$2k + 1 = \bar{k}$; $\delta = \bar{\delta}$; $\left(\alpha + \frac{1}{2} \bar{k} \right) = \bar{\alpha}$. This implies that also k, δ and α are uniquely determined by the ∞-jet of X.

3. The proof of theorem 1.

To derive theorem 1 from proposition (2,3) it is enough to prove the following.

Proposition (3,1). — *Let* X *and* $\tilde{X}$ *be two* C^∞-*vector-fields on* $\mathbf{R}^1$ *of the form* $X = x^k F(x) \frac{\partial}{\partial x}$, *resp.* $\tilde{X} = x^k \tilde{F}(x) \frac{\partial}{\partial x}$, *with* $k \geqslant 2$, $F(0) \neq 0$ *and* $\tilde{F}(0) \neq 0$. *If the* $(2k-1)$*-jets of* X *and* $\tilde{X}$ *are the same, then there is a* C^∞*-orientation preserving diffeomorphism* $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *such that* $\varphi_*(X) = X$ *holds in a neighbourhood of* $0 \in \mathbf{R}^1$.

Proof. — We define on $\mathbf{R}^2$ the vectorfield $\overline{X}$ by $\overline{X} = (x^k F(x) + y . x^k . (\tilde{F}(x) - F(x)) \frac{\partial}{\partial x}$. In lemma (3,2) below we show that there is a vectorfield Y on $\mathbf{R}^2$ of the form $Y = G(x, y) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ such that $G(0, x) = 0$ and such that, in a neighbourhood of $I_x = \{(x, y) | x \in [0,1], y = 0\}$, $[Y, \overline{X}] = 0$. We show now, assuming the existence of such a vectorfield Y, that there is a diffeomorphism as in the conclusion of proposition (3,1). We take φ such that, for x near $0 \in \mathbf{R}^1$, $(x, 0)$ and $(\varphi(x), 1)$ are on the same integral curve of Y, or $(\varphi(x), 1) = \mathcal{D}_{Y,1}(x, 0)$. Because $[Y, \overline{X}] = 0$ in a neighbourhood of I_x, $\mathcal{D}_{Y,t}$ and $\mathcal{D}_{\overline{X},t}$, commute as long as all the integral curves in question are close to I_x. This means in particular that for x close to zero and t small, $\mathcal{D}_{Y,1} \; \mathcal{D}_{\overline{X},t}(x, 0) = \mathcal{D}_{\overline{X},t} \; \mathcal{D}_{Y,1}(x, 0)$. This, together with the fact that $\overline{X} | \{y = 0\} = X$ and $\overline{X} | \{y = 1\} = \tilde{X}$, implies that $\varphi_*(X) = X$ in a neighbourhood of $0 \in \mathbf{R}^1$.

Lemma (3,2) (see also [6]).

$$\textit{Let } \overline{X} = (x^k . F(x) + y . x^{2k} . H(x)) \frac{\partial}{\partial x}$$

be a vectorfield on $\mathbf{R}^2$, $k \geqslant 2$ *and* $F(0) \neq 0$. *Then there is a vectorfield* $Y = x^k . K(x, y) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ *such that* $[X, Y] = 0$ *on a neighbourhood of* $I_X = \{(x, y) | x \in [0,1], y = 0\}$.

Proof. — Writing out $[Y, \overline{X}] = 0$, we obtain

$$\begin{aligned} x^k . K(x, y) . [k . x^{k-1} . F(x) + x^k . F'(x) + 2k . y . x^{2k-1} . H(x) \\ + y . x^{2k} . H'(x)] + x^{2k} . H(x) - (x^k . F(x) \\ + y . x^{2k} . H(x)) . \left(k . x^{k-1} . K(x, y) + x^k . \frac{\partial K}{\partial x}(x, y) \right) = 0. \end{aligned}$$

The termes with x^{2k-1} cancel, so we can devide by x^{2k} and obtain:

$$- \frac{\partial K}{\partial x}(x, y)(F(x) + y . x^{2k} . H(x)) + K(x, y)(F'(x) \\ +2k . y . x^{k-1} . H(x) + y . x^k . H'(x) - k . y . x^{k-1} . H(x)) + H(x) = 0.$$

We have to solve K from this equation; the functions F and H are given. For each fixed y the above equation is an ordinary differential equation without « singularities » near $x = 0$ (because $F(0) \neq 0$ and hence $F(x) + y . x^k . H(x) \neq 0$ for x near 0). By the existence (and smoothness) theorem for solutions of differential equations depending on a parameter there is a solution (near I_x) of the above equation; this solution can be made unique by requiring that $K(0, y) = 0$ for all $y \in \mathbf{R}$.

4. Existence and uniqueness of solutions of a certain functional equation.

In this § we want to prove the following.

THEOREM (4,1). — *Let* $A: \mathbf{R}^1 \to \mathbf{R}^1$ *be a* C^∞*-diffeomorphism, depending on real parameters* $\mu_1, \ldots, \mu_r$, *which is of the form* $A(x; \mu_1, \ldots, \mu_r) = x + x^k . F(x; \mu_1, \ldots, \mu_r)$ *with*:

(i) $k \geqslant 2$;

(ii) $F(0; \mu_1, \ldots, \mu_r) \neq 0$ *for all* $\mu_1, \ldots, \mu_r$;

(iii) F *is a* C^∞*-function of* $(x; \mu_1, \ldots, \mu_r)$ *and its* ∞*-jet, as function of* x, *is the same for all points* $(0; \mu_1, \ldots, \mu_r)$, i.e., F *can be written as*

$$F(x; \mu_1, \ldots, \mu_r) = F_1(x) + F_2(x; \mu_1, \ldots, \mu_r),$$

with F_2 *flat in all points* $(0; \mu_1, \ldots, \mu_r)$.

Let $Y = g(x; \mu_1, \ldots, \mu_r) \frac{\partial}{\partial x}$ *be a* C^∞*-vectorfield, depending smoothly on* $\mu_1, \ldots, \mu_r$, *with* g *flat in all points* $(0; \mu_1, \ldots, \mu_r)$.

Then there is, for each positive C, *an* $\varepsilon > 0$ *and a unique* C^∞*-vectorfield* Z *on* $\mathbf{R}^1 \cap \{|x| \leqslant \varepsilon\}$, *flat in* $0 \in \mathbf{R}^1$ *and* *depending smoothly on* $\mu_1, \ldots, \mu_r$, *such that* $A_*(Z) + Y = Z$ *on* $\mathbf{R}^1 \cap \{|x| \leqslant \varepsilon\}$ *for each* $(\mu_1, \ldots, \mu_r)$, *with* $|\mu_i| \leqslant C$ *for all* $i = 1, \ldots, r$, *for some extention* $\tilde{Z}$ *of* Z *to the whole real line.*

Before we come to the real proof of the theorem (4,1)' we have to introduce some notation, apply some coordinate changes and reformulate the theorem.

a) Some notation and definitions.

From now on we shall denote $(\mu_1, \ldots, \mu_r)$ by μ; we shall also sometimes denote $A(x; \mu)$ by $A_\mu(x)$. On $T(\mathbf{R}^1)$, the tangent space of $\mathbf{R}^1$, we introduce coordinates (x, ν); x giving the position on $\mathbf{R}^1$ and ν measuring the length of vectors. A induces in a natural way a diffeomorphism $A_* : T(\mathbf{R}^1) \to T(\mathbf{R}^1)$, depending on μ, which has the form

$$A_*(x, \nu; \mu) = (x + x^k . F(x; \mu),$$
$$[1 + k . x^{k-1} . F(x; \mu) + x^k . F'(x; \mu)] . \nu),$$

where $F'(x; \mu) = \dfrac{\partial F}{\partial x}(x; \mu)$. A_{μ_*} is defined by

$$A_{\mu_*}(x, \nu) = A_*(x, \nu; \mu).$$

Next we choose a vectorfield X on $\mathbf{R}^1$ (independent of μ) such that $\hat{\mathfrak{D}}_{X,1} = \hat{A}_\mu$ for all μ (we use the symbol $\hat{}$ to indicate that only the ∞-jets in $\{x = 0\}$ are equal); such a vectorfield X exists because the ∞-jet of A_μ in $\{x = 0\}$ is independent of μ and because for each orientation preserving diffeomorphism $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ there is a vectorfield X on $\mathbf{R}^1$ such that $\hat{\mathfrak{D}}_{X,1} = \hat{\varphi}$. It is clear that X can be written in the form $X = x^k . \bar{F}(x) \dfrac{\partial}{\partial x}$ with $\bar{F}(x) \neq 0$. Also X induces a vectorfield on $T(\mathbf{R}^1)$ which is given by $X_* = x^k . \bar{F}(x) \dfrac{\partial}{\partial x} + (k . x^{k-1} . \bar{F}(x) + x^k . \bar{F}'(x)) . \nu \dfrac{\partial}{\partial \nu}$. Clearly we have $\hat{\mathfrak{D}}_{X_*,1} = \hat{A}_{\mu_*}$ (i.e., ∞-jets are equal in all points of $\{x = 0\}$) for all μ. (The only reason why we introduced the vectorfields X and X_* is that they will turn out to be very helpfull to carry out one of the necessary coordinate transformations.)

