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SPECTRAL STUDY OF HOLOMORPHIC FUNCTIONS WITH BOUNDED GROWTH

par Ivan CNOP

1. Introduction.

We prove a spectral property for some algebras of holomorphic functions of several complex variables with rather general growth conditions. It was obtained in the author's Ph. D. thesis [3] and announced in [2]. The algebras involved arise in the holomorphic functional calculus (symbolic calculus) for b-algebras (complete algebras) constructed by L. Waelbroeck [14], and have also been studied in [12], [13]. L. Waelbroeck has remarked that properties of the algebra of holomorphic functions with bounded growth can be obtained if the weight function is « spectral » for the identity mapping, and he has given necessary conditions in order that this be verified [14, p. 124]. Here we prove that the necessary conditions are also sufficient. The proof uses the existence theorems with L^2 estimates for the $\bar{\partial}$ operator, obtained by L. Hörmander [11]. By applying the holomorphic functional calculus, we also obtain a characterisation of the spectrum of arbitrary elements. Applications to algebras of holomorphic functions are given, or referred to in the last paragraph. We start by recalling basic facts from the spectral theory and holomorphic functional calculus, following [14], but we only consider the case of commutative algebras.

2. Complete bounded structures and spectral theory.

A set $\mathcal{B}$ of subsets of a set E is called a bounded structure on E, when $\mathcal{B}$ contains singletons, and all subsets of finite unions of its elements, which are then called bounded sets. When E is a vector space, an absolutely convex subset B is completant if the semi-norm induced by B on the vector subspace E_B spanned by B in E, turns E_B into a Banach space. A *b*-space is a vector space equipped with a bounded structure which is compatible with the vector space operations, and such that each bounded set is contained in a completant bounded set. A *b*-space E is in a natural way a filtrating union of the Banach spaces E_B (B bounded completant), a subset of E being bounded if and only if it is contained and bounded in one of the Banach spaces E_B. A *b*-subspace F of the *b*-space E is a vector subspace equipped with the bounded structure of a *b*-space, such that all bounded sets in F are bounded in E (a set which is bounded in E and contained in F need not be bounded in F). More about *b*-spaces, and in particular their relation to topological structures, can be found in [1] and [17].

A *b*-algebra A is a *b*-space equipped with a bounded multiplication: B.B′ is bounded whenever B and B′ are. A *b*-subalgebra of a *b*-algebra A is a *b*-subspace of A which is also a *b*-algebra; a *b*-ideal α in a commutative *b*-algebra A is a *b*-subspace of A which is an ideal and such that the product of a bounded set in α and a bounded set in A is bounded in α. All *b*-algebras encountered here will be commutative and have a unit: we are going to consider only *b*-algebras of holomorphic functions with bounded growth, which occur in the holomorphic functional calculus for *b*-algebras with unit.

The growth condition is given by weight functions δ, which will always be lower semi-continuous bounded nonnegative functions of the space $\mathbf{C}^n$ of n complex variables. Special such δ are the functions

$$\delta_0(s) = (1 + |s|^2)^{-1/2}$$
$$\delta_D(s) = \min \left\{ \delta_0(s), d\left(s, \complement D\right) \right\},$$

where $s = (s_1, \ldots, s_n)$ is the variable in $\mathbf{C}^n$, $|s|$ its Euclidian norm and $d\left(s, \complement D\right)$ the Euclidian distance to the complement of the set D in $\mathbf{C}^n$. When E is a b-space, $\mathcal{C}(\delta, E)$ is the b-space of all E-valued functions u on Ω, the open set where δ does not vanish, for which there exists a positive integer N such that $\{u(s)\delta^N(s)|s \in \Omega\}$ is bounded in E; a subset B of $\mathcal{C}(\delta, E)$ is bounded if for some N, $\{u(s)\,\delta^N(s)|s \in \Omega,\ u \in B\}$ is bounded in E. If E is a b-algebra, $\mathcal{C}(\delta, E)$ also is. In particular, $\mathcal{C}(\delta) = \mathcal{C}(\delta, \mathbf{C})$ is a b-algebra. We say that δ and δ' are equivalent if a positive real number ε and a positive integer N can be found such that $\delta \geqslant \varepsilon\delta'^N$, $\delta' \geqslant \varepsilon\delta^N$. Equivalent weight functions give the same algebras.

$\mathcal{O}(\delta)$ is the subset of holomorphic elements of $\mathcal{C}(\delta)$. It is a b-subalgebra with the induced structure. If δ is Lipschitz, equivalent to a function decreasing more rapidly than δ_0 at infinity and equivalent to a function whose logarithm is plurisuperharmonic, these algebras coïncide with the algebras A_p ($p = -\log\delta$) considered in [12] and [13] (see [6, p. 16] or [10]).

