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A POINCARÉ DUALITY TYPE THEOREM FOR POLYHEDRA

by Gerald Leonard GORDON (1)

1. Introduction.

Let X be a locally finite n-dimensional polyhedron. Then if we embed $X \subset M$ where M is an orientable $(n+s)$-dimensional C^∞-manifold with $s \geqslant 1$, we shall define groups $H_p(X)_\Delta$ (resp. $H^p(X)_\Delta$) called *tubular cycles* (resp. *cocycles*) where $H_p(X)_\Delta$ (resp. $H^p(X)_\Delta$) are the homology groups (resp. cohomology) of a subcomplex of X, but with a different boundary operator. The subcomplex shall depend upon the embedding. Then in section 2 we will give a geometric proof of

THEOREM 2.3. — *There are natural isomorphisms*

$$H_p(X) \simeq H^{n-p}(X)_\Delta \qquad \textit{and} \qquad H^p(X) \simeq H_{n-p}(X)_\Delta$$

which induce a natural intersection pairing

$$H_p(X) \otimes H_q(X)_\Delta \to H_{p+q-n}(X)_\Delta.$$

If X is a topological manifold, then the isomorphism is Poincaré duality if X is orientable, and twisted coefficient duality if X is unorientable.

In section 4 we show, using spectral sequences, that $H_p(X)_\Delta$ and $H^p(X)_\Delta$ are intrinsically defined on X; i.e., they depend only on X and are independent of the embedding or the ambient space M. In fact, the spectral sequences will be isomorphic to the ones considered by Zeeman [4], and we

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shall give a geometric interpretation of these spectral sequences which will allow us to prove a conjecture of Zeeman about them.

Finally, throughout this paper $H_*(Y)$ (resp. $H^*(Y)$) will denote $H_*(Y; G)$ (resp. $H^*(Y; G)$) for any arbitrary coefficient group G and any space Y. We can take the supports to be either closed or compact. We also note that if G is any sheaf, then the results will still be true after appropriate modification in the various duality theorems.

2. A duality theorem.

Let X be an n-dim locally finite polyhedron and $X \subset M$, where M is an orientable $(n+s)$-dimensional C^∞ manifold with $s \geqslant 1$ such that X is a subcomplex of M under some triangulation.

By the regular neighbourhood theorem we can construct a regular neighbourhood of X in M, denoted by $T(X)$; i.e., $T(X)$ is a C^∞ submanifold of M whose boundary $\tau(X)$ is an $(n+s-1)$-dim orientable submanifold of M. Furthermore, we can decompose $X = \cup M_{i,j}$ where $M_{i,j}$ is an orientable i-dim submanifold of M such that the $M_{i,j}$ are locally finite, the boundary of each $M_{i,j}$ is a subcollection of the $M_{i',j}$ for $i' \leqslant i$ and if $T(M_{i,j})$ is a tubular neighbourhood of $M_{i,j}$ in M, then $T(M_{i,j})$ intersects each of the $M'_{i,j}$ transversely. E.G., $M_{i,j}$ could be chosen to be the open i-simplices of X. $\{M_{i,j}\}$ is called a *stratification* of X.

Let $\pi : TX \to X$ be the retraction and

$$\pi^{-1} \cap \tau(X) = \tau : X \to \tau(X).$$

If we set $\tilde{M}_{i,j} = M_{i,j} - \bigcup_{i'<i} T(M_{i',j})$, then

$$\pi^{-1}(\tilde{M}_{i,j}) = T(\tilde{M}_i) = T(M_i)|\tilde{M}_i.$$

Then on the chain level, $\tau : C_p(\tilde{M}_{i,j}) \to C_{p+n+s-i-1}(\tau(X))$ where $C_p(\tilde{M}_{i,j})$ are the p-chains of $\tilde{M}_{i,j}$.