Finally we define the diffeomorphism $\Phi : T(\mathbf{R}^1) \to T(\mathbf{R}^1)$, depending on μ, by

$$\Phi(x, \nu; \mu) = A_*(x, \nu; \mu) + (0, g(x; \mu)) = (x + x^k.F(x; \mu),$$

$[1 + k.x^{k-1}.F(x; \mu) + x^k.F'(x; \mu)].\nu + g(x; \mu))$, where g is the function defining Y (see the statement of theorem (4,1)). $\Phi_\mu(x, \nu)$ is defined by $\Phi_\mu(x, \nu) = \Phi(x, \nu; \mu)$;

$$\tilde{\Phi} : T(\mathbf{R}^1) \times \mathbf{R}^r \to T(\mathbf{R}^1) \times \mathbf{R}^r$$

is defined by $\tilde{\Phi}(x, \nu; \mu) = (\Phi(x, \nu; \mu); \mu)$. Clearly we have $\hat{\mathfrak{D}}_{X_{*}}, \hat{\Phi}_1 = {}_\mu$ for all μ.

b) First restatement of the theorem.

Using the notation, introduced in a), we can give the following restatement of theorem (4,1):

For each $C > 0$ there is an $\varepsilon > 0$ and an unique smooth C^∞-submanifold W of dimension $r + 1$ in the $(x, \nu; \mu)$-space such that:

(i) W is of the form $W = \{(x, \nu; \mu) | \nu = h(x; \mu), |\mu_i| \leqslant C, i = 1, \ldots, r$ and $|x| \leqslant \varepsilon\}$ for some C^∞-function h which is flat in all points $(0; \mu)$;

(ii) there is an extension $\tilde{W}$ of W, i.e.,

$$\tilde{W} = \{(x, \nu; \mu) | \nu = \tilde{h}(x; \mu)\}$$

with $\tilde{h}(x; \mu) | \{|x| \leqslant \varepsilon, |\mu_i| \leqslant C\} = h$, such that

$$(\tilde{\Phi}(\tilde{W})) \cap \{|x| \leqslant \varepsilon, |\mu_i| \leqslant C\} = \tilde{W}$$
$$= (\tilde{\Phi}^{-1}(\tilde{W})) \cap \{|x| \leqslant \varepsilon, |\mu_i| \leqslant C\}.$$

c) The first coordinate transformation.

As new coordinates on $T(\mathbf{R}^1)$ we take $\bar{\nu} = x^{-k}.\nu$ and $\bar{x} = x$. This change of coordinates is of coarse singular along $x = 0$, but Φ and X_*, expressed in these coordinates, are still smooth; this follows from the explicit form of Φ_* and X_*, with respect to these coordinates, which we shall calculate now.

From the definition of $\bar{x}$ and $\bar{\nu}$, we obtain directly

$$\nu = \bar{x}^{+k}.\bar{\nu},\ x = \bar{x},\ \frac{\partial}{\partial \nu} = \bar{x}^{-k}.\frac{\partial}{\partial \bar{\nu}} \text{ and } \frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}} - k.\bar{x}^{-1}.\bar{\nu}\,\frac{\partial}{\partial \bar{\nu}}.$$

Using this we see that X_* has, with respect to the $(\bar{x}, \bar{\nu})$ coordinates, the form

$$X_* = \bar{x}^k.\bar{F}(\bar{x})\left(\frac{\partial}{\partial\bar{x}} - k.\bar{x}^{-1}.\bar{\nu}\frac{\partial}{\partial\bar{\nu}}\right) + (k.\bar{x}^{k-1}.\bar{F}(\bar{x})$$
$$+ \bar{x}^k.\bar{F}'(\bar{x})).\bar{x}^k.\bar{\nu}.\bar{x}^{-k}\frac{\partial}{\partial\bar{\nu}} = \bar{x}^k.\bar{F}(\bar{x})\frac{\partial}{\partial\bar{x}} + \bar{x}^k.\bar{\nu}.\bar{F}'(\bar{x})\frac{\partial}{\partial\bar{\nu}};$$

this is still C^∞.

For Φ, expressed in the $(\bar{x}, \bar{\nu})$ coordinates we obtain

$$\Phi(\bar{x}, \bar{\nu}; \mu) = \Bigg(\bar{x} + \bar{x}^k.\bar{F}(\bar{x}; \mu),$$
$$\frac{(1 + k.\bar{x}^{k-1}.F(\bar{x}; \mu) + \bar{x}^k.F'(\bar{x}; \mu)).\bar{x}^k.\bar{\nu} + g(\bar{x}; \mu)}{(\bar{x} + \bar{x}^k.F(\bar{x}; \mu))^k}\Bigg).$$

It is clear that $\tilde{g}(\bar{x}; \mu) = \dfrac{g(\bar{x}; \mu)}{(\bar{x} + \bar{x}^k.F(\bar{x}; \mu))^k}$ is still C^∞ and flat in all points of $\{\bar{x} = 0\}$ for all μ. Furthermore, $(\bar{x} + \bar{x}^k.F(\bar{x}; \mu))^k = \bar{x}^k(1 + k.\bar{x}^{k-1}.F(\bar{x}; \mu) +$ terms of order $\geqslant k$ in $\bar{x}$). Hence there is some C^∞-function $\tilde{F}(\bar{x}; \mu)$ such that $\bar{x}^k.\tilde{F}(\bar{x}; \mu) = \dfrac{(1 + k.\bar{x}^{k-1}.F(\bar{x}; \mu) + \bar{x}^k.F'(\bar{x}; \mu)).\bar{x}^k.\bar{\nu}}{(\bar{x} + \bar{x}^k.F(\bar{x}; \mu))^k}$; so we obtain $\Phi(\bar{x}, \bar{\nu}; \mu) = (\bar{x} + \bar{x}^k.\bar{F}(\bar{x}; \mu), \bar{x}^k\tilde{F}(\bar{x}; \mu).\bar{\nu} + \tilde{g}(\bar{x}; \mu))$. From the definition of $\tilde{F}$ and the condition (iii) imposed upon F in theorem (4,1) it follows that the ∞-jet of $\tilde{F}(\bar{x}; \mu)$, as function of x, is the same in all points $(0; \mu)$.

Next we show that $\hat{\mathfrak{D}}_{X_*,1} = \hat{\Phi}_\mu$ still holds after the coordinate change (this is not evident because along

$$\{x = 0\} = \{\bar{x} = 0\},$$

our coordinate change is singular). We first observe that the ∞-jet of Φ along $\{\bar{x} = 0\}$, in the $(\bar{x}, \bar{\nu})$ coordinates, is completely determined by the ∞-jet of F along $\{x = 0\}$. Next we write $\mathcal{D}_{X_*,1}$ in the form

$$\mathcal{D}_{X_*,1}(x, \nu) = (x + x^k.\dot{F}(x), [1 + k.x^{k-1}.\dot{F}(x) + x^k.\dot{F}'(x)].\nu)$$

for some function $\dot{F}$. From the way X was chosen and X_* was defined, it is clear that such $\dot{F}$ exists; furthermore the ∞-jet of $\dot{F}$ in $\{x = 0\}$ equals the ∞-jet of $F(x; \mu)$, as function of x, in $\{x = 0\}$ for all μ. From this it follows that if we now write $\mathcal{D}_{X_*,1}$ in the $(\bar{x}, \bar{\varphi})$-coordinates we find along $\{\bar{x} = 0\}$ the same ∞-jets as for Φ along $\{\bar{x} = 0\}$. Hence we have also in the $\bar{x}, \bar{\varphi}$ coordinates $\hat{\mathcal{D}}_{X_*,1} = \hat{\Phi}_\mu$.