All polynomials, and in particular the coordinate functions z_i belong to $\mathcal{O}(\delta)$ if and only if δ is equivalent to a function which is smaller than δ_0. If this is the case, we can find, for all p, $1 \leqslant p \leqslant \infty$, nonnegative constants N′ and K such that for all measurable functions f in $\mathcal{C}(\delta)$:

$$\|f\delta^{N+N'}\|_p \leqslant K\|f\delta^N\|_\infty$$

if the right hand side makes sense, and the nonnegative constants N′ and K only depend on the constants in the equivalence, p and the dimension n (Apply Hölder's inequality). When δ is a Lipschitz function, this means satisfying a Lipschitz condition of order 1 with constant 1, we can find a constant M_p such that for all f in $\mathcal{O}(\delta)$:

$$\|f\delta^{N-n}\|_\infty \leqslant M_p\|f\delta^N\|_p,$$

if the right hand side makes sense. This is a standard application of the Cauchy integral formula on suitably chosen polydiscs and an application of the Hölder inequality. The same argument proves that a complex derivation D^k, $k = (k_1, \ldots, k_n)$

is a bounded mapping from $\mathcal{O}(\delta)$ into itself (see for instance [3]).

Given k elements $a_1, \ldots, a_k$ in a commutative b-algebra A with unit 1, and a b-ideal α in A, we say that a nonnegative function δ on $\mathbf{C}^k$ is spectral for $a_1, \ldots, a_k$ in A modulo α, or belongs to the spectrum $\Delta(a_1, \ldots, a_k; \mathrm{A}|\alpha)$ if we have a decomposition

$$1 = \delta(s)u_0(s) + \sum_1^k (a_i - s_i)u_i(s) + \varphi(s)$$

where $s = (s_1, \ldots, s_k)$ is the variable in $\mathbf{C}^k$, the $u_0, u_1, \ldots, u_k$ are A-valued functions on $\mathbf{C}^k$ belonging to $\mathcal{C}(\delta_0, \mathrm{A})$, and φ is an α-valued function on $\mathbf{C}^k$ belonging to $\mathcal{C}(\delta_0, \alpha)$. This notion of spectrum generalises in some sense the joint spectrum of elements in a commutative Banach algebra with unit, and has been studied in [14], [15] and [16]. We recall that the spectrum is an ideal in the lattice of nonnegative functions, which possesses a base consisting of Lipschitz functions and which does not contain the function identically zero. This last result generalises in some sense the result which says that the joint spectrum of some elements in a commutative Banach algebra with unit is never empty. The holomorphic functional calculus states that if δ is a nonnegative bounded Lipschitz function on $\mathbf{C}^k$, which decreases more rapidly than δ_0 at infinity, and which belongs to $\Delta(a_1, \ldots, a_k; \mathrm{A}|\alpha)$, then we can find a bounded homomorphism of algebras sending $\mathcal{O}(\delta)$ into A, defined modulo α, sending 1 onto 1 and the i^{th} coordinate projection onto a_i.

The problem consists in finding spectral functions. We will solve this in the case of the algebras $\mathcal{O}(\delta)$, for suitable δ.

3. Spectrum of the coordinate projections.

In this section, we prove the following

THEOREM 1 — *Let δ' be a bounded nonnegative Lipschitz function on $\mathbf{C}^n$. Let δ be a Lipschitz function on $\mathbf{C}^n$, which is equivalent to a function which is smaller than δ_0, with*

$\Omega = \{\delta > 0\}$ *pseudoconvex, and such that we can find a function* $\delta^\wedge$ *on* $\mathbf{C}^n$ *with the properties:*

(i) $-\log \delta^\wedge$ *is plurisubharmonic on* Ω;

(ii) δ *majors some function equivalent with* $\delta^\wedge$;

(iii) $\delta^\wedge$ *majors some function equivalent with* δ'.

Then δ *belongs to* $\Delta(z_1, \ldots, z_n;\ 0(\delta'))$.

Proof. — In the proof, we use the notations of [12]. When δ is any nonnegative function on $\mathbf{C}^n$, and t and r are nonnegative integers, $L_r^t(\delta)$ is the set of all systems $h = \{h_I\}$, antisymmetric in the indices I, where for each I, index set of length t containing numbers i_j in $\{0, \ldots, n\}$, h_I is a differential form of order $(0, r)$ on Ω, with C^∞ coefficients, and such that for some positive integer N:

$$\int_\Omega |h_I|^2 \delta^N < \infty,$$

where $|h_I|^2$ is the sum of squares of the absolute values of the coefficients of h_I. Next, $\bar{\partial}$ is the unbounded operator $L_r^t(\delta) \rightarrow L_{r+1}(\delta)$ defined on those h in $L_r^t(\delta)$ such that $\{\bar{\partial} h_I\}$ belongs to $L_{r+1}^t(\delta)$ (here, the derivative is taken in the distribution sense). Of course $\bar{\partial}\bar{\partial} = 0$. Scalar multiplication by $(\delta(s), z_1 - s_1, \ldots, z_n - s_n)$:

$$(P_s h)_I = h_{I,0}\, \delta(s) + \sum_1^n h_{I,i}\, (z_i - s_i)$$

defines a map P_s: $L_r^{t+1}(\delta) \rightarrow L_r^t(\delta)$, depending on the parameter $s = (s_1, \ldots, s_n)$ in $\mathbf{C}^n$. For all s we put $P_s(L_r^0(\delta)) = \{0\}$. Antisymmetry implies $P_s P_s = 0$, and since for constant s the functions $\delta(s)$, $z_i - s_i$ are analytic: $P_s \bar{\partial} = \bar{\partial} P_s$.