If $C_*(X) = \Sigma C_p(X)$, where $C_p(X)$ are the p-chains with closed or compact support, then

DEFINITION 2.1. — $C_p(X)_\Delta = \{c \in C_*(X) \mid \tau(c) \in C_{p+s-1}(\tau(X))\}$ *Define* $\partial_\Delta : C_p(X)_\Delta \to C_{p-1}(X)_\Delta$ *by the following diagram:*

$$\begin{array}{ccc} C_p(X)_\Delta & \xrightarrow{\tau} & C_{p+s-1}(\tau(X)) \\ \downarrow \partial_\Delta & & \downarrow \partial \\ C_{p-1}(X)_\Delta & \xleftarrow{\pi_\#} & C_{p+s-2}(\tau(X)) \end{array}$$

where $\pi_\#$ *is the induced map on chains.*

Example. — $M = R^3 = (x, y, z)$ and

$$X = \{z = 0\} \cup \{x = 0, y = 0, z \geqslant 0\},$$

then if $c = \{(0, 0, 0)\}$, $c' = \{x^2 + y^2 = 1, z = 0\}$ and $c'' = \{x^2 + y^2 \leqslant 1, z = 0\}$, then $\partial_\Delta c'' = c' - c$ with c, $c' \in C_1(X)_\Delta$, since in this case τ can be constructed so that

$$\tau(c) = \{x^2 + y^2 = 1/2, z = \pm 1\},$$
$$\tau(c') = \{x^2 + y^2 = 1, z = \pm 1\}$$
and $$\tau(c'') = \{1/2 \leqslant x^2 + y^2 \leqslant 1, z = \pm 1\}.$$

Then $(C_p(X)_\Delta, \partial_\Delta)$ is a chain complex and let $H_p(X; G)_\Delta = H_p(X)_\Delta$ (resp. $H^p(X; G)_\Delta = H^p(X)_\Delta$) be the homology (resp. cohomology) of the complex with coefficients in G.

Note. — In section 4 we shall show that $H_*(X)_\Delta$ and $H^*(X)_\Delta$ are independent of the embedding and the ambient space M, but only depend on the abstract complex X.

DEFINITION 2.2. — $H_p(X)_\Delta(H^p(X)_\Delta)$ *are called the tubular p-cycles (cocycles).*

THEOREM 2.3. — *There exist natural isomorphisms* $H_p(X) \simeq H^{n-p}(X)_\Delta$, $H^p(X) \simeq H_{n-p}(X)_\Delta$ *which induce a natural intersection pairing* $H_p(X) \otimes H_q(X)_\Delta \to H_{p+q-n}(X)_\Delta$.

Proof. — Consider the following diagram

$$\begin{array}{ccccccccc} H^p(M, X) & \longrightarrow & H^p(M) & \longrightarrow & H^p(X) & \longrightarrow & H^{p+1}(M, X) & \longrightarrow & H^{p+1}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+s-p}(M-X) & \xrightarrow{i_*} & H_{n+s-p}(M) & \xrightarrow{I} & H_{n-p}(X)_\Delta & \xrightarrow{\tau} & H_{n+s-p-1}(M-X) & \xrightarrow{i_*} & H_{n+s-p-1}(M) \end{array}$$

Diagram I.

where the top row is the exact sequence in cohomology for the pair (M, X), the vertical isomorphisms are Poincaré-Lefschetz duality for homology with closed or compact support (see Swan [3, p. 136]). Also, τ is the map induced from the τ defined above, i_* is induced from the inclusion, and I is the map induced from geometric intersection on the chain level; see, e.g., Lefschetz [2]. Equivalently, I is cap product with the dual class of $[\tau(X)] \in H_{n+s-1}(X)$, and then project down by π_*.

In [1, Corollary 4.17] the author shows that the bottom row is also exact. The result there is only stated for X a complex subvariety of M, a complex manifold, but all that is needed is that the stratum $M_{i,j}$ obey the transversality condition, as noted after the proof of Theorem 4.12. Essentially, it is the Thom-Gysin sequence for $\tau(X)$ in M.

Then Diagram I induces a natural map $H^p(X) \to H_{n-p}(X)_\Delta$ which is an isomorphism by the 5-Lemma.

By looking at the corresponding homology sequence of the pair (M, X) or by the above diagram and the universal coefficient theorem, we also get $H_p(X) \simeq H^{n-p}(X)_\Delta$.