Finally, let $W = \{(x, \varphi; \mu) | \varphi = h(x; \mu), |x| \leqslant \varepsilon, |\mu_i| \leqslant C\}$ be a submanifold of the $(x, \varphi; \mu)$-space as in the conclusion of restatement $b.$. Then the same manifold W is given in the $(\bar{x}, \bar{\varphi})$-coordinates by

$$W = \{(\bar{x}, \bar{\varphi}; \mu) | \bar{\varphi} = \bar{x}^{-k}.h(\bar{x}; \mu), |\bar{x}| \leqslant \varepsilon, |\mu_i| \leqslant C\}.$$

As h was flat in all points of $\{x = 0\}$, $\bar{x}^{-k}.h(\bar{x}; \mu)$ is still smooth and flat in all points of $\{\bar{x} = 0\}$.

Resuming, we have the following: by the coordinate transformation $\bar{\varphi} = x^{-k}.\varphi$, $\bar{x} = x$, we changed the form of Φ but apart from that the problem to be solved (finding W as in $b.$) remained the same.

d) The second coordinate change.

This coordinate change will be obtained by using the vectorfield X_*. As we have seen under c., the vectorfield X_* has, with respect to the $(\bar{x}, \bar{\varphi})$ coordinates, the form

$$X_* = \bar{x}^k.\bar{F}(\bar{x}) \frac{\partial}{\partial \bar{x}} + \bar{x}^k.\bar{\varphi}.\bar{F}'(\bar{x}) \frac{\partial}{\partial \bar{\varphi}}.$$

Hence, if we devide by $\bar{x}^k$, we get again a smooth vectorfield $\bar{x}^{-k}.X_* = \bar{F}(\bar{x}) \frac{\partial}{\partial \bar{x}} + \bar{\varphi}.\bar{F}'(\bar{x}) \frac{\partial}{\partial \bar{\varphi}}$. Because $\bar{F}(0) \neq 0$, $\bar{x}^{-k}.X_*$ has no zero-points for $|\bar{x}|$ small. Using integral curves of $\bar{x}^{-k}.X_*$ it is easy to see that there is a regular coordinate change $\bar{\bar{\varphi}} = \bar{\bar{\varphi}}(\bar{x}, \bar{\varphi})$, $\bar{\bar{x}} = \bar{x}$ with $\bar{\bar{\varphi}}(0, \bar{\varphi}) = \bar{\varphi}$, such that $\bar{x}^{-k}X_* = \bar{F}(\bar{x}) \frac{\partial}{\partial \bar{\bar{x}}}$ and hence

$$X_* = \bar{x}^k.\bar{F}(\bar{x}) \frac{\partial}{\partial \bar{\bar{x}}} = \bar{\bar{x}}^k.\bar{F}(\bar{\bar{x}}) \frac{\partial}{\partial \bar{\bar{x}}}$$

(at least for $|\bar{x}|$, or $|\bar{\bar{x}}|$, small). From this it follows that Φ, in the $(\bar{\bar{x}}, \bar{\bar{\varphi}})$ coordinates, has the form

$$\Phi(\bar{\bar{x}}, \bar{\bar{\varphi}}, \mu) = (\bar{\bar{x}} + \bar{\bar{x}}^k.F(\bar{\bar{x}}; \mu), \bar{\bar{\varphi}} + G(\bar{\bar{x}}, \bar{\bar{\varphi}}; \mu))$$

with F as in $b.$ and G flat in all points of $\{\bar{\bar{x}} = 0\}$.

e) The second restatement of the theorem.

From *b*) and the properties of the coordinate transformations described under *c*) and *d*) it follows that theorem (4,1) is a consequence of

PROPOSITION (4,2). — *Let* $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ *be a* C^∞*-diffeomorphism, depending on* $\mu = (\mu_1, \ldots, \mu_r)$, *which is of the form* $\Phi(x, \nu; \mu) = (x + x^k.F(x; \mu), \nu + G(x, \nu; \mu))$ *with* $F(x; \mu) = F_1(x) + F_2(x; \mu)$, F_2 *and* G *flat along* $\{x = 0\}$ *and* $F_1(0) \neq 0$. $\tilde{\Phi} : \mathbf{R}^2 \times \mathbf{R}^r \to \mathbf{R}^2 \times \mathbf{R}^r$ *denotes the map* $\tilde{\Phi}(x, \nu; \mu) = (\Phi(x, \nu; \mu); \mu)$.

Then there is for each $C > 0$ *an* $\varepsilon > 0$ *and a unique submanifold* W *of* $\mathbf{R}^{2+r}$ *such that*

(i) W *is of the form* $W = \{(x, \nu; \mu) | \nu = h(x; \mu),\ |x| \leqslant \varepsilon$ *and* $|\mu_i| \leqslant C,\ i = 1, \ldots, r\}$ *for some* C^∞*-function* h *which is flat in all points* $(0; \mu)$;

(ii) *there is an extension* $\tilde{W}$ *of* W, i.e.,

$$\tilde{W} = \{(x, \nu; \mu) | \nu = \tilde{h}(x; \mu)\}$$

with $\tilde{h}(x; \mu) = h(x; \mu)$ *if* $|x| \leqslant \varepsilon$ *and* $|\mu_i| \leqslant C$, *such that*

$$\tilde{\Phi}(\tilde{W}) \cap \{|x| \leqslant \varepsilon, |\mu_i| \leqslant C\} = W = \tilde{\Phi}^{-1}(\tilde{W}) \cap \{|x| \leqslant \varepsilon, |\mu_i| \leqslant C\}.$$

Notice that we simplified our notation (symbols) compared with d..

f) Reduction to the expansion case.

If Φ satisfies the conditions in the assumption of proposition (4,2), then also Φ^{-1} satisfies them. If $\tilde{W}$ is invariant under Φ (in the sense of (ii) above), then $\tilde{W}$ is also invariant under Φ^{-1}. With this in mind, one can carry out the following simplification.

For any $C > 0$, there is an $\varepsilon > 0$ such that if $|\mu_i| \leqslant C$ for $i = 1, \ldots, r$ and $0 < |x| \leqslant \varepsilon$, then $|(x + x^k.F(x; \mu))| \neq |x|$. Let now U_+, resp. U_-, be the set of points $(x; \mu)$, $|x| \leqslant \varepsilon$, $|\mu_i| \leqslant C$, such that

$$|x + x^k.F(x; \mu)| \geqslant |x|, \quad \text{resp.} \quad |x + x^k.F(x; \mu)| \leqslant |x|.$$

Depending on the sign of $F(0; \mu)$ and the value of k (even or odd), we have one of the following four situations:

$$\left.\begin{array}{l}1.\ U_- \text{ is everything}\\ 2.\ U_+ \text{ is everything}\\ 3.\ U_-=\{|x|\leqslant 0\},\ U_+=\{|x|\geqslant 0\}\\ 4.\ U_-=\{|x|\geqslant 0\},\ U_+=\{|x|\leqslant 0\}\end{array}\right\}\text{ as subsets of } |x|\leqslant\varepsilon,\ |\mu_i|\leqslant C.$$

Below we shall prove that there is an invariant manifold W_+ of the form

$$W_+ = \{(x, \nu; \mu) | (x; \mu) \in U_+, |x| \leqslant \varepsilon', \nu = h_+(x; \mu)\}$$

which satisfies the conditions (i), (ii) in proposition (4,2) if everything is restricted to the set of points $(x, \nu; \mu)$ with $(x; \mu) \in U_+$. The same construction, using Φ^{-1} instead of Φ, gives a manifold W_-. In case 1, resp. 2, above W_-, resp. W_+, is already the required manifold. In case 3. or 4., one has to take $W = W_- \cup W_+$; as W_- and W_+ have, along $\{x = \nu = 0\}$ ∞-order contact with $\{\nu = 0\}$,

$$W = W_- \cap W_+$$

is a smooth manifold which has all the required properties. Hence in the proof of proposition (4,2) we may restrict our attention to those points $(x, \nu; \mu)$ with $(x; \mu) \in U_+$.

g) Some definitions.

