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# ON NONBORNOLOGICAL BARRELLED SPACES [1]

by Manuel VALDIVIA

L. Nachbin [5] and T. Shirota [6], give an answer to a problem proposed by N. Bourbaki [1] and J. Dieudonné [2], giving an example of a barrelled space, which is not bornological. Posteriorly some examples of nonbornological barrelled spaces have been given, e.g. Y. Kōmura, [4], has constructed a Montel space which is not bornological. In this paper we prove that if E is the topological product of an uncountable family of barrelled spaces, of nonzero dimension, there exists an infinite number of barrelled subspaces of E, which are not bornological. We obtain also an analogous result replacing « barrelled » by « quasi-barrelled ».

We use here nonzero vector spaces on the field K of real or complex number. The topologies on these spaces are separated.

If E is a separated locally convex space, we denote, as usual, by E′, $\sigma(E', E)$ and $\beta(E', E)$, the topological dual of E, the weak topology on E′, and the strong topology on E′, respectively. If A is a bounded, closed and absolutely convex set of E, we denote by $E_A$ the linear hull of A equipped with the norm associated to A.

We shall need the following result of J. Dieudonné [3]:

*a) Let* E *be a bornological space. If* F *is a subspace of* E, *of finite codimension, then* F *is bornological.*

THEOREM 1. — *If* E *is the topological product of the barrelled spaces* $E_i$, $i \in I$, *where* I *is an uncountable set, there exists*

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*an infinite family* $\mathcal{F}$ *of barrelled dense subspaces of* E, *which are not bornological.*

*Proof.* — Let G be the subspace of E, whose points have all components zero except a countable set. Let $\mathcal{F}$ be the family of all the subspaces of E, such that $F \in \mathcal{F}$ if and only if $G \subset F$ and the codimension of G in F is finite and different from zero. Obviously $\mathcal{F}$ is infinite. If $F \in \mathcal{F}$, then F is barrelled since G is barrelled. (It can be proved that G is barrelled taking any $\sigma(G', G)$-bounded set A of $G' = E'$, and noticing that there exists a finite set $\{i_1, i_2, \ldots, i_n\} \subset I$, such that $A \subset \prod_{p=1}^{n} E'_p$, hence A is $\sigma(E',E)$-bounded and, therefore, equicontinuous respect to E and respect to G also.) Now we suppose that F is bornological. According to *a*) we can find a bornological space L, such that $G \subset L \subset F$, being G an hyperplane of L. In L let $\mathcal{B}$ be the family of all the absolutely convex, closed and bounded sets. Since L is the inductive limit of $\{E_B : B \in \mathcal{B}\}$ and G is a dense hyperplane of L, there exists a $M \in \mathcal{B}$, such that $G \cap E_M$ is dense in $E_M$ and $G \not\supset E_M$. Therefore we can find in $E_M$ a sequence $\{x_n\}_{n=1}^{\infty} \subset G$, which converges to $x \notin G$. That is in contradiction with being G sequentially closed in E and also in L. Q.E.D.

Theorem 2. — *If* E *is the topological product of the bornological barrelled spaces* $E_i$, $i \in I$, *where* I *is an uncountable set, there exists a family* $\mathcal{F}$ *of barrelled dense subspaces of* E, *which are not bornological, so that if* $F \in \mathcal{F}$, *there exists a subspace* H *of* F, *of finite codimension, such that* H *is bornological.*

*Proof.* — It is enough to prove that the space G defined in the proof of Theorem 1 is bornological. Let $\mathcal{M}$ be the family of the parts of I, which have a countable infinity of elements. For each $M \in \mathcal{F}$, we denote by E(M) the subspace of E, whose points have all the components zero except at most those with indices in M. It is inmediate that E(M) is bornological. Since G is the inductive limit of the family of spaces $\{E(M) : M \in \mathcal{M}\}$ then G is bornological. (We can prove that G is the inductive limit of $\{E(M) : M \in \mathcal{M}\}$ of the following way : let $u$ be any linear form on G, such that its restriction $u_M$ to E(M) is continuous, $M \in \mathcal{M}$. Let $v_M$ be the continuous extension of $u_M$ to G, such that if $x \in G$ and $x(M)$ is the projection of $x$ on E(M), then $v_M(x) = u_M(x(M))$. Obviously the net $\{v_M : M \in \mathcal{M}, \subset\}$ converges weakly to $u$. Furthemore, if $x \in G$, it is easy to prove that $\{v_M(x) : M \in \mathcal{M}\}$ is a bounded set in K, as since G is barrelled, it results that $\{v_M : M \in \mathcal{M}\}$ is equicontinuous set, hence $u$ is continuous in G. Therefore, the space G and the inductive limit of $\{E(M) : M \in \mathcal{M}\}$ have the same topological dual and since, the topology of G is the Mackey one, both spaces are the same.) Q.E.D.

*Note.* — From the anterior prooj it can be deduced that if there exists the strongly inaccessible cardinal $\beta$, then there exists a bornological space G, whose completion $\hat{G}$ is not bornological. It is enough to carry out the topological product E of nonzero Frechet spaces, in number equal to $\beta$, and to take the subspace G formed by all points of E, whose components are nulle, except a countable set. Then G is bornological and its completion $\hat{G} = E$ is not it.

Theorem 3. — *If* E *is the topological product of the quasi-barrelled spaces* $E_i$, $i \in I$, *where* I *is an uncountable set, there exists an infinite family of quasi-barrelled dense subspaces, which are not bornological.*

*Proof.* — The proof is analogous to that of Theorem 1, replacing barrelled by quasi-barrelled. (The proof of being G quasi-barrelled can be done, taking any set A of $G' = E'$, $\beta(G', G)$-bounded, and taking into account that there exists a finite set $\{i_1, i_2, \ldots, i_n\} \subset I$ so that $A \subset \prod_{p=1}^{n} E'_{i_p}$, hence it is easy to deduce that A is bounded for the topology $\beta(E', E)$, and since E is quasi-barrelled it results that A is equicontinuous respect to E, and also respect to G.)

Q.E.D.

Theorem 4. — *If* E *is the topological product of the quasi-barrelled spaces* $E_i$, $i \in I$, *where* I *is an uncountable set, and there exists a* $i_0 \in I$, *such that* $E_{i_0}$ *is not barrelled, then there* *exists a infinite family* $\mathcal{F}$ *of quasi-barrelled dense subspaces of* E, *which are not bornological nor barrelled.*

*Proof.* — It is enough to prove if in the Theorem 3, $F \in \mathcal{F}$, then F is not barrelled. Indeed, if F is barrelled, then its closure in E, which is equal to E, is a barrelled space. In contradiction with the fact that $E_{i_0}$ is not barrelled. Q.E.D.

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