To see the intersection pairing, if we think of the Poincaré-Lefschetz duality theorems as between homology groups as done in Lefschetz [2], then the above proofs show that given a p-cycle γ in X we can form its dual cycle in $H_{n-p}(X)_\Delta$ (or in $H_{n-p-1}(X)_\Delta$ if γ is a torsion element). By definition of $H_*(X)_\Delta$, this defines a $n-p+s-1$ (or $n-p+s-2$) cycle in $\tau(X)$ which by Poincaré duality yields a p-cycle in $\tau(X)$. We can assume it is a p-cycle by choosing $s > n$; i.e., the $(n-p+s-2)$-cycle will be a torsion element, hence its dual is a p-cycle, while we can assume the $(n-p+s-1)$-cycle is not a torsion cycle since the torsion would measure the obstruction to getting a « section » over γ in $\tau(X)$. So by choosing M sufficiently large, which is always possible, we have an injection $H_p(X) \to H_p(\tau(X))$. Thus to get a pairing between $H_p(X)$ and $H_q(X)_\Delta$, we lift them both to $\tau(X)$, do the intersecting there, and then project back to X via π_*.

Q.E.D.

Example. — $X \subset R^3$ given by

$$\{x^2 + y^2 + z^2 = 1\} \cup \{x^2 + (y-1)^2 = 1,\ y \geq 1/2,\ z = 0\},$$

i.e., unit sphere with a $_1$-dim handle attached at the points $P, Q = \left\{\left(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)\right\}$. Then $H_1(X; Z) \simeq Z$ and if α is a small circle about P in S^2, then $[\alpha] \in H_1(X; Z)_\Delta$ will be the dual cycle to the generator of $H_1(X; Z)$. But as an element of $H_1(X; Z)_\Delta$, $\alpha \sim P$; i.e., $[P] = [\alpha]$. What this intersection pairing states is that given a non-torsion p-cycle γ in X, there is a representative of its dual Γ in $H_{n-p}(X)_\Delta$ which has dim $(n-p)$ and $\gamma . \Gamma = +1$ in the sense of the Kronecker pairing.

We remark that if X is oriented topological manifold, then the isomorphism is just the usual Poincaré duality and if X is an unoriented topological manifold, then the isomorphism is the usual one between $H_p(X; G)$ and $H^{n-p}(X; \underset{\sim}{G})$ where $\underset{\sim}{G}$ means twisted coefficients (see, e.g., Swan [3, p. 136]).

Also if $\gamma \in H_p(X)_\Delta$, then γ transversely meets S, the singular locus of X, i.e., S is the subcomplex where X fails to be a topological manifold. That is, $\tau(\gamma)$ is a $(p+s-1)$-dim cycle which bounds in M, say its boundary is c and c meets X transversely; i.e., $X = \cup M_{i,j}$ and then c intersects each $M_{i,j}$ transversely in a $p+s+i-n$-dim chain and $\gamma = c \cap X$.

3. A special case.

In this section we shall consider a special case when $X = \bigcup_{i \in I} X_i$ and the X_i are submanifolds of M and they intersect transversely; i.e., if X_i is of codim α_i in M, then $\bigcap_{i \in \alpha} X_i$ is a submanifold of M of codim $\sum_{i \in \alpha} \alpha_i$ for all $\alpha \subset I$. This seems like a very restrictive case, but in algebraic geometry if X is an arbitrary analytic subvariety of M, where M is a closed complex submanifold of either C^N or CP_N, then one can resolve the singularities of X; i.e., there exists a closed complex submanifold M' of C^k or CP_k (as M is) and a proper holomorphic map $\pi : M' \to M$ such that if $X' = \pi^{-1}(X)$, then X' is the transverse intersection of complex submanifolds of M' and $\pi | M' - X \to M - X$ is a bianalytic homeomorphism. In [1] it was conjectured that ker $\{H_p(M - X) \to H_p(M)\}$ is invariant under this resolution, and proved in case X was ample hypersurface, so that in this case to study $H_p(X)_\Delta$, it would suffice to consider X as the transverse intersection of submanifolds, except for the one cycle coming from $H_p(CP)_N$, if p is even.