We define $\Psi : \mathbf{R}^2 \to \mathbf{R}^2$ by $\Psi(x, \nu) = (x + x^k.F_1(x), \nu)$ and $\tilde{\Psi} : \mathbf{R}^2 \times \mathbf{R}^r \to \mathbf{R}^2 \times \mathbf{R}^r$ by $\tilde{\Psi}(x, \nu; \mu) = (\Psi(x, \nu); \mu)$; F_1 is as in the statement of proposition (4,2). It is clear that $\hat{\Psi} = \hat{\Phi}_\mu$ for each μ, i.e., the ∞-jets are equal along $\{x = 0\}$.

For a, $\varepsilon > 0$, we define $D_{a,\varepsilon} = \{(x, \nu; \mu) | \nu = 0, |x| \leqslant \varepsilon, |\mu_i| \leqslant a$ for $i = 1, \ldots, r$, $|x + x^k.F_1(x)| \geqslant |x|$ and $|x + x^k.F(x; \mu)| \geqslant |x|\}$; note that for given a and ε small enough, $D_{a,\varepsilon} = \{(x, \nu; \mu) | \nu = 0, |x| \leqslant \varepsilon, |\mu_i| \leqslant a$ for $i = 1, \ldots, r$ and $|x + x^k.F(x; \mu)| \geqslant |x|\}$. Points of $D_{a,\varepsilon}$ are sometimes denoted by $(x; \mu)$ instead of $(x, 0; \mu)$.

Let $\mathscr{F}_{a,\varepsilon}$ be the set of real valued functions on $D_{a,\varepsilon}$. The maps $\Gamma_\Phi, \Gamma_\Psi : \mathscr{F}_{a,\varepsilon} \to \mathscr{F}_{a,\varepsilon}$ are defined, for each $a, \varepsilon > 0$, by:

$$[\tilde{\Phi}(\{x, f(x; \mu); \mu\})] \cap \{(x, \nu; \mu) | (x, 0; \mu) \in D_{a,\varepsilon}\} = \{x, (\Gamma_\Phi f)(x; \mu); \mu\},$$

$$[\tilde{\Psi}(\{x, f(x; \mu); \mu\})] \cap \{(x, \nu; \mu) | (x, 0; \mu) \in D_{a,\varepsilon}\} = \{x, (\Gamma_\Psi f)(x; \mu); \mu\}$$

for all $f \in \mathscr{F}_{a,\varepsilon}$. Using the above definitions and the reduction described under *f.*, it follows that, in order to prove proposition (4,2), it is enough to prove.

PROPOSITION (4,3). — *For each* a, *there is an* $\varepsilon > 0$ *such that* $\Gamma_\Phi : \mathscr{F}_{a,\varepsilon} \to \mathscr{F}_{a,\varepsilon}$, *restricted to* C^∞*-functions which are flat along* $\{x = 0\}$, *has a unique fixedpoint.*

For the proof of this proposition we need a sequence of lemmas which is stated and proved below.

h) Some lemmas and the proof of proposition (4,3).

For any vector $\vec{a} \in T_{(x,0;\mu)}\{\nu = 0\}$ of the form $a = a_x \frac{\partial}{\partial x} + \sum_{i=1}^{r} a_i \frac{\partial}{\partial \mu_i}$ we define $\|\vec{a}\| = \sqrt{a_x^2 + a_1^2 + \cdots + a_r^2}$. If D is some l-linear function on $T_{(x,0;\mu)}\{\nu = 0\}$, we define $\|D\|$ by

$$\|D\| = \sup_{\substack{\vec{a}_1, \ldots, \vec{a}_l \neq 0 \\ \vec{a}_i \in T_{(x,0;\mu)}\{\nu=0\}}} \|\vec{a}_1\|^{-1} \ldots \|\vec{a}_l\|^{-1} . |D(\vec{a}_1, \ldots, \vec{a}_l)|$$

LEMMA (4,4). — *For each* $a > 0$ *and each pair of positive integers* h *and* m, *there is an* $\varepsilon > 0$ *such that for any* C^h*-function* $f \in \mathscr{F}_{a,\varepsilon}$ *with* $\|D^h f(x;\mu)\| \leqslant |x|^m$, *and*

$$\|D^i f(x;\mu)\| \leqslant |x|^{m+k}$$

for all $i = 1, \ldots, h-1$ *in each point* $(x;\mu) \in D_{a,\varepsilon}$, $\|D^h(\Gamma_\Psi f)(x;\mu)\| \leqslant |x|^m . (1 - 4^{-1} . |x|^{k-1} . |F_1(0)|)$ for all

$$(x;\mu) \in D_{a,\varepsilon};$$

the norm of the i^{th} *derivative* $D^i f(x;\mu)$ *is the norm as i-linear function.*

Proof. — We define $\overline{F}_1(x;\mu) = (x + x^k . F_1(x); \mu)$. From the definition of L_Ψ, we obtain that $\Gamma_\Psi f = f\overline{F}_1^{-1}$. For such compositions it is easy to obtain the following estimate

$$\|D^h(f\overline{F}_1^{-1})(x;\mu)\| \leqslant \|D^h(f)(\overline{F}_1^{-1}(x;\mu))\| . \|D^1(\overline{F}^{-1})(x;\mu)\|^h + \sum_{i=1}^{-1} C_i \|D^i(f)(\overline{F}_1^{-1}(x;\mu))\|,$$

where $C_1, \ldots, C_{h-1}$ depend on the derivatives of order

$1, \ldots, h-1$ of $\overline{F}_1^{-1}$ in $(x; \mu)$. We now choose, for some $\varepsilon_1 > 0$, the constants $C_1, \ldots, C_{h-1}$ such that the above formula is valid for any $(x; \mu) \in D_{a,\varepsilon_1}$.

From the definition of $D_{a,\varepsilon}$ it follows that for some ε_2, $0 < \varepsilon_2 \leqslant \varepsilon_1$ we have $\|D^1(\overline{F}_1^{-1})(x; \mu)\| = 1$ whenever $(x; \mu) \in D_{a,\varepsilon_2}$.

ε_3, $0 < \varepsilon_3 \leqslant \varepsilon_2$, is then choosen so that, for $(x; \mu) \in D_{a,\varepsilon_3}$, the x-coordinate of $\overline{F}_1^{-1}(x; \mu)$ is, in absolute value, $\leqslant |x| - \frac{1}{2}|x^k . F_1(0)|$. Now we come back to the estimate for the norm of $D^h(f\overline{F}_1^{-1})$. Applying the above formula in points $(x; \mu) \in D_{a,\varepsilon_3}$ to any C^h-function $f \in \mathscr{F}_{a,\varepsilon_3}$, satisfying the assumptions of lemma (4,4), we obtain:

$$\begin{aligned}\|D^h(f\overline{F}_1^{-1})(x; \mu)\| &\leqslant (|x\text{-coordinate of } \overline{F}_1^{-1}(x; \mu)|)^m \\ &\quad + \sum_{i=1}^{h-1} C_i|x|^{m+k} \leqslant \left(1 - \frac{1}{2}|F_1(0)| . |x|^{k-1}\right)^m . |x|^m \\ &\quad + \sum_{i=1}^{h-1} C_i|x|^{m+k} = |x|^m . \left(1 - \frac{1}{2}\, m . |F_1(0)| . |x|^{k-1}\right) \\ &\quad + \text{terms of order } \geqslant m + k \text{ in } x.\end{aligned}$$

Finally we take ε, $0 < \varepsilon \leqslant \varepsilon_3$, so small that for $|x| \leqslant \varepsilon$,

$$\left(1 - \frac{1}{2}|F_1(0)| . |x|^{k-1}\right)^m . |x|^m + \sum_{i=1}^{h-1} C_i|x|^{m+k} \leqslant (1 - 4^{-1} . |F_1(0)| . |x|^{k-1}) . |x|^m.$$

It is then clear that for any $(x; \mu) \in D_{a,\varepsilon}$ and any C^h-function $f \in \mathscr{F}_{a,\varepsilon}$, satisfying the assumptions of lemma (4,4), we have $\|D^h(f\overline{F}_1^{-1})(x; \mu)\| \leqslant |x|^m . (1 - 4^{-1} . |F_1(0)| . |x|^{k-1}$.