With these notations, we have to find a system $u = (u_0, \ldots, u_n)$ of $n + 1$ functions of $2n$ complex variables $(s, z) = (s_1, \ldots, s_n, z_1, \ldots, z_n)$ in $\mathbf{C}^n \times \Omega'$, such that for all fixed s in $\mathbf{C}^n$:

$$\begin{aligned} P_s u(s, z) &= 1 \\ \bar{\partial} u(s, z) &= 0 \end{aligned}$$

on Ω, and with the estimate: there exist positive integers N and M such that:

$$|u_i(s, z)|\ (\delta_0(s)\delta'(z))^N \leqslant M$$

on $C^n \times \Omega'$. For fixed s in C^n, it is easy to find a system of functions ${}_0h(s) = ({}_0h_0(s), \dots, {}_0h_n(s))$ in $L_r^t(\delta)$, such that $P_s\, {}_0h(s, z) = 1$:

LEMMA — *If* g(s) *is a system in* $L_r^t(\delta)$ *which satisfies*

$$P_s\, g(s) = \bar{\delta} g(s) = 0,$$

we can find a system h(s) *in* $L_r^{t+1}(\delta)$ *and in the domain of* $\bar{\delta}$, *such that* $P_s\,\mathrm{h(s)} = \mathrm{g(s)}$.

Such a system is given by

$$h_{i_0, i_1, \dots, i_t}(s) = (-1)^t g_{i_1, \dots, i_t}(s)\delta(s)|P_s|^{-2} + \sum_{j=1}^{t} (-1)^{t-j} g_{i_0, i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_t}(\bar{z}_{i_j} - \bar{s}_{i_j})|P_s|^{-2}$$

where $|P_s|^2 = \delta^2(s) + \sum_1^n |z_i - s_i|^2$. This solution is in general not unique, and by a double induction on the integers t and r, we are going to improve the ${}_0h(s)$ obtained in the lemma such that it becomes analytic and satisfies the required estimate.

Increasing induction. — For fixed s in C^n, we first construct by induction on k some systems of differential forms ${}_kh(s)$ in $L_k^{k+1}(\delta)$, which satisfy $P_s\, {}_kh(s) = \bar{\delta}\, {}_{k-1}h(s)$: suppose we already have ${}_0h, \dots, {}_{k-1}h$; since $\bar{\delta}\bar{\delta}\, {}_{k-1}h(s) = 0$ and

$$\begin{aligned} P_s \bar{\delta}\, {}_{k-1}h(s) = \bar{\delta}\, P_s\, {}_{k-1}h(s) &= \bar{\delta}\, 1 = 0 \quad \text{if} \quad k = 1; \\ &= \bar{\delta}\bar{\delta}\, {}_{k-2}h(s) = 0 \quad \text{if} \quad k \neq 1, \end{aligned}$$

we can apply the lemma to obtain ${}_kh(s)$ in $L_k^{k+1}(\delta)$ with $\bar{\delta}\, {}_kh(s)$ in $L_{k+1}^{k+1}(\delta)$. This construction gives coefficients of the monomials in ${}_kh(s)$ which are a constant multiple of

$$\begin{aligned} &|P_s|^{-2p_0}({}_0h_0(s))^{q_0}({}_0\bar{h}_1(s))^{p_1}({}_0h_1(s))^{q_1} \dots ({}_0\bar{h}_n(s))^{p_n}({}_0h_n(s))^{q_n}, \\ &\quad = |P_s|^{-2\gamma}\delta^{q_0}(s)(z_1 - s_1)^{p_1}(\bar{z}_1 - \bar{s}_1)^{q_1} \dots (z_n - s_n)^{p_n}(\bar{z}_n - \bar{s}_n)^{q_n}, \end{aligned}$$

where the p_i, q_i and γ are nonnegative integers satisfying:

$$p_0 + q_0 + p_1 + \dots + p_n + q_n = \gamma \leqslant 2k + 2.$$

Indeed, we already know this is the case for ${}_0h(s)$; we suppose it is true for ${}_{k-1}h(s)$. The derivation $\bar\partial$ gives two kinds of coefficients: q_j times the same coefficient with q_j replaced by $q_j - 1$ (or $0.|P_s|^{-2\gamma}$ if $q_j = 0$), or -1 times the same coefficient with p_j replaced by $p_j + 1$ and γ by $\gamma + 1$. Application of the lemma multiplies these expressions by $\pm\ \delta(s)|P_s|^{-2}$ or $\pm\ (\bar z_j - \bar s_j)|P_s|^{-2}$. So γ increases at most by 2, while p_0 remains constant or increases. Since δ is Lipschitz, we can estimate the obtained coefficients: when $|z - s| \leqslant \frac{1}{2}\delta(z)$ then $\frac{1}{2}\delta(z) \leqslant \delta(s) \leqslant \frac{3}{2}\delta(z)$ and