If $X = \bigcup_{i \in I} X_i$, let $M_{i,\alpha} = X_\alpha - \bigcap_{j \notin \alpha} X_j \cap X_\alpha$ where $\alpha \subset I$, $|\alpha| = i$ and if $\alpha = \{j_1, \ldots, j_i\}$, then

$$X_\alpha = X_{j_1} \cap \ldots \cap X_{j_i}.$$

$M_{i,\alpha}$ are the *i-tuple points* of X and $\bigcup_{i,\alpha} M_{i,\alpha}$ gives the stratification of X because of the transverse intersection. Let $M_i = \bigcup_{|\alpha|=i} M_{i,\alpha}$ and $\overline{M}_i = \bigcup_{|\alpha|=i} \overline{M}_{i,\alpha} = \bigcup_\alpha X_\alpha$. Also let $M_{|i(j)|, i(j)}$, $j \in I(i)$ be the components of $\overline{M}_i$, where dim $M_{|i(j)|, i(j)} = |i(j)|$, so that $|i(j)| \leqslant n - i$.

Given α, we set $\alpha' = \{\alpha\} \cup \{j\}$ and can form $\bigcup_{\alpha'} X_{\alpha'}$ which is a transverse intersection of manifolds in $X_{\alpha'}$ so we have $\tau_{|\alpha|} : H_*(\cup X_{\alpha'})_\Delta \to H_{*+q-1}(X_\alpha)$ where q is the codimension of $\cup X_{\alpha'}$ in X_α. Since $\tau_{|\alpha|}(\cup X_{\alpha'}) \subset M_{|\alpha|, \alpha'}$ we have iterated maps $\tau_{|\alpha|-k} : M_{|\alpha|, \alpha} \to X_{\alpha''}$ where $\alpha'' \subset \alpha$, $|\alpha''| = |\alpha| - k$ which induce maps on homology, also called $\tau_{|\alpha|-k}$, which increase the dimension by one less than the codimension of X_α in $X_{\alpha''}$.

In [1, Corollary 2.8] we have shown that

$$H_p(X)_\Delta = \sum_{q=1}^{p} \bigoplus_{j \in I(q)} \tau_1 \ldots \tau_{q-1} H_{p-n-|q(j)|+1}(\overline{M}_{q(j), |q(j)|})_\Delta.$$

Note that $\overline{M}_{|1(j)|, 1(j)}$ are the components of X. (It is actually proved when all the X_i are of codimension 2, but the exact same proof goes through with the obvious modifications.) What this says is that if a p-cycle γ of X is such that $\tau(\gamma) \subset M - X$ but $\gamma \sim 0$ in X, then $\gamma = \tau(\gamma')$ for some $\gamma' \subset \overline{M}_2$ and if $\gamma' \sim 0$ in $\overline{M}_2$, then $\gamma' = \tau(\gamma'')$ for some $\gamma'' \subset \overline{M}_3$, etc.

Let $H_{p-q+1} = \bigoplus_{j \in I(q)} \tau_1 \ldots \tau_{q-1} H_{p-n-|q(j)|+1}(\overline{M}_{|q(j)|, q(j)})_\Delta$. Then we can say $H_p(X)_\Delta = \sum_{q=0}^{p-1} H_{p-q}$.

Given $X = \bigcup_{i \in I} X_i$, we can compute the cohomology of X via the Maier-Victoris sequence; e.g., if $|I| = 3$, we have

$$\begin{array}{ccccc}
H^{p-1}(X_1 \cap X_2) \oplus H^{p-1}(X_1 \cap X_3) & & & & H^p(X_2) \oplus H^p(X_1) \\
\uparrow{\scriptstyle i_2} & & & & \uparrow{\scriptstyle i_1} \\
H^{p-1}((X_1 \cap X_2) \cup (X_1 \cap X_3)) & \xrightarrow{\delta_1} & H^p(X_1 \cup X_2 \cup X_3) & \xrightarrow{i_1} & H^p(X_1) \oplus H^p(X_2 \cup X_3) \\
\uparrow{\scriptstyle \delta^2} & & & & \uparrow{\scriptstyle \delta^1} \\
H^{p-2}(X_1 \cap X_2 \cap X_3) & & & & H^{p-1}(X_2 \cap X_3)
\end{array}$$

Diagram II.