Lemma (4,5). — *For each $a > 0$ and positive integer l, there is an $\varepsilon > 0$, such that for any C^1-function $f \in \mathscr{F}_{a,\varepsilon}$ with $\|D^i f(x; \mu)\| \leqslant |x|^{1+k.(l-i)}$ for all $i = 1, \ldots, l$ and all $(x; \mu) \in D_{a,\varepsilon}$,*

$$\|D^i(\Gamma_\Psi f)(x; \mu)\| \leqslant |x|^{1+k.(l-i)} . (1 - 4^{-1} . |x|^{k-1} . |F_1(0)|),$$

for all $i = 1, \ldots, l$ and $(x; \mu) \in D_{a,\varepsilon}$.

Proof. — This lemma follows by induction from lemma (4,4): First we take ε_1, $0 < \varepsilon_1 < 1$ such that the conclusion of lemma (4,4) holds with $h = l$ and $m = 1$. Then we have for any C^l-function $f \in \mathscr{F}_{a,\varepsilon_1}$ with $\|D^i f(x;\mu)\| \leqslant |x|^{1+k.(l-i)}$ for all $i = 1, \ldots, l$ and all $(x;\mu) \in D_{a,\varepsilon_1}$ that

$$\|D^l(\Gamma_\Psi f)(x;\mu)\| \leqslant |x|^1.(1 - 4^{-1}.|x|^{k-1}.|F_1(0)|)$$

for all $(x;\mu) \in D_{a,\varepsilon_1}$. Then we take ε_2, $0 < \varepsilon_2 \leqslant \varepsilon_1$ as in the conclusion of lemma (4,4) with $h = l - 1$ and $m=1+k$. Then the conclusion of lemma (4,5) holds for functions $f \in \mathscr{F}_{a,\varepsilon_2}$, as far as D^l and D^{l-1} of $(\Gamma_\Psi f)$ are concerned. Going on this way we find the required ε in l steps.

Lemma (4,6). — *For any $a > 0$ and positive integer l, there is an $\varepsilon > 0$ such that for any $f \in \mathscr{F}_{a,\varepsilon}$ with $f\,|\,\{x=0\}=0$ and $\|D^i f(x;\mu)\| \leqslant |x|^{1+k.(l-i)}$ for all $i = 1, \ldots, l$ and $(x;\mu) \in D_{a,\varepsilon}$,*

$$\|D^i(\Gamma_\Phi f)(x;\mu)\| \leqslant |x|^{1+k.(l-i)} \textit{ for all } i = 1, \ldots, l$$

and $(x;\mu) \in D_{a,\varepsilon}$.

Proof. — We define $\mathscr{F}^l_{a,\varepsilon} \subset \mathscr{F}_{a,\varepsilon}$ to be the set of C^l-functions $f \in \mathscr{F}_{a,\varepsilon}$ with $f|\{x = 0\} = 0$ and

$$\|(D^i f)(x;\mu)\| \leqslant |x|^{1+k.(l-i)} \text{ for all } i = 1, \ldots, l$$

and $(x;\mu) \in D_{a,\varepsilon}$. From the definition of Γ_Φ and Γ_Ψ it is clear that there is a continuous flat function $\alpha : \mathbf{R}_+ \to \mathbf{R}_+ (\mathbf{R}_+ = [0, \infty))$ (flat means here that for any n there is an $\varepsilon_n > 0$ such that $\alpha(x) \leqslant x^n$ if $x \leqslant \varepsilon_n$) such that for any $f \in \mathscr{F}^l_{a,\varepsilon}$, $\|D^i((\Gamma_\Psi f) - (\Gamma_\Phi f))(x;\mu)\| \leqslant \alpha(x)$ for all $i = 1, \ldots, l$; we do not exclude that α also depends on a.

From this and lemma (4,5) it follows that, for ε small enough, $\Gamma_\Phi(\mathscr{F}^l_{a,\varepsilon}) \subset \mathscr{F}^l_{a,\varepsilon}$.

Definition (4,7). — *$\overline{\mathscr{F}}{}^{l-1}_{a,\varepsilon}$ denotes the closure of $\mathscr{F}^l_{a,\varepsilon}$ (defined in the proof of lemma (4,6)) in the C^{l-1}-topology.*

Remark (4,8). — By lemma (4,6) there is, for each $a > 0$, and positive integer l, an $\varepsilon_{a,l} > 0$ such that for any $0 < \varepsilon < \varepsilon_{a,l}$, $\Gamma_\Phi(\overline{\mathscr{F}}{}^{l-1}_{a,\varepsilon}) \subset \overline{\mathscr{F}}{}^{l-1}_{a,\varepsilon}$. Furthermore, $\overline{\mathscr{F}}{}^{l-1}_{a,\varepsilon}$ is compact with respect to the C^{l-1}-topology. This follows from the fact that the family of partial derivatives of order $(l-1)$ of all $f \in \overline{\mathscr{F}}^{l-1}_{a,\varepsilon}$ is equicontinuous (compactness then follows from Ascoli's theorem [1]).

LEMMA (4,9). — *For each $a > 0$ there is an $\varepsilon > 0$ and a unique continuous function $\bar{f} \in \mathscr{F}_{a,\varepsilon}$ such that for any C^∞-function $f \in \mathscr{F}_{a,\varepsilon}$, which is flat in all points of $\{x = 0\}$ and such that $|f(x;\mu)| \leqslant |x|$ for all $(x;\mu) \in D_{a,\varepsilon}$, $\lim_{i\to\infty} ((\Gamma_\Phi)^i f) = \bar{f}$ (this is the limit in the C^0-topology).*

Proof. — With the same methods as we used in the proofs of lemma (4,4), (4,5) and (4,6) it easily follows that there is an $\varepsilon > 0$ such that for any pair of functions $f_1, f_2 \in \mathscr{F}_{a,\varepsilon}$ with $|f_i(x;\mu)| \leqslant |x|$ and $|f_1(x;\mu) - f_2(x;\mu)| \leqslant C.|x|$ for all $(x;\mu) \in D_{a,\varepsilon}$ (C is a constant $\leqslant 2$ depending on f_1, f_2) we have $|(\Gamma_\Phi f_i)(x;\mu)| \leqslant |x|$ and

$$|\Gamma_\Phi f_1(x;\mu) - \Gamma_\Phi f_2(x;\mu)| \leqslant C.|x|.\left(1 - \frac{1}{2}|F(0)|.|x|^{k-1}\right)$$

for all $(x;\mu) \in D_{a,\varepsilon}$.

We shall show that, with ε as above, the conclusion of lemma (4,9) holds. Let $f_0 \in \mathscr{F}_{a,\varepsilon}$ be a smooth function, flat in all points of $\{x = 0\}$ and with $|f_0(x;\mu)| \leqslant |x|$ for all $(x;\mu) \in D_{a,\varepsilon}$. Let $l \geqslant 2$ be some integer; then, for some $0 < \varepsilon' < \varepsilon$, $f_0|D_{a,\varepsilon'} \in \overline{\mathscr{F}}^{l-1}_{a,\varepsilon'}$. We may, and do, assume that $\varepsilon' < \varepsilon_{a,l}$ (see remark (4,8)) so that we have for $f_i = (\Gamma_\Phi)^i f_0$, $f_i|D_{a,\varepsilon'} \in \overline{\mathscr{F}}^{l-1}_{a,\varepsilon'}$. From the definition of $\overline{\mathscr{F}}^{l-1}_{a,\varepsilon'}$, it follows that $|f_i(x;\mu)| \leqslant |x|^{l+1}$ for $|x| \leqslant \varepsilon'$. Next we define $0 \leqslant C_{i,j} \leqslant 2$ to be the smallest constant such that

$$|f_i(x;\mu) - f_j(x;\mu)| \leqslant C_{i,j}|x|$$

for all $(x;\mu) \in D_{a,\varepsilon}$; from our choice of ε it follows that $C_{i+1,j+1} \leqslant C_{i,j}$. In order to show that $\lim_i f_i$ exists, it is enough to show that $\lim_{i,j\to\infty} C_{i,j} = 0$. Suppose the contrary: let $C = \lim_{N\to\infty}\left(\sup_{i,j\geqslant N} C_{i,j}\right) > 0$; we shall derive a contradiction from this.

