ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 22, nº 4 (1972), p. 265-269
<http://www.numdam.org/item?id=AIF_1972__22_4_265_0>

Ann. Inst. Fourier, Grenoble
22, 4 (1972), 265-269.

ON ABSOLUTE STABILITY

by Roger C. McCANN

It is well known that absolute stability of a compact subset M of a locally compact metric space can be characterized by the presence of a fundamental system of absolutely stable neighbourhoods, and also by the existence of a continuous Liapunov function V defined on some neighbourhood of $M = V^{-1}(0)$, [1]. Here we characterize the absolute stability of M in terms of the cardinality of the set of positively invariant neighbourhoods of M.

Throughout this paper R and R^+ will denote the reals and non-negative reals respectively. A rational number r is called dyadic if and only if there are integers n and j such that $n \geqslant 0, 1 \leqslant j \leqslant 2^n$, and $r = \frac{j}{2n}$.

A dynamical system on a topological space X is a mapping π of $X \times R$ into X satisfying the following axioms (where $x\pi t = \pi(x, t)$) :

(1) $x\pi 0 = x$ for $x \in X$.

(2) $(x\pi t)\pi s = x\pi(t + s)$ for $x \in X$ and $t, s \in R$.

(3) π is continuous in the product topology.

If $M \subset X$ and $N \subset R$, then $M\pi N$ will denote the set $\{x\pi t : x \in M, t \in N\}$. A subset M of X is called positively invariant if and only if $M\pi R^+ = M$. A point $x \in X$ is called a critical point if and only if $x\pi R = \{x\}$. A subset M of X is called stable if and only if every neighbourhood of M contains a positively invariant neighbourhood of M.

A Liapunov function for a positively invariant compact subset M of X is a continuous mapping V of a neighbourhood W of M into R^+ such that $V^{-1}(0) = M$ and $V(x\pi t) \leqslant V(x)$ for $x \in W$ and $t \in R^+$.

Absolute stability is defined in terms of a prolongation and is characterized by the following theorem, [1].

THEOREM. — *Let* M *be a compact subset of a locally compact metric space. Then the following are equivalent:*

(i) *There is a Liapunov function* V *for* M.

(ii) M *possesses a fundamental system of absolutely stable neighbourhoods.*

(iii) M *is absolutely stable.*

LEMMA 1. — *Let* $A \subset R$ *be uncountable. Then there exists an* $x \in A$ *such that every neighbourhood of* x *contains uncountably many elements of* A.

Proof. — [4, 6,23, III].

The following is a consequence of Lemma 1.

LEMMA 2. — *Let* $A \subset R$ *be uncountable. Then there exists an* $x \in A$ *such that the sets* $\{y \in A : y < x\}$ *and* $\{y \in A : x < y\}$ *are uncountable.*

LEMMA 3. — *Let* S *and* T *be relatively compact sets of a locally compact connected metric space* X *and* $\mathfrak{D}$ *a family of open sets of* X *such that*

(i) *for every* $U \in \mathfrak{D}$, $\overline{S} \subset U \subset \overline{U} \subset T$,

(ii) *if* $U, V \in \mathfrak{D}$, *then either* $\overline{U} \subset V$ *or* $\overline{V} \subset U$.

Then there is a $W \in \mathfrak{D}$ *such that the sets* $\{U \in \mathfrak{D} : U \subset W\}$ *and* $\{U \in \mathfrak{D} : W \subset U\}$ *are uncountable.*

Proof. — Since X is connected, the boundary ∂U of $U \in \mathfrak{D}$ is nonempty. If $U \in \mathfrak{D}$, then ∂U is compact since T is relatively compact. Let d be a metric on X and define f: $\mathfrak{D} \to R^+$ by $f(U) = d(\overline{S}, \partial U)$. If $U, V \in \mathfrak{D}$ with $\overline{U} \subset V$, then $f(U) < f(V)$. Let A be the image of $\mathfrak{D}$ under f.

Then f is a one-to-one order preserving mapping of $\mathfrak{D}$ onto A. A is uncountable since $\mathfrak{D}$ is such. By Lemma 2 there is an $x \in A$ such that the sets $\{y \in A : x < y\}$ and

$\{y \in A : y < x\}$ are uncountable. Set $W = f^{-1}(x)$. It is easily verified that

$$\{U \in \mathfrak{D} : U \subset W\} = \{f^{-1}(y) : y < x\},$$
$$\{U \in \mathfrak{D} : W \subset U\} = \{f^{-1}(y) : x < y\},$$

and that both sets are uncountable.

THEOREM 4. — *A nontrivial compact subset* M *of a locally compact connected metric space is absolutely stable if and only if* M *possesses a fundamental system* $\mathcal{F}$ *of open positively invariant neighbourhoods such that*

(i) for each $U \in \mathcal{F}$, the set $\{V \in \mathcal{F} : V \subset U\}$ is uncountable,

(ii) if $U, V \in \mathcal{F}$, then either $\overline{U} \subset V$ or $\overline{V} \subset U$.