Let $\tilde{H}^{p-q} = \{\Gamma \in H^p(X) | i_q(\Gamma) = 0\}$; i.e.,

$$i_q : H^{p-q-1}\left(X_\alpha \cup \bigcup_{\alpha'} X_{\alpha'}\right) \to H^{p-q-1}(X_\alpha) \oplus H^{p-q-1}\left(\bigcup_{\alpha'} X_{\alpha'}\right)$$

with $|\alpha| = |\alpha'| = q$, e.g., in Diagram II if $i_1(\Gamma) = 0$, then $\Gamma = \delta'(\Gamma')$ and $i_2\Gamma' \neq 0$ implies $\Gamma \in \tilde{H}^{p-1}$. Also if $i_1(\Gamma) \neq 0$ then $i_1(i_1\Gamma) = 0$ will imply $\Gamma \in \tilde{H}^{p-1}$. Let

$$H^{p-q} = \tilde{H}^{p-q} \Big/ \sum_{j<q} \tilde{H}^{p-j}$$

where $\sum_{j<q} \tilde{H}^{p-j}$ means the subgroup generated by the $\tilde{H}^{p-j} \subset \tilde{H}^{p-q}$.

Proposition 3.1. — *In the isomorphism of Theorem 2.3,* $H_{p-q} \simeq H^{p-q}$, $q = 0, \ldots, p-1$.

Proof. — The proof is a diagram chase and a multiple use of the Poincaré duality in the various X_α. E.g., in Diagram II if $\Gamma \in H^{p-1}$ with $\Gamma = \delta^1(\Gamma')$ and $\Gamma' \notin \operatorname{im} \delta^2$, then we have $\Gamma' = (\Gamma^{12}, \Gamma^{13})$ for $0 \neq \Gamma^{ij} \in H^{p-1}(X_i \cap X_j)$. (We have each $\Gamma^{ij} \neq 0$ because $\Gamma \in H^{p-1}$.) Then applying Poincaré duality in each of the $X_i \cap X_j$'s, we get $(\gamma_{12}, \gamma_{13})$ and we claim $(\gamma_{12}, \gamma_{13}) \in H_{p-1}$. In fact, if $X_1 \cap X_2 \cap X_3 \neq \emptyset$, then one can show that $\gamma_{12} \cap V_3 = -\gamma_{13} \cap V_2$, so that we can plumb together $\gamma_{12} + \gamma_{13}$ to form $0 \neq \tau_1(\gamma_{12} + \gamma_{13}) \in H_p(\overline{M}_1)_\Delta$. $\tau_1(\gamma_{12} + \gamma_{13}) \neq 0$ if $\Gamma \neq 0$ from a diagram chase combining Diagrams I and II, which will also show that the Γ goes to $\tau_1(\gamma_{12} + \gamma_{13})$ under the map induced from Diagram I.

The general case is proved in the same way.

Q.E.D.

What this proposition says is that if under the isomorphism $H^p(X) \simeq H_{n-p}(X)_\Delta$ we have $\Gamma \in H^p(V)$ going to an element of H_{n-p-q}, $q \geqslant 0$, then $\dim(\operatorname{supp} \Gamma) = n - p - q$. By $\operatorname{supp} \Gamma$, we mean if Γ is a cocycle, where we consider singular cohomology, then $x \notin \operatorname{supp} \Gamma$ if x has a neighbourhood $U_x \subset X$ with $\Gamma(\sigma) = 0$ for all singular p-chains σ with $\operatorname{supp} \sigma \subset U_x$. Then for a cohomology class Γ, $\dim \operatorname{supp} \Gamma = \min_{\alpha \in \Gamma} (\dim \operatorname{supp} \alpha)$. Equivalently, in the simplicial cohomology, since $\Gamma : C_p(X) \to G$, we say $\dim \operatorname{supp} \Gamma \leqslant k$ if we have that the p-chains on which Γ does not vanish lie in an arbitrarily small neighbourhood of the fixed k-skeleton of X in the original complex; i.e., we can take a sufficiently fine subdivision of X and find a representative of Γ in this subdivision which vanishes on p-chains away from the fixed k-skeleton of the complex. We always have $k \leqslant n - p$ [4, page 177] and for topological manifolds, $k = n - p$ [4, page 178].