Take i, j such that $C_{i,j} \in \left[\frac{1}{2}C, 2C\right]$; we shall compute an upperbound for $C_{i+1,\,j+1}$. From the properties of ε and ε', we know that

$$|f_{i+1}(x;\mu) - f_{j+1}(x;\mu)| \leqslant C_{i,j}.|x|.\left(1 - \frac{1}{2}|F(0)|.|x|^{k-1}\right)$$

for $(x;\mu) \in D_{a,\varepsilon}$ and $|f_{i+1}(x;\mu) - f_{j+1}(x;\mu)| \leqslant |x|^{l+1}$ for $(x;\mu) \in D_{a,\varepsilon'}$. It is now easy to see that there is some $\delta > 0$, depending on C only, such that, for any $C_{i,j} \in \left[\frac{1}{2}C, 2C\right]$:

$$C_{i,j}.|x| \cdot \left(1 - \frac{1}{2}|F(0)|.|x|^{k-1}\right) \leqslant (C_{i,j} - \delta).|x|$$

for $\varepsilon' \leqslant |x| \leqslant \varepsilon$ and

$$\min\left\{C_{i,j}.|x|.\left(1 - \frac{1}{2}|F(0)|.|x|^{k-1}\right), |x|^{l+1}\right\} \leqslant (C_{i,j} - \delta).|x|$$

for $0 \leqslant |x| \leqslant \varepsilon'$. Hence for each i, j with $C_{i,j} \in \left[\frac{1}{2}C, 2C\right]$ we have $C_{i+1,j+1} \leqslant C_{i,j} - \delta$.

Now we take some $N(\delta)$ such that if $i,\ j \geqslant N(\delta)$, then $C_{i,j} \leqslant \min.\left\{2C, C + \frac{1}{2}\delta\right\}$. From the above argument it follows that, for any

$$i',\ j' \geqslant N(\delta) + 1,\ C_{i',j'} \leqslant \max.\left\{\frac{1}{2}C, C - \frac{1}{2}\delta\right\} < C.$$

This however leads to $C = \lim_{N\to\infty}\left(\sup_{i,j\geqslant N} C_{i,j}\right) < C$ which is the required contradiction.

Finally we have to show that if we take another smooth function $f'_0 \in \mathscr{F}_{a,\varepsilon}$, which is flat in all points of $\{x = 0\}$ and such that $|f'(x;\mu)| \leqslant |x|$ for all $(x;\mu) \in D_{a,\varepsilon}$, then $\lim_{i\to\infty} f'_i = \lim_{i\to\infty} f_i$, where $f'_i = (\Gamma_\Phi)^i f'_0$ and $f_i = (\Gamma_\Phi)^i f_0$.

For this one defines C_i to be the smallest constant such that $|f'_i(x;\mu) - f_i(x;\mu)| \leqslant C_i|x|$ for all $(x;\mu) \in D_{a,\varepsilon}$. From the properties of ε it follows that $C_{i+1} \leqslant C_i$. The assumption $\lim_i C_i = \tilde{C} > 0$ leads, with the same arguments as above for C, to a contradiction (in this case we have to replace ε' by $\varepsilon'' < \varepsilon_{a,l}$, which is such that both $f_0|D_{a,\varepsilon''}$ and $f'_0|D_{a,\varepsilon''}$ are in $\mathscr{F}^{(l-1)}_{a,\varepsilon''}$). Hence we have $\lim_{i\to\infty} f'_i = \lim_{i\to\infty} f_i$.

LEMMA (4,10) (= PROPOSITION (4,3)). — *For each* $a > 0$ *there is an* $\varepsilon > 0$ *such that there is a unique* C^∞*-function* $f \in \mathcal{F}_{a,\varepsilon}$ *which is flat along* $\{x = 0\}$ *and such that* $\Gamma_\Phi(f) = f$.

Proof. — We denote the map $(x; \mu) \longmapsto (x + x^k.F(x; \mu); \mu)$ (see proposition (4,2)) by $\overline{\Phi}$. We take our ε such that the conclusion of lemma (4,9) is valid and such that for any $0 < \varepsilon' < \varepsilon$ there is an integer $N(\varepsilon')$ such that

$$\overline{\Phi}^{N(\varepsilon')}(D_{a,\varepsilon'}) \supset D_{a,\varepsilon}.$$

It is clear that if there is a function $f \in \mathcal{F}_{a,\varepsilon}$ with the required properties, it is unique (because of lemma (4,9)). Hence it is enough to show that the « limit-function » $\bar{f} \in \mathcal{F}_{a,\varepsilon}$, the existence of which was asserted in lemma (4,9), is C^∞ (it is clear that $\Gamma_\Phi(\bar{f}) = \bar{f}$). To show this we prove that for any given integer $l \geqslant 1$, $\bar{f}$ is C^{l-1}.

Let $f_0 \in \mathcal{F}_{a,\varepsilon}$ be some C^∞-function, flat in all points of $\{x = 0\}$ and such that $|f_0(x; \mu)| \leqslant |x|$ for all $(x; \mu) \in D_{a,\varepsilon}$. Let $0 < \varepsilon' < \min(\varepsilon, \varepsilon_{a,l})$ be such that $f_0|D_{a,\varepsilon'} \in \overline{\mathcal{F}}^{l-1}_{a,\varepsilon'}$. We shall denote $\Gamma^i_\Phi(f_0)$ by f_i and $\Gamma^i_\Phi(f_0|D_{a,\varepsilon'})$ by $\tilde{f}_i \in \overline{\mathcal{F}}^{l-1}_{a,\varepsilon'}$ (clearly $\tilde{f}_i = f_i|D_{a,\varepsilon'}$). As $\overline{\mathcal{F}}^{l-1}_{a,\varepsilon'}$ is compact, $\{\tilde{f}_i\}_{i=0}^\infty$ must have at least one accumulation point (with respect to the C^{l-1}-topology). Because f_i converges in the C^0-topology, $\{\tilde{f}_i\}_{i=0}^\infty$ must have exactly one accumulation point in the C^{l-1}-topology (if there would be two different accumulation points, they would have C^0-distance zero which is impossible). Hence $\{\tilde{f}_i\}_{i=0}^\infty$ converges in the C^{l-1}-topology and hence $\bar{f}\,|D_{a,\varepsilon'}$ is C^{l-1}. Finally $\bar{f}$ is C^{l-1} because $\Gamma_\Phi \bar{f} = \bar{f}$ and hence the graph of $\bar{f}$ can be obtained by applying $\overline{\Phi}^{N(\varepsilon')}$ to the graph of $\bar{f}\,|D_{a,\varepsilon'}$.

5. The proofs of the theorems 2, 3 and 4.

Proof of theorem 2 *(orientation preserving case)*. — From the results in § 2 it follows that we only have to show that for any diffeomorphism $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$, which is of the form $\Psi(x) = x + \delta x^k + \alpha x^{2k-1} + g(x)$, where g is a flat function, $\delta = \pm 1$, $\alpha \in \mathbf{R}$ and $k \geqslant 2$, there is an orientation preserving diffeomorphism $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ such that $\varphi\Psi\varphi^{-1}(x) = x + \delta x^k + \alpha x^{2k-1}$.