$$|{}_0h_0(s, z)| \leqslant \delta^{-1}(s) \leqslant 2\delta^{-1}(z),$$

$$|{}_0h_i(s, z)| \leqslant |z - s|\delta^{-2}(s) \leqslant 2\delta^{-1}(z) \quad \text{for} \quad 1 \leqslant i \leqslant n;$$

when $|z - s| \geqslant \frac{1}{2}\delta(z)$, then

$$|{}_0h_0(s, z)| \leqslant \delta(s)(2|z - s|\delta(s))^{-1} \leqslant 2\delta^{-1}(z),$$

$$|{}_0h_i(s, z)| \leqslant |z - s|^{-1} \leqslant 2\delta^{-1}(z) \quad \text{for} \quad 1 \leqslant i \leqslant n;$$

finally:

$$|P_s|^{-2} \leqslant 2(|z - s| + \delta(s))^{-2} \leqslant 2\delta^{-2}(z).$$

So the absolute value of each coefficient in ${}_kh$ is dominated by

$$(1) \qquad c.(\delta(z))^{-(p_0+\gamma)} \leqslant c.\delta^{-(4k+4)}(z).$$

This estimate does not depend on s.

Decreasing induction. — Next we remark that $L^t_r(\delta) \subset L^t_r(\delta^\wedge)$, with bounded identity mapping. In these spaces the following lemma (Lemma 4 in [12], obtained from theorem 2.2.1'. in [11]) is valid:

LEMMA. — *Let* Ω *be a pseudoconvex open set in* $\mathbf{C}^n$ *and let* $\delta^\wedge$ *be a nonnegative function on* Ω *with* $-\log\delta^\wedge$ *plurisubharmonic. For every* (p, q) *form* g, $q \geqslant 1$, *with locally square integrable coefficients on* Ω, *satisfying* $\bar\partial g = 0$ *and*

$$\int_\Omega |g|^2\delta^\wedge\, d\lambda < \infty,$$

where $d\lambda$ *is the Lebesgue measure on* $\mathbf{C}^n$, *one can find a*

$(p, q-1)$ *from* f *satisfying* $\bar{\delta} f = g$ *and*

$$\int_\Omega |f|^2 \delta^\wedge \delta_0^4 \, d\lambda \leqslant \int_\Omega |g|^2 \delta^\wedge \, d\lambda.$$

We construct, by this lemma and by decreasing induction on k, $1 \leqslant k \leqslant n$, systems of differential forms ${}_k h'(s)$ in $L_k^{k+2}(\delta^\wedge)$ satisfying

$$\bar{\delta}\, {}_{k-1}h'(s) = {}_k h(s) - P_s\, {}_k h'(s)$$

where ${}_n h'(s) = 0$ by definition of $L_n^{n+2}(\delta^\wedge)$. This construction is possible since for fixed s and for all k, $1 \leqslant k \leqslant n$:

$$\bar{\delta}({}_k h(s) - P_s\, {}_k h'(s)) = \bar{\delta}\, {}_k h(s) - P_s\, {}_{k+1}h(s) - P_s P_s\, {}_{k+1}h(s) = 0,$$

and the lemma applies. Moreover, we obtain the following estimate: for each k, and each index family I:

$$(2) \qquad \|{}_{k-1}h'_I(s) \delta^{\wedge N} \delta_0^2\|_2 \leqslant \|({}_k h_I(s) - P_s\, {}_k h'_I(s)) \delta^{\wedge N}\|_2$$

if N is such that the right hand side is finite.

Finally we put

$$u(s) = {}_0 h(s) - P_s\, {}_0 h'(s).$$

This system u satisfies:

$$\bar{\delta}\, u(s) = 0,$$
$$P_s u(s) = P_s\, {}_0 h(s) = 1.$$

We now have to estimate u. Since δ is equivalent to a function which is smaller than δ_0, we can replace the L^∞ estimate (1) on ${}_k h(s)$ by an L^2 estimate, and since δ majors some function equivalent with $\delta^\wedge$, there exist positive integers N_0 and M_0 such that for each k and each index family I:

$$\|{}_k h_I(s) \delta^{\wedge N_0}\|_2 < M_0.$$

Since $\delta^\wedge$ and δ are smaller than δ_0 up to equivalence, there exist positive integers N_1 and M_1 such that

$$\delta^{\wedge N_1}(z) \delta_0(s) |z - s| \leqslant M_1 \text{ and } \delta^{\wedge N_1}(z) \delta_0(s) \delta(s) \leqslant M_1,$$

and for each k and each index set I:

$$\delta^{\wedge N + N_1}(z) \delta_0^{3(n-k)}(s) |P_s\, {}_k h'(s, z)| \leqslant M_1 \delta^{\wedge N}(z) \delta_0^{3(n-k)-1}(s) \sum_{i=0}^{n} |{}_k h'_{I,i}(s, z)|;$$

if N is such that the L^2 norm of the right hand side is bounded, we have the same inequality for the L^2 norms. Using (2) and by decreasing induction on k one finally obtains: there exist positive integers N′ and M′ such that for all i, $0 \leqslant i \leqslant n$:

$$\| u_i(s, z)\delta^{\wedge N'}(z)\delta_0^{3n}(s)\|_2 \leqslant M'$$

and the same holds with $\delta^\wedge$ replaced by δ', which is Lipschitz, and therefore we can find positive integers N″ and M″ such that for all i, $0 \leqslant i \leqslant n$:

$$\| u_i(s, z)\delta'^{N''}(z)\delta_0^{3n}(s)\|_\infty \leqslant M'' \tag{3}$$

which is the required estimate.