So Proposition 3.1 is a special case of a conjecture of Zeeman, which we shall prove in the next section.

4. Zeeman's dihomology.

Recall, Zeeman [4] has defined spectral sequences for any polyhedron (in fact, any topological space) with

$$E_2^{p,q} = H^p(X, \mathfrak{L}_q) \Longrightarrow H_{q-p}(X)$$
$$\hat{E}_2^{p,q} = H_p(X, \hat{\mathfrak{L}}^q) \Longrightarrow H^{q-p}(X)$$

where $\mathfrak{L}_q$ (resp. $\hat{\mathfrak{L}}_q$) is the sheaf associated to the presheaf which sends $U \to H_q(X, X - U)$ (resp. $U \to H^q(X, X - U)$. Further, if X is a local homology manifold, then these spectral sequences collapse at $E_2^{p,q}$ yielding Poincaré duality if X is oriented or twisted integer coefficient duality if X is unoriented.

Because X is a polyhedron, we have by excision that $H_q(X, X - U) \simeq H_q(\bar{U}, \partial U)$ (resp. $H^q(X, X - U) \simeq H^q(\bar{U}, \partial U)$).

Suppose we embed X into M as a subcomplex for M a C^∞-manifold. Then we have the map $M - X \subset M$ and we can look at the induced Leray spectral sequence

$$E_2^{p,q} \simeq H^p(M, R^q) \Longrightarrow H^{p+q}(M - X)$$

where R^q is the sheaf associated to the presheaf $U \to H^q(U - U \cap X)$ (see Swan [3, chapter 10], especially Lemma 5).

Let us consider the sheaf $R^q|X$; i.e., let U be open in X and $\pi : T(X) \to X$ be the projection of a regular tubular neighbourhood. Let $\pi^{-1}(U) = U'$. Then $R^q|X$ is the sheaf associated to the presheaf $U \to H^q(U' - U)$. But $U' - U$ retracts onto $\partial U' - \partial U$, because $\tau(X)$ is regular neighbourhood, so we can use transverse intersection of U' with X to show this. Then by Lefschetz duality, we have

$$H^q(\partial U' - \partial U) \simeq H_{n+s-q-1}(\partial U', \partial U)$$

where $\dim M = n + s$. But $\partial U'$ has the same homotopy type as $\overline{U}$; i.e., we can retract $\partial U'$ onto $\overline{U}$ by contracting along the fibres of $T(X)$ keeping ∂U fixed. Thus we have that $R^q|X$ is the sheaf associated to $U \to H_{n+s-q-1}(\overline{U}, \partial U)$; i.e., $R^q|X \simeq \mathcal{L}_{n+s-q-1}$.

Similarly, if R^q is the sheaf associated to the presheaf $U \to H_q(U - U \cap X)$, then $R_q|X \simeq \mathcal{L}^{n+s-q-1}$ and

$$\hat{E}_2^{p,q} = H_p(M, R_q) \Longrightarrow H_{p+q}(M - X)$$

and $\hat{E}$ is adjoint to E.

Notice that $R_0 \simeq G_M$ where G_M is the constant sheaf; i.e., $G_M \simeq M \times G$ where G has the discrete topology. Also, for $P \in M - X$, $(R^q)_p = \{\text{stalk of } R^q \text{ at } P\} = 0$ for $q \neq 0$. Hence for $q \geqslant 1$, we have that $R^q|X \simeq R^q$ by extending $R^q|X$ to be zero off X.

In [1, section 4] it was shown that $\hat{E}_2^{p,0} \Longrightarrow$ coimage $\{H_p(M - X) \to H_p(M)\}$ and for

$$q \geqslant 1, \ \hat{E}_2^{p,q} \Longrightarrow \ker \{H_{p+q}(M - X) \to H_{p+q}(M)\}.$$

In fact, if $\hat{E}_r^{p,q} \simeq \hat{E}_\infty^{p,q}$ but $\hat{E}_{r-1}^{p,q}\ \hat{E} \not\simeq_r^{p,q}$ for $q \geqslant 1$; i.e., spectral sequence collapses at $\hat{E}_r$, then $\hat{E}_{r-1}^{p,q} \subseteq H_{p+q-s-1}(X)_\Delta$ where $s =$ codimension of X in M and $d_{r-1}(\gamma) = 0$ for $\gamma \in \hat{E}_{r-1}^{p,q}$ determines whether $\tau(\gamma) \not\sim 0$ in $M - X$; i.e., $d_{r-1}(\gamma) \neq 0$ if and only if $\tau(\gamma) \sim 0$ in $M - X$ if and only if $\gamma = \Gamma \cap X$ for $\Gamma \in H_{p+q+1}(M)$. Thus the $\hat{d}_2$ are given by transverse intersections, i.e., Gysin maps with appropriate identifications.