Proof. — Since X is connected, no nontrivial subset of X is both open and closed. If M is absolutely stable, then there is a continuous Liapunov function V for M. Set $\mathcal{F} = \{V^{-1}([0, r)] : r$ in the range of $V\}$. It is easily verified that $\mathcal{F}$ possesses the desired properties. Now assume that $\mathcal{F}$ is a fundamental system of open positively invariant neighbourhoods of M with properties (i) and (ii). For each dyadic rational we will construct a set $U(r) \in \mathcal{F}$ such that $U(r) \subset U(S)$ whenever $r < s$. We first obtain from $\mathcal{F}$ a fundamental system of neighbourhoods $\left\{U\left(\frac{1}{2^n}\right) : n\right.$ a non-negative integer$\}$ such that $U\left(\frac{1}{2^{n+1}}\right) \subset U\left(\frac{1}{2^n}\right)$ and the set $\left\{A \in \mathcal{F} : U\left(\frac{1}{2^{n+1}}\right) \subset A \subset U\left(\frac{1}{2^n}\right)\right\}$ is uncountable. This is done by induction in the following manner. Let N_i be a countable fundamental system of neighbourhoods of M. Let $U(1) \subset N_1$ be an element of $\mathcal{F}$ which is relatively compact. Suppose that $U\left(\frac{1}{2^n}\right)$ has been defined. By Lemma 3 and property (ii), there is a $B \in \left\{W \in \mathcal{F} : W \subset U\left(\frac{1}{2^n}\right)\right\}$ such that $B \subset N_{n+1}$ and both $\{W \in \mathcal{F} : V \subset B\}$ and

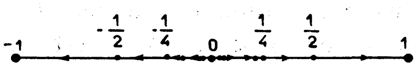
$$\left\{W \in \mathcal{F} : B \subset V \subset U\left(\frac{1}{2^n}\right)\right\}$$

are uncountable. Set $U\left(\frac{1}{2^{n+1}}\right) = B$. Now extend this system to one with the desired properties. For example, we chose $U\left(\frac{3}{4}\right)$ to be any element C of $\mathcal{F}$ such that the sets $\left\{W \in \mathcal{F} : U\left(\frac{1}{2}\right) \subset V \subset C\right\}$ and $\{W \in \mathcal{F} : C \subset V \subset U(1)\}$ are uncountable. This is possible by the properties of the sets $U\left(\frac{1}{2^n}\right)$ and Lemma 3. Now define $V : U(1) \to R^+$ by $V(x) = \inf\{r : x \in U(r)\}$. Evidently $V(x) = 0$ if and only if $x \in M$. If $x \in U(r)$ and $t \in R^+$, then $x\pi t \in U(r)$ since $U(r)$ is positively invariant. Therefore,

$$V(x) = \inf\{r : x \in U(r)\} \geqslant \inf\{r : x\pi t \in U(r)\} = V(x\pi t).$$

The continuity of V is proved as in the proof of Urysohm's lemma. Thus we have constructed a Liapunov function for M. M is absolutely stable.

Example. — Let $X = [-1, 1]$, $M = \{0\}$, and π be the dynamical system indicated by the following diagram where the points $\pm 2^{-n}$, n a non-negative integer, are critical points.



Clearly M is stable. The only open positively invariant neighbourhoods of M are X and intervals of the form $(-2^{-m}, 2^{-n})$ where m and n are non-integers. There are only countably many such neighbourhoods. Hence, M is not absolutely stable.

Proposition 5. — *Let* X *be the plane and* p *an isolated critical point. If each neighbourhood of* p *contains uncountably many periodic trajectories* (cycles), *then* p *is absolutely stable.*

Proof. — Let W be a disc neighbourhood of p which contains no critical points other than p. A cycle C is a Jordan curve and, hence, decomposes the plane into two components, one bounded (denoted by int C) and the other unbounded. If C is a cycle, then int C contains a critical point, [3, VII, 4.8]. Hence, if C is a cycle in W, then C is the boundary of a neighbourhood (necessarily invariant) of p. It can be shown (the proof is almost identical with that of Proposition 1.10 of [6]) that if C_1 and C_2 are distinct cycles in W, then either $\overline{\text{int } C_1} \subset \text{int } C_2$ or $\overline{\text{int } C_2} \subset \text{int } C_1$. Theorem 4 may now be applied to obtain the desired result.

Another characterization of absolute stability of compact sets is found in [5]. Non-compact absolutely stable sets are characterized in [3].

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Manuscrit reçu le 7 février 1972,
accepté par M. Reeb.

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