This ends the proof of theorem 1.

Remark 1. — The constants arising in the proof and in the final estimate are universal, i.e. they only depend on the dimension n and the constants occuring in the equivalence (and on the Lipschitz constants, but we put those equal to one), but do not depend on the systems obtained in the induction.

Remark 2. — In fact, we have obtained that for all i, $1 \leqslant i \leqslant k$: u_i belongs to $\mathcal{C}(\delta_0, A')$, Where A′ is the b-subalgebra of $\mathcal{O}(\delta')$ generated by the set B′ of functions f such that $|f|\delta'^{N''} \leqslant 1$ (where N″ is chosen as in formula (3)) and δ therefore belongs to $\Delta(z_1, \ldots, z_n; A')$. The holomorphic functional calculus defines a morphism

$$\mathcal{O}(\delta') \rightarrow A',$$

which can only be the identity since it sends 1 onto 1 and the z_i onto the z_i:

$$\mathcal{O}(\delta') \subset A'.$$

In particular, when δ is equivalent to δ', we obtain:

Proposition 2. — *Let δ and δ' be as in theorem 1, δ and δ' equivalent. Any set which contains the set $\{f \mid |f|\delta^N \leqslant 1\}$ with N sufficiently large, generates $\mathcal{O}(\delta)$ as a b-algebra.*

PROPOSITION 3. — *Let* δ *be a Lipschitz function on* $\mathbf{C}^n$, *which is equivalent to a function which is smaller than* δ_0 *and with* $\Omega = \{\delta > 0\}$ *pseudoconvex. Then* $\Delta(z_1, \ldots, z_n; \mathcal{O}(\delta))$ *possesses a base consisting of Lipschitz functions* ψ *with* $-\log \psi$ *plurisubharmonic.*

What we really show is that, if $-\log \delta^\wedge$ is plurisubharmonic, δ Lipschitz and $\delta^\wedge \geqslant \delta$, then a function δ_1 exists, which is Lipschitz, plurisubharmonic, and such that $\delta^\wedge \geqslant \delta_1 \geqslant \frac{\delta}{2}$.

Proof. — Let first $\delta^\wedge$ be any lower semi-continuous function with $\Omega^\wedge = \{\delta^\wedge > 0\}$ pseudoconvex and with $-\log \delta^\wedge$ plurisubharmonic on $\Omega^\wedge$. The set Ω_1 of the couples (w, z) in $\Omega^\wedge \times \mathbf{C}$ such that $|w|\delta^{\wedge-1}(z) < 1$, is pseudoconvex; the restriction δ_1 of the distance function δ_{Ω_1} to $\Omega^\wedge$ is smaller or equal than $\delta^\wedge$, and $-\log \delta_1$ is plurisubharmonic. Moreover

$$\delta_1(s) \geqslant \inf_{z \in \mathbf{C}^n} (\delta^{\wedge 2}(s) + |s - z|^2)^{1/2}.$$

Let now $\delta^\wedge$ be the smallest function larger than δ, with $-\log \delta^\wedge$ plurisubharmonic on Ω. We have

$$\delta^\wedge \geqslant \delta_1 \geqslant \frac{\delta}{2},$$

δ_1 belongs to $\Delta(z_1, \ldots, z_n; \mathcal{O}(\delta_1))$ which is contained in $\Delta(z_1, \ldots, z_n; \mathcal{O}(\delta))$, and the same goes through with δ replaced by δ^N, for any positive integer N. If φ belongs to $\Delta(z_1, \ldots, z_n; \mathcal{O}(\delta))$, then [14, p. 124] we can find a function δ' with $-\log \delta'$ plurisubharmonic, and a positive integer N such that

$$\varphi \geqslant \delta' \geqslant \varepsilon\delta^N,$$

and therefore $\varphi \geqslant \varepsilon.(\delta^N)^\wedge \geqslant \varepsilon.(\delta^N)_1$. Thus the functions $\psi_N = \varepsilon.(\delta^N)_1$ form a basis of the spectrum $\Delta(z_1, \ldots, z_n; \mathcal{O}(\delta))$.

By the preceding construction, theorem 1 is equivalent to the following apparently weaker theorem:

THEOREM 1′ [3]. — *Let* δ *be a nonnegative Lipschitz function on* $\mathbf{C}^n$, *with* $\Omega = \{\delta > 0\}$ *pseudoconvex,* $-\log \delta$

plurisubharmonic on Ω, *and which is equivalent to a function smaller than* δ_0. *Then* δ *belongs to* $\Delta(z_1, \ldots, z_n;\ \mathcal{O}(\delta))$.