Similarly $E_2^{p,0} \Longrightarrow$ image $\{H^p(M) \to H^p(M - X)\}$ and for $q \geqslant 1$, $E_2^{p,q} \Longrightarrow$ coker $\{H^{p+q}(M) \to H^{p+q}(M - X)\}$, whit $\hat{E}_{r-1}^{p,q} \subseteq H^{p+q-s+1}(X)_\Delta$.

That is, we have

$$\begin{array}{ccc} E_2^{p,q} = H^p(X, R^q|X) & \Longrightarrow & H^{p+q-s+p1}(X)_\Delta \\ \wr\!\downarrow & & \\ H^p(X, \mathfrak{L}_{n+s-q-1}) & \Longrightarrow & H_{n+s-q-1-p}(X) \\ \hat{E}_2^{p,q} = H_p(X, R_q|X) & \Longrightarrow & H_{p+q-s+1}(X)_\Delta \\ \wr\!\downarrow & & \\ H_p(X, \mathfrak{L}^{n+s-q-1}) & \Longrightarrow & H_{n+s-q-p-1}(X) \end{array}$$

where $n = \dim X$, so that the isomorphism in Theorem 2.3 is just given by looking at Zeeman's Dihomology. So we have

PROPOSITION 4.1. — $H_*(X)_\Delta$ *and* $H^*(X)_\Delta$ *are intrinsic properties of* X.

As stated above, the $\hat{d}_2$ are intersection maps. To see this, let us work out an example of the cone over an elliptic curve (as Zeeman does cone over quadratic curve); i.e., $G = Z$, $X = \{x^3 + y^3 + z^3 = 0\}$ in CP_3 with homogeneous coordinates $[x, y, z, w]$. X is the Thom space of normal bundle of an elliptic curve C in $CP_2 = [x, y, z]$ where C is given by $\{x^3 + y^3 + z^3 = 0\}$, a torus. Here $n = 4$, $s = 2$. Let P_∞ be the singular point of X. Then $R_1|X \simeq Z_X$, the constant sheaf, and supp $R_i|X = \{P_\infty\}$ for $i = 2,3$ whit $(R_2)_{P_\infty} = Z \oplus Z \oplus Z_3$ and $(R_3)_{P_\infty} = Z \oplus Z$ because for a sufficiently small neighbourhood U of P_∞, $\partial U \approx$ circle bundle of degree 3 over C.

So $\hat{d}_2^{2,1} : \hat{E}_2^{2,1} = H_2(X; Z) \to H_0(P_\infty; Z \oplus Z \oplus Z_3) = \hat{E}_2^{0,2}$ is the intersection map of $H_2(X; Z) \to H_1(\tau(P_\infty, X); Z)$ where $\tau(P_\infty, X) = \tau(P_\infty) \cap X$ is the 3-dim manifold gotten by intersecting X with $\tau(P_\infty)$ the boundary of a regular tubular neighbourhood in M of the complex P_∞. That is, $\hat{d}_2$ is the Gysin map and if F is the generator of $H_2(X; Z)$, i.e., the fibre of normal bundle, then $\hat{d}_2(F)$ is the generator of the Z_3 term, since this is the fibre over a point; i.e., $F \cap \tau(P_\infty, X) =$ circle which is the fibre of ∂U. Thus $\ker \hat{d}_2 = 3F \sim C$ in X. This is seen if ν is the normal bundle of C in CP_2 and ν^*

the projectification; i.e., replace each fibre C^1 by CP_1 yielding C_∞, the curve at ∞, then $H_2(\nu^\star) = Z \oplus Z$ with generators F and C and $C_\infty = 3F - C$ (degree $C = 3$). Since $X = \nu^\star/C_\infty$, the result follows. Thus we have $C \in H_2(X)_\Delta$ as the abutment of $F \in H_2(X)$, i.e., C is the dual of $F^\star \in H^2(X; Z)$ and $C \cdot F = +1$ in X.