We define $\tilde{\Psi} : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\tilde{\Psi}(x, \mu) = (x + \delta x^k + \alpha x^{2k-1} + \mu g(x), \mu);$$

$\tilde{\Psi}_\mu$ is defined by $(\tilde{\Psi}_\mu(x), \mu) = \tilde{\Psi}(x, \mu)$. Now we want a vectorfield $\tilde{Z} = Z(x, \mu)\frac{\partial}{\partial x} + \frac{\partial}{\partial \mu}$, with Z flat in all points $\{x = 0\}$ and such that $\tilde{\Psi}_*(\tilde{Z}) = \tilde{Z}$ holds in a neighbourhood of $I = \{(x, \mu) | x = 0, 0 \leqslant \mu \leqslant 1\}$. The existence of such a vectorfield $\tilde{Z}$ follows from theorem (4,1):

We take $\tilde{Z}_\mu$ to be the vectorfield on $\mathbf{R}^1$, depending on μ, defined by $\tilde{Z}_\mu = Z(x, \mu)\frac{\partial}{\partial x}$ The vectorfield $\tilde{Y}$ is defined by $\tilde{Y} = \tilde{\Psi}_*\left(\frac{\partial}{\partial \mu}\right) - \frac{\partial}{\partial \mu} = Y(x, \mu)\frac{\partial}{\partial x}$. It is clear that the function Y is flat in all points of $\{x = 0\}$. $\tilde{Y}_\mu$ is defined to be the vectorfield $Y(x, \mu)\frac{\partial}{\partial x}$ on $\mathbf{R}^1$ depending on μ. It follows that $\tilde{\Psi}_*(\tilde{Z}) = (\tilde{Z})$ is equivalent with $\tilde{\Psi}_{\mu_*}(\tilde{Z}_\mu) + \tilde{Y}_\mu = \tilde{Z}_\mu$ for all μ; this last equation is just the equation which was solved in theorem (4,1).

The diffeomorphism $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ is now obtained from Z by requiring that, for $|x|$ small, $(x, 1)$ and $(\varphi(x), 0)$ are on the same integral curve of $\tilde{Z}$. The fact that $\tilde{\Psi}^*(\tilde{Z}) = \tilde{Z}$ implies that $\varphi(\tilde{\Psi}_1(x)) = \tilde{\Psi}_0(\varphi(x))$. From this and the definition of $\tilde{\Psi}$ it follows that φ is such that

$$\varphi\Psi\varphi^{-1}(x) = x + \delta x^k + \alpha x^{2k-1}.$$

To prove theorem 2 for the non-orientation preserving case it is convenient to use theorem 4, which can be proved using the orientation preserving case of theorem 2. This is the reason that we are now first going to prove theorem 4.

Proof of theorem 4. — We have again $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ of the form $\Psi(x) = x + x^k . F(x)$ with $F(0) \neq 0$ and $k \geqslant 2$. According to theorem 2 (orientation preserving case) just proved, there is an orientation preserving diffeomorphism φ_1 such that $\varphi_1\Psi\varphi_1^{-1} = x + \delta x^k + \alpha x^{2k-1}$ for some $\delta = \pm 1$ and $\alpha \in \mathbf{R}$. Let $\tilde{X}$ be the vectorfield

$$\tilde{X} = \left(\delta x^k + \left(\alpha - \frac{1}{2}k\right)x^{2k-1}\right)\frac{\partial}{\partial x}$$

where δ and α are the same as in the formula for $\varphi_1\Psi\varphi_1^{-1}(x)$. It follows from lemma (2,7) that the $(2k-1)$-jets of $\varphi_1\Psi\varphi_1^{-1}$ and $\mathcal{D}_{\tilde{X},1}$ are the same. Hence we can apply theorem 2 again and obtain φ_2 such that $\varphi_2\mathcal{D}_{\tilde{X},1}\varphi_2^{-1} = \varphi_1\Psi\varphi_1^{-1}$ or $\varphi_1^{-1}\varphi_2\mathcal{D}_{\tilde{X},1}(\varphi_1^{-1}\varphi_2)^{-1} = \Psi$ or $\mathcal{D}_{(\varphi_1^{-1}\varphi_2)_*\tilde{X},1} = \Psi$. Hence $X = (\varphi_1^{-1}\varphi_2)_*\tilde{X}$ is the required vectorfield.

We shall use the following corollary of theorem 4:

Proposition (5,1). — *Let* $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *be a diffeomorphism of the form* $\Psi(x) = x + x^k.F(x)$ *with* $k \geq 2$ *and* $F(0) \neq 0$. *Then the only germ of a diffeomorphism* $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ *which has in* $0 \in \mathbf{R}^1$ *the same* ∞*-jet as the identity and which commutes with* Ψ, *is the germ of the identity mapping.*

Proof. — Let X be a vectorfield on $\mathbf{R}^1$ such that $\mathcal{D}_{X,1} = \Psi$ (at least in a neighbourhood of the origin). Let φ be a (germ of) a diffeomorphism $\varphi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ which has the same ∞-jet as the identity. Then there is a (germ of) a continuous function $f_\varphi : \mathbf{R}^1 \to \mathbf{R}$ such that, near the origin, $\varphi(x) = \mathcal{D}_X(x, f_\varphi(x))$. Clearly $\varphi(0) = 0$. φ commutes with Ψ if and only if f_φ is invariant under Ψ, i.e., $f_\varphi(\Psi(x)) = f_\varphi(x)$. As, for each $x \in \mathbf{R}^1$ close enough to the origin, $\lim_{i\to-\infty} \Psi^i(x)$ or $\lim_{i\to+\infty} \Psi^i(x)$ is the origin, f_φ is invariant under Ψ if and only if $f_\varphi = 0$. Hence if φ has, in the origin, the same ∞-jet as the identity mapping and if $\varphi\Psi = \Psi\varphi$ holds in a neighbourhood of the origin, then φ = identity (in a neighbourhood of the origin).

Remark (5,2). — We proved (proof of theorem 2) that if $\Psi : (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ has the form

$$\Psi(x) = x + \delta x^k + \alpha x^{2k-1} + g(x)$$

with $\delta = \pm 1$, $k \geq 2$, $\alpha \in \mathbf{R}$ and g a flat function, then there is a C^∞-orientation preserving diffeomorphism $\varphi: (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ such that $\varphi\Psi\varphi^{-1}(x) = x + \delta x^k + \alpha x^{2k-1}$ and such that the ∞-jet of φ in 0 is the ∞-jet of the identity. Using proposition (5,1) it follows that the germ of φ is uniquely determined by the above two properties: namely if there were another such φ, say $\bar{\varphi}$, $\bar{\varphi}^{-1}\varphi$ would commute with Ψ and hence it's germ would be the germ of the identity.

Proof of theorem 2 *(orientation reversing case).* — We have now $\Psi(x) = -x + \delta x^k + \alpha x^{2k-1} + g(x)$ with $k \geqslant 2$, k odd, $\delta = \pm 1$, $\alpha \in \mathbf{R}$ and g a flat function. We define Ψ_0 by $\Psi_0(x) = -x + \delta x^k + \alpha x^{2k-1}$. We want to find a diffeomorphism φ such that $\varphi\Psi\varphi^{-1} = \Psi_0$. From the orientation preserving case and remark (5,2) we know that there is a unique germ of a diffeomorphism $\varphi: (\mathbf{R}^1, 0) \to (\mathbf{R}^1, 0)$ such that $\varphi\Psi^2\varphi^{-1} = \Psi_0^2$ and such that φ has in 0 the same ∞-jet as the identity. $\bar{\varphi} = \Psi_0^{-1}\varphi\Psi$ has also these two properties. Hence, in some neighbourhood of $0 \in \mathbf{R}^1$, $\bar{\varphi} = \varphi$, and hence $\varphi\Psi\varphi^{-1} = \Psi_0$.

The next lemma will be applied in the proof of theorem 3.

LEMMA (5,3). — *Let* $f: \mathbf{R}^2 \to \mathbf{R}$ *be some function. If* $f(r.\cos\varphi, r.\sin\varphi)$, *as function of* r *and* φ, *is* C^∞, *and flat in all points of* $\{r = 0\}$, *then* f, *as function on* $\mathbf{R}^2$, *is* C^∞, *and flat in the origin.*

Proof. — The fact that f, as a function of r and φ, is C^∞, and flat in all points of $\{r = 0\}$, is equivalent with: $f|\mathbf{R}^2\backslash 0$ is C^∞ and $(D_r)^i(D_\varphi)^j f$ is continuous and flat for each i,j; D_r is defined by $D_r f = x_1 \dfrac{\partial f}{\partial x_1} + x_2 \dfrac{\partial f}{\partial x_2}$ and D_φ is defined by $D_\varphi f = x_1 \dfrac{\partial f}{\partial x_2} - x_2 \dfrac{\partial f}{\partial x_1}$; a continuous function g is flat in the origin if there is for each integer $l \geqslant 0$ an $\varepsilon_1 > 0$ such that $|g(x)| \leqslant |x|^l$ if $|x| < \varepsilon_1$. Using the fact that $(x_1^2 + x_2^2)\dfrac{\partial}{\partial x_1} = x_1 D_r - x_2 D_\varphi$ and

$$(x_1^2 + x_2^2)\frac{\partial}{\partial x_2} = x_2 D_r + x_1 D_\varphi$$

one easily obtains that on $\mathbf{R}^2\setminus 0$

$$\frac{\partial^{k+l}}{\partial x_1^k \partial x_2^l} = \sum_{i,j\geq 0}^{i+j\leq k+l} \frac{Q_{k,l,i,j}(x_1, x_2)}{(x^2+x_2^2)^n_{k,l,i,j}} \cdot D_r^i D_\varphi^j, \text{ where } Q_{k,l,i,j}(x_1, x_2)$$

are polynomials in x_1, x_2. This implies that also $\dfrac{\partial^{k+l}}{\partial x_1^k \partial x_2^l} f$ is continuous and flat for all $k, l \geqslant 0$; this implies that f is C^∞, and flat in the origin.