Indeed, let δ and δ' satisfy the weaker condition in theorem 1. We may suppose that $\delta^\wedge \geqslant \delta'$. We can then find a function δ_1 which is Lipschitz, with $-\log \delta_1$ plurisubharmonic and $\delta^\wedge \geqslant \delta_1 \geqslant \frac{\delta'}{2}\cdot$ Theorem $1'$ says that δ_1 belongs to $\Delta(z_1, \ldots, z_n;\ \mathcal{O}(\delta_1))$, which is contained in $\Delta(z_1, \ldots, z_n;\ \mathcal{O}(\delta'))$. Therefore δ belongs to $\Delta(z_1, \ldots, z_n;\ \mathcal{O}(\delta'))$.

An argument of L. Waelbroeck [14] combined with this theorem gives:

Proposition 4. — *Let* δ *be as in theorem* $1'$, $n' \leqslant n$ *and* δ' *on* $\mathbf{C}^{n'}$ *defined by*

$$\delta'(z_1, \ldots, z_{n'}) = \delta(z_1, \ldots, z_n, 0, \ldots, 0).$$

Then $\mathcal{O}(\delta')$ *is the quotient of* $\mathcal{O}(\delta)$ *modulo the ideal* α *generated in* $\mathcal{O}(\delta)$ *by the functions* $z_{n'+1}, \ldots, z_n$.

Proof. — By theorem $1'$, δ belongs to $\Delta(z_1, \ldots, z_n; \mathcal{O}(\delta))$, and we can recall the argument in [14]. Since δ is spectral, there exist functions $u_0, u_1, \ldots, u_n$ in $\mathcal{C}(\delta_0, \mathcal{O}(\delta))$ satisfying a relation

$$1 = \delta(s)u_0(s, z) + \sum_1^n (z_i - s_i)u_i(s, z).$$

If we put $s_{n'+1} = \cdots = s_n = 0$ in this relation, we have $\delta' \in \Delta(z_1, \ldots, z_{n'};\ \mathcal{O}(\delta)|\alpha)$. If we put further $z_{n'+1} = \cdots = z_n = 0$, we see that $\delta' \in \Delta(z_1, \ldots, z_{n'};\ \mathcal{O}(\delta'))$. Let T be the canonical mapping of $\mathbf{C}^{n'}$ in

$$\mathbf{C}^{n'} \times \{0\} \times \ldots \times \{0\} \subset \mathbf{C}^n,$$

and $\check{\mathrm{T}}$ the map defined by

$$\check{\mathrm{T}}\mathrm{F}' = \mathrm{F}'(\mathrm{T}(z_1, \ldots, z_{n'}))$$

for all F' in $\mathcal{O}(\delta')$. By the holomorphic functional calculus we have a mapping

$$\mathcal{O}(\delta') \to \mathcal{O}(\delta) \text{ modulo } \alpha,\ \mathrm{F}' \to \mathrm{F}'\,[z_1, \ldots, z_n].$$

$F'[z_1, \ldots, z_{n'}]$ belongs to $\mathcal{O}(\delta)$ modulo α. A straightforward computation which uses the invariance of the holomorphic functional calculus under linear transformation (see also [14]) shows that the image under $\check{T}$ of an element of the equivalence class $F'[z_1, \ldots, z_{n'}]$ is F', and that the kernel of $\check{T}$ is α.

4. Spectrum of arbitrary elements.

We use the following result [15, Appendix p. 8]: Let A be a commutative b-algebra, $a_1, \ldots, a_n$ elements of A, and δ in the spectrum $\Delta(a_1, \ldots, a_n; A)$. If δ_* belongs to the spectrum $\Delta(f_1, \ldots, f_k; \mathcal{E}(\delta))$ of some elements $f_1, \ldots, f_k$ in $\mathcal{O}(\delta)$, then δ_* belongs to the spectrum

$$\Delta(f_1[a_1, \ldots, a_n], \ldots, f_k[a_1, \ldots . a_n]; A)$$

where $f_i[a_1, \ldots, a_n]$ is the image of $a_1, \ldots, a_n$ through the homomorphism $\mathcal{O}(\delta) \rightarrow A$ defined by the holomorphic functional calculus. This, together with theorem 1 yields:

Corollary. — *Let* δ *and* δ' *be as in theorem* 1 *and* $f_1, \ldots, f_k$ *elements of* $\mathcal{O}(\delta)$. *Then*

$$\Delta(f_1, \ldots, f_k; \mathcal{E}(\delta)) \subset \Delta(f_1, \ldots, f_k; \mathcal{O}(\delta')).$$

This is straightforward since $f[z_1, \ldots, z_n] = f$ for all f in $\mathcal{O}(\delta)$.

We now suppose that δ and δ' are equivalent. This gives us a complete characterisation of the spectrum of some elements of $\mathcal{O}(\delta)$:

Proposition 5. — *Let* δ *be as in theorem* 1′. *A bounded nonnegative Lipschitz function* δ_* *on* $\mathbf{C}^k$ *belongs to the spectrum of some elements* $f_1, \ldots, f_k$ *in* $\mathcal{O}(\delta)$ *if and only if for some positive number* ε *and some positive integer* N:

$$\delta_*(f_1(z), \ldots, f_k(z)) \geqslant \varepsilon\delta^N(z) \tag{4}$$

for all z *in* $\mathbf{C}^n$.