Also, $\hat{E}_2^{0,2} = H_0(P_\infty; R_2|X) = H_1(\tau(P_\infty, X); Z) \to \hat{E}_2^{-2,1} = 0$ and $\hat{E}_{2|}^{3,1} \to \hat{E}_2^{0,2}$ has image Z_3, so that $\hat{d}_3^{0,2} = \hat{d}_\infty^{0,2} = Z \oplus Z$ with the generators being sections over α, β generators of $H_1(C)$. Hence $H_1(X)_\Delta$ has generators α, β and $\alpha \rightsquigarrow (\beta \times F)^\star \in H^3(X; Z)$ (similarly for β and $\alpha . (\beta \times F) = +1$ in X).

In [1, Corollary 4.13] we showed that if $\gamma \in H_p(X)_\Delta$ and $\gamma \sim 0$ in X, then $\gamma \in H_p(\tau(S, X))$ where S is some subcomplex of X and $\tau(S, X) = \tau(S) \cap X$ where $\tau(S)$ is the boundary of a regular tubular neighbourhood of S in M. That is, γ is the tube over a lower dim cycle γ' sitting in S and if $\gamma' \sim 0$ in S, then $\gamma' \in H_*(\tau(S', X))$, etc. Hence, what the above example illustrates is that the $\hat{d}_2$ are essentially Gysin maps, i.e., transverse intersection in appropriate interpretation, and $\gamma \in$ image $\hat{d}_2$ means γ is intersection of some cycle, so that $\tau(\gamma) \sim 0$; i.e., $\gamma \notin H_*(X)_\Delta$.

Thus $\gamma \in H_p(X, R_q|X)$ should be interpreted as some p-cycle whose support lies in supp $(R_q|X)$ where $P \notin$ supp $(R_q|X)$ if there is a neighbourhood U of P in X such that $Q \in U$ implies $(R_q|X)_Q = 0$.

Notice that $H^\star(X)$ has a natural filtration on it induced from the spectral sequence; i.e., we say $\Gamma \in H^q(X)$ has *filtration degree* p if there is an element of

$$\hat{E}_2^{p,p+q} = H_p(X, \hat{\mathfrak{L}}^{p+q})$$

which maps onto Γ via the spectral sequence. E.g., if X is a topological manifold, then $\Gamma \in H^q(X)$ always has filtration degree $n - q$ as $\hat{\mathfrak{L}}^r = 0$ unless $r = n$. But $H_p(X, \hat{\mathfrak{L}}^{p+q}) \simeq H_p(X, R_{n+s-p-q-1}) \Longrightarrow H_{n-q}(X)_\Delta$. Thus Γ has filtration degree p if its dual $(n - q)$ cycle in $H_{n-q}(X)_\Delta$ has a representative γ where $0 \neq \gamma \in H_p(S)$ for some subcomplex S of X.

But this p is what Zeeman calls the *codim of an arbitrary* *cohomology class*, where cohomology is taken to be singular cohomology; i.e., Γ is of codim p if $p = \min_{\xi \in \Gamma} (\dim \operatorname{supp} \xi)$, i.e., Γ is of codim p if its dual cycle γ is the tube over some p-cycle, but by taking a sufficiently fine subdivision, the tube can lie in an arbitrarily small neighbourhood of the p-skeleton of S.

Zeeman [4, p. 178] conjectures this to be true for all polyhedra, and we have proved this for all finite dimensional locally finite polyhedra. However, from the statements in the paper, polyhedron there means finite polyhedron.

PROPOSITION (ZEEMAN) 4.2. — X *is a locally finite, finite dimensional polyhedron and* $\Gamma \in H^s(X)$, *then* codim Γ = *filtration degree* Γ.

We note here that in [1], we showed that S is always an analytic subvariety of X in case X is an analytic variety (real or complex), i.e., supp $R^q|X$ is an analytic subvariety.

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