The proof of theorem 3. — From proposition (2,8) it follows that we only have to prove that if the vectorfield X on $\mathbf{R}^2$ is of the form

$$\begin{aligned} X = f(x_1, x_2) \Big[& 2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} \\ & + \{\delta(x_1^2+x_2^2)^k + \alpha(x_1^2+x_2^2)^{2k}\} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)\Big] \\ & + \tilde{X}_1 \frac{\partial}{\partial x_1} + \tilde{X}_2 \frac{\partial}{\partial x_2}, \end{aligned}$$

with f, $\tilde{X}_1$, $\tilde{X}_2$ C^∞-functions, $\tilde{X}_1$, $\tilde{X}_2$ flat in $0 \in \mathbf{R}^2$, $f(0) = 1$, $\delta = \pm 1$, $k \geqslant 1$ and $\alpha \in \mathbf{R}$, then there is a C^∞-diffeomorphism $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that

$$\begin{aligned} \varphi_*(X) = \bar{f}(x_1, x_2) \Big[& 2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} \\ & + \{\delta(x_1^2+x_2^2)^k + \alpha(x_1^2+x_2^2)^{2k}\} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)\Big] \end{aligned}$$

for some smooth function $\bar{f}$ with $\bar{f}(0) = 1$. We define g: $\mathbf{R}^2 \to \mathbf{R}$ by $g(x_1, x_2) = x_1\tilde{X}_2 - x_2\tilde{X}_1$ and h: $\mathbf{R}^2 \to \mathbf{R}$ by $h(x_1, x_2) = x_1\tilde{X}_1 + x_2\tilde{X}_2$. Because g and h are C^∞ and flat in the origin, also $x \longmapsto \|x\|^{-2} g(x)$ and $x \longmapsto \|x\|^{-2} h(x)$ are C^∞ and flat in the origin; $\|x\|^2 = x_1^2 + x_2^2$. It is clear that

$$\begin{aligned} \tilde{X}_1 \frac{\partial}{\partial x_1} + \tilde{X}_2 \frac{\partial}{\partial x_2} = \|x\|^{-2} \Big[& h(x_1, x_2) \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) \\ & + g(x_1, x_2) \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)\Big]. \end{aligned}$$

Hence we can write X in the form

$$X = \tilde{f}(x_1, x_2)\Big[2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} + \{\delta(x_1^2 + x_2^2)^k + \alpha(x_1^2 + x_2^2)^{2k} + \tilde{g}(x_1, x_2)\}\Big(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\Big)\Big],$$

where g is C^∞, and flat in the origin. Now we define $\tilde{X}$ to be the vectorfield $\tilde{f}^{-1}.X$, or

$$\tilde{X} = 2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} + \{\delta(x_1^2 + x_2^2)^k + \alpha(x_1^2 + x_2^2)^{2k} + \tilde{g}(x_1, x_2)\}\Big(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\Big)$$

and define

$$\tilde{X}_t = 2\pi x_1 \frac{\partial}{\partial x_2} - 2\pi x_2 \frac{\partial}{\partial x_1} + \{\delta(x_1^2 + x_2^2)^k + \alpha(x_1^2 + x_2^2)^{2k} + t.\tilde{g}(x_1, x_2)\}\Big(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\Big).$$

Now we want to show that there is a unique germ of a C^∞-diffeomorphism $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that:

(i) the ∞-jet of φ in the origin is the ∞-jet of the identity;
(ii) for each x: x, 0 and $\varphi(x)$ lie on a straight line;
(iii) $\varphi \mathscr{D}_{\tilde{X}_1, 1}\varphi^{-1} = \mathscr{D}_{\tilde{X}_0, 1}$.

The existence of such a φ is proved using polar coordinates r, φ ($x_1 = r.\cos\theta$, $x_2 = r.\sin\theta$). For any θ_0, we have that $\mathscr{D}_{\tilde{X}_1, 1}\{\theta = \theta_0\} = \mathscr{D}_{\tilde{X}_0, 1}\{\theta = \theta_0\} = \{\theta = \theta_0\}$. Hence, restricted to each $\{\theta = \theta_0\}$, we can apply the method of proof of theorem 2 (orientation preserving case). To show that φ, thus obtained, is smooth with respect to the polar coordinates, we only have to take θ as the second parameter μ_2 on the moment where we apply theorem (4,1). φ is obtained by integrating a vectorfield of the form

$$\tilde{Z}(r, t, \theta) = Z(r, t, \theta)\frac{\partial}{\partial r} + \frac{\partial}{\partial t}$$

on $\mathbf{R}^2 \times \mathbf{R}$ with $\check{Z}$ a C^∞-function of (r, t, θ) and flat along $\{r = 0\}$. Hence Z is also C^∞ with respect to the x_1, x_2 coordinates and flat in $\{x_1 = x_2 = 0\}$ (see lemma (5,3)). Hence φ has all the required properties.

The fact that the germ of φ is uniquely determined by (i), (ii) and (iii) follows from proposition (5,1) and remark (5,2).

Finally we want to show that $\varphi_* \tilde{X}_1 = \tilde{X}_0$. For any t we define $\varphi_t = \mathscr{D}_{\tilde{X}_0, t}^{-1} \varphi \mathscr{D}_{\tilde{X}_1, t}$, where φ is the diffeomorphism constructed above. From a simple calculation we see that $\varphi_t \mathscr{D}_{\tilde{X}_1, 1} \varphi_t^{-1} = \mathscr{D}_{\tilde{X}_0, 1}$ hence φ_t satisfies (iii) above. Both $\mathscr{D}_{\tilde{X}_1, t}$ and $\mathscr{D}_{\tilde{X}_0, t}$, for any t, map straight lines through the origin to straight lines through the origin; hence φ_t maps straight lines through the origin to straight lines through the origin. As the ∞-jets of X_1 and X_0 are the same in the origin, the ∞-jet of φ_t in the origin is the ∞-jet of the identity; hence φ_t satisfies (i) above. From (i) and the fact that φ_t maps staight lines through the origin to straight lines through the origin, it follows that φ_t satisfies (ii). Hence the germs of φ and φ_t are equal. From this it follows that $\varphi_t = \mathscr{D}_{\tilde{X}_0, t}^{-1} \varphi_t \mathscr{D}_{\tilde{X}_1, t}$ for all t and hence

$$\varphi_{t*} \tilde{X}_1 = \varphi_* \tilde{X}_1 = \tilde{X}_0.$$

BIBLIOGRAPHY

[1] J. Dieudonné, Foundations of Modern Analysis, Acad. Press, New York, 1960.

[2] S. Kobayashi and K. Nomizu, Foundations of differential geometry vol. 1, Interscience , New York, 1963.

[3] R. Narasimhan, Analysis on real and complex manifolds, North-Holland, Amsterdam, 1968.

[4] S. Sternberg, Local contractions and a theorem of Poincaré, *Amer. J. Math. vol.* LXXIX (1957), 809-824.

[5] F. Takens, Singularities of vectorfields, to appear in Publ. I. H. E. S..

[6] F. Takens, Derivations of vectorfields, to appear in Comp. Math..

Floris Takens,
Mathematisch Institut,
postbus 800,
Groningen,
The Netherlands.