Proof. — δ_* belongs to $\Delta(f_1, \ldots, f_k; \mathcal{C}(\delta))$ if and only if (4) is valid. The corollary applies to one direction, while the other direction is trivial. The characterisation is complete since $\Delta(f_1, \ldots, f_k; \mathcal{O}(\delta))$ possesses a base consisting of Lipschitz functions.

In particular, any bounded Lipschitz function which majors the function

$$(\sup \delta\, f^{-1})(s) = \sup \{\delta(z) | z \in \mathbf{C}^n, f_1(z) = s_1, \ldots, f_k(z) = s_k\}$$

belongs to $\Delta(f_1, \ldots, f_k; \mathcal{O}(\delta))$.

5. A « Nullstellensatz » for bounded families in $\mathcal{O}(\delta)$.

The following theorem generalises the result of J. J. Kelleher and B. A. Taylor [13] to bounded families $(f_\lambda)_{\lambda \in \Lambda}$ of functions f_λ in $\mathcal{O}(\delta)$, Λ being an arbitrary index set.

THEOREM 6. — *Let δ be a nonnegative Lipschitz function on $\mathbf{C}^n$, equivalent with a function smaller than δ_0 and with a function $\delta\hat{}$ with* $-\log \delta\hat{}$ *plurisubharmonic. Let $(f_{1\lambda}), \ldots, (f_{k\lambda})$ and (g_λ) be bounded families of $\mathcal{O}(\delta)$ such that there exists a positive number ε and a positive integer* N *such that for all λ in Λ:*

$$|f_{1\lambda}(s)| + \cdots + |f_{k\lambda}(s)| \geqslant \varepsilon \delta^{N}(s) |g_\lambda(s)|$$

for all s in $\mathbf{C}^n$. Then it is possible to find a positive integer m such that for all λ in Λ, g_λ^m belongs to the ideal generated by $f_{1\lambda}, \ldots, f_{k\lambda}$ in $\mathcal{O}(\delta)$, the coefficients being bounded uniformly with respect to λ in Λ.

Proof. — We first prove the following lemma [18], using the existence of spectral functions in its first part, and the fact that the zero function does not belong to the spectrum.

LEMMA. — *Let* A *be a commutative b-algebra with unit, α a b-ideal and $a_1, \ldots, a_n, a$ elements of* A. *If δ belongs to $\Delta(a_1, \ldots, a_n; \mathrm{A}|\alpha)$, and if a belongs to the sum of $\mathcal{C}(\delta, \alpha)$ and the ideal generated by $a_1 - s_1, \ldots, a_n - s_n$ in $\mathcal{C}(\delta, \mathrm{A})$ (the s_i being complex variables), then a is nilpotent modulo α in* A.

Proof of Lemma. — First, if δ is spectral, then for any positive integer N, δ^N also is :

$$1 = \delta^N(s)u_0(s) + \sum_1^n (a_i - s_i)u_i(s) + v(s)$$

with coefficients u belonging to $\mathcal{E}(\delta_0, A)$ and v belongs to $\mathcal{E}(\delta_0, \alpha)$. Multiplying this equation with

$$a = \sum_1^n (a_i - s_i)U_i(s) + V(s),$$

where the coefficients U belong to $\mathcal{E}(\delta, A)$ and V belongs to $\mathcal{E}(\delta, \alpha)$, and rearranging the terms gives coefficients with polynomial growth, such that a belongs to the sum of $\mathcal{E}(\delta_0, \alpha)$ and the ideal generated by $a_1 - s_1, \ldots, a_n - s_n$ in $\mathcal{E}(\delta_0, A)$. Next, consider the b-algebra $A[x]$ of polynomials in one variable x with coefficients in A, and its b-ideal $\beta = \alpha[x] + (1 - ax)A[x]$. Using the last assertion, one easily obtains the decomposition

$$1 = \sum_1^n (a_i - s_i)U'_i(s)x + V'$$

where the coefficients U' belong to $\mathcal{E}(\delta_0, A[x])$ and V' belongs to $\mathcal{E}(\delta_0, \beta)$. But the zero function is not spectral, so we must have $\beta = A[x]$, and there exists a positive integer m such that a^m belongs to α, which completes the proof of the lemma.

We let A be the b-algebra $\mathcal{O}(\delta)_\Lambda$ of systems in $\mathcal{O}(\delta)$, indexed by Λ and bounded in $\mathcal{O}(\delta)$, and α its b-ideal generated by the systems $(f_{1\lambda}), \ldots, (f_{k\lambda})$. We know that δ belongs to the spectrum of the coordinate functions in $\mathcal{O}(\delta)$, therefore δ belongs to the spectrum of the coordinate functions in A and *a fortiori* in the bigger algebra $\mathcal{O}(\delta_s \otimes \delta_z)_\Lambda$ defined by the weight function $\delta_s \otimes \delta_z$ on $\mathbf{C}^n \times \mathbf{C}^n$, which is δ on the first n complex variables s, and which is δ on the last n complex variables z. We apply the holomorphic functional calculus modulo the b-ideal γ generated by $z_1 - s_1, \ldots, z_n - s_n$ in $\mathcal{O}(\delta_s \otimes \delta_z)_\Lambda$, to obtain :

$$(f_{i\lambda})(z) - (f_{i\lambda})(s) = (f_{i\lambda})[z] - (f_{i\lambda})[s] \in \gamma \text{ for all } i,\ 1 \leqslant i \leqslant k;$$
$$(g_\lambda)(z) - (g_\lambda)(s) = (g_\lambda)[z] - (g_\lambda)[s] \in \gamma,$$

since the homomorphism is defined modulo γ. The growth restriction imposed on (g_λ) is equivalent with: $(g_\lambda)(s)$ belongs to the ideal generated by $(f_{1\lambda})(s), \ldots, (f_{k\lambda})(s)$ in $\mathcal{C}(\delta_s)_\Lambda$, which is contained in the ideal generated by the same systems in the bigger algebra $\mathcal{C}(\delta_s, \mathrm{A})$. Combining these results, one finds out that $(g_\lambda)(z)$ belongs to the sum of $\mathcal{C}(\delta_s, \alpha)$ and the ideal generated by $z_1 - s_1, \ldots, z_n - s_n$ in $\mathcal{C}(\delta_s, \mathrm{A})$. By the lemma, we can find a positive integer m such that

$$(g_\lambda^m) \in \alpha.$$

The coefficients are in A and thus bounded uniformly with respect to λ. By this method of proof, we do not obtain an estimate on the number m, as in [13].

6. The spectrum with respect to a closure; approximation.

In this section we do not give complete results, but only a summary of approximation theorems for algebras $\mathcal{O}(\delta)$ which have been obtained using theorem 1 and the holomorphic functional calculus. Ferrier defines the « closure » $\overline{\mathrm{F}}$ of a subspace F in a b-space E as the union of closures of the subspaces $\mathrm{F} \cap \mathrm{E}_\mathrm{B}$ of the Banach spaces E_B, B bounded completant set in E, the union being equipped with its natural bounded structure. Elements of F « approximate » the elements in E if $\overline{\mathrm{F}} = \mathrm{E}$.

In particular, one can consider the closure $\overline{\mathcal{O}(\delta')}$ of $\mathcal{O}(\delta')$ in $\mathcal{O}(\delta)$, when δ and δ' are nonnegative bounded Lipschitz functions on $\mathbf{C}^n$, such that $\delta \leqslant \delta' \leqslant \delta_0$ up to equivalence and $\Omega' = \{\delta' > 0\}$ is pseudoconvex. The spectrum of the coordinate functions with respect to $\overline{\mathcal{O}(\delta')}$ has been characterised in [5]:

THEOREM 7. — *Let δ and δ' be as above. A nonnegative function φ on $\mathbf{C}^n$ belongs to $\Delta(z_1, \ldots, z_n; \overline{\mathcal{O}(\delta')})$ if and only if the restriction of $1/\varphi$ to Ω' is smaller than the upper envelope of a family (π_α), bounded in $\mathcal{C}(\delta)$, of nonnegative functions in $\mathcal{C}(\delta')$ with* log π_α *plurisubharmonic.*

Its proof uses theorem 1 and a fundamental lemma by Waelbroeck [14, p. 69], [16, p. 314].

If φ is a bounded nonnegative Lipschitz function, $\varphi \leqslant \delta'$ and φ belongs to $\Delta(z_1, \ldots, z_n; \overline{\mathcal{O}(\delta')})$, the holomorphic functional calculus defines a morphism

$$\mathcal{O}(\varphi) \rightarrow \mathcal{O}(\delta')$$

which has to be the identity since it sends 1 onto 1 and the z_i onto the z_i:

$$\mathcal{O}(\varphi) = \overline{\mathcal{O}(\delta')}.$$

In particular:

Corollary. — *Let δ and δ' be as above. If $1/\delta$ is the upper envelope on Ω' of a family π_α of nonnegative functions in $\mathcal{C}(\delta')$ with plurisubharmonic logarithms, then $\overline{\mathcal{O}(\delta')} = \mathcal{O}(\delta)$.*

Such approximation results in one dimension were obtained by J.-P. Ferrier (see [6]). He has generalised these approximation theorems to several dimensions in [7], [8]. In these papers, he uses a double induction similar to the one in the proof of theorem 1 to obtain a uniform decomposition of functions in certain sets in $\mathcal{O}(\delta')$ which are bounded in $\mathcal{O}(\delta)$; this in turn gives that δ is spectral with respect to $\overline{\mathcal{O}(\delta')}$ and the holomorphic functional calculus applies. In [8, appendix 1] he mentions a way to avoid the rather cumbersome induction by using theorem 1 and the holomorphic functional calculus modulo the ideal generated by $z_1 - s_1, \ldots, z_n - s_n$, as we did in the proof of the « Nullstellensatz ». These computations can be found in [3]. A more elegant proof, which uses only theorem 1 and the fundamental lemma of Waelbroeck, is in [9]. In this paper, Ferrier generalises his approximation results to inductive systems of algebras $\mathcal{O}(\delta)$; this covers some well-known classical results on holomorphic approximation. The lectures [10] form a nice survey of applications of spectral theory and the holomorphic functional calculus to algebras $\mathcal{O}(\delta)$.

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