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On nonbornological barrelled spaces


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L. Nachbin [5] and T. Shirota [6], give an answer to a problem proposed by N. Bourbaki [1] and J. Dieudonné [2], giving an example of a barrelled space, which is not bornological. Posteriorly some examples of nonbornological barrelled spaces have been given, e.g. Y. Komura, [4], has constructed a Montel space which is not bornological. In this paper we prove that if $E$ is the topological product of an uncountable family of barrelled spaces, of nonzero dimension, there exists an infinite number of barrelled subspaces of $E$, which are not bornological. We obtain also an analogous result replacing « barrelled » by « quasi-barrelled ».

We use here nonzero vector spaces on the field $K$ of real or complex number. The topologies on these spaces are separated.

If $E$ is a separated locally convex space, we denote, as usual, by $E'$, $\sigma(E', E)$ and $\beta(E', E)$, the topological dual of $E$, the weak topology on $E'$, and the strong topology on $E'$, respectively. If $A$ is a bounded, closed and absolutely convex set of $E$, we denote by $E_A$ the linear hull of $A$ equipped with the norm associated to $A$.

We shall need the following result of J. Dieudonné [3]:

a) Let $E$ be a bornological space. If $F$ is a subspace of $E$, of finite codimension, then $F$ is bornological.

**Theorem 1.** — If $E$ is the topological product of the barrelled spaces $E_i$, $i \in I$, where $I$ is an uncountable set, there exists

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an infinite family \( \mathcal{F} \) of barrelled dense subspaces of \( E \), which are not bornological.

**Proof.** — Let \( G \) be the subspace of \( E \), whose points have all components zero except a countable set. Let \( \mathcal{F} \) be the family of all the subspaces of \( E \), such that \( F \in \mathcal{F} \) if and only if \( G \subseteq F \) and the codimension of \( G \) in \( F \) is finite and different from zero. Obviously \( \mathcal{F} \) is infinite. If \( F \in \mathcal{F} \), then \( F \) is barrelled since \( G \) is barrelled. (It can be proved that \( G \) is barrelled taking any \( \sigma(G', G) \)-bounded set \( A \) of \( G' = E' \), and noticing that there exists a finite set \( \{i_1, i_2, \ldots, i_n\} \subseteq I \), such that \( A \subseteq \prod_{p=1}^{n} E'_{p} \), hence \( A \) is \( \sigma(E', E) \)-bounded and, therefore, equicontinuous respect to \( E \) and respect to \( G \) also.) Now we suppose that \( F \) is bornological. According to \( a) \) we can find a bornological space \( L \), such that \( G \subseteq L \subseteq F \), being \( G \) an hyperplane of \( L \). In \( L \) let \( \mathcal{B} \) be the family of all the absolutely convex, closed and bounded sets. Since \( L \) is the inductive limit of \( \{E_{B} : B \in \mathcal{B}\} \) and \( G \) is a dense hyperplane of \( L \), there exists a \( M \in \mathcal{B} \), such that \( G \cap E_{M} \) is dense in \( E_{M} \) and \( G \not\subseteq E_{M} \). Therefore we can find in \( E_{M} \) a sequence \( \{x_{n}\}_{n=1}^{\infty} \subseteq G \), which converges to \( x \notin G \). That is in contradiction with being \( G \) sequentially closed in \( E \) and also in \( L \). Q.E.D.

**Theorem 2.** — If \( E \) is the topological product of the bornological barrelled spaces \( E_{i} \), \( i \in I \), where \( I \) is an uncountable set, there exists a family \( \mathcal{F} \) of barrelled dense subspaces of \( E \), which are not bornological, so that if \( F \in \mathcal{F} \), there exists a subspace \( H \) of \( F \), of finite codimension, such that \( H \) is bornological.

**Proof.** — It is enough to prove that the space \( G \) defined in the proof of Theorem 1 is bornological. Let \( \mathcal{M} \) be the family of the parts of \( I \), which have a countable infinity of elements. For each \( M \in \mathcal{F} \), we denote by \( E(M) \) the subspace of \( E \), whose points have all the components zero except at most those with indices in \( M \). It is immediate that \( E(M) \) is bornological. Since \( G \) is the inductive limit of the family of spaces \( \{E(M) : M \in \mathcal{M}\} \) then \( G \) is bornological. (We can prove that \( G \) is the inductive limit of \( \{E(M) : M \in \mathcal{M}\} \) of the following
way: let $u$ be any linear form on $G$, such that its restriction $u_M$ to $E(M)$ is continuous, $M \in \mathcal{M}$. Let $\nu_M$ be the continuous extension of $u_M$ to $G$, such that if $x \in G$ and $x(M)$ is the projection of $x$ on $E(M)$, then $\nu_M(x) = u_M(x(M))$. Obviously the net $\{\nu_M : M \in \mathcal{M}, \subset\}$ converges weakly to $u$. Furthermore, if $x \in G$, it is easy to prove that $\{\nu_M(x) : M \in \mathcal{M}\}$ is a bounded set in $K$, as since $G$ is barrelled, it results that $\{\nu_M : M \in \mathcal{M}\}$ is equicontinuous set, hence $u$ is continuous in $G$. Therefore, the space $G$ and the inductive limit of $\{E(M) : M \in \mathcal{M}\}$ have the same topological dual and since, the topology of $G$ is the Mackey one, both spaces are the same. Q.E.D.

Note. — From the anterior proof it can be deduced that if there exists the strongly inaccessible cardinal $\beta$, then there exists a bornological space $G$, whose completion $\hat{G}$ is not bornological. It is enough to carry out the topological product $E$ of nonzero Frechet spaces, in number equal to $\beta$, and to take the subspace $G$ formed by all points of $E$, whose components are null, except a countable set. Then $G$ is bornological and its completion $\hat{G} = E$ is not it.

Theorem 3. — If $E$ is the topological product of the quasi-barrelled spaces $E_i$, $i \in I$, where $I$ is an uncountable set, there exists an infinite family of quasi-barrelled dense subspaces, which are not bornological.

Proof. — The proof is analogous to that of Theorem 1, replacing barrelled by quasi-barrelled. (The proof of being $G$ quasi-barrelled can be done, taking any set $A$ of $G' = E'$, $\beta(G', G)$-bounded, and taking into account that there exists a finite set $\{i_1, i_2, \ldots, i_n\} \subset I$ so that $A \subset \prod_{\beta=1}^{n} E_{i_{\beta}}$, hence it is easy to deduce that $A$ is bounded for the topology $\beta(E', E)$, and since $E$ is quasi-barrelled it results that $A$ is equicontinuous respect to $E$, and also respect to $G$.) Q.E.D.

Theorem 4. — If $E$ is the topological product of the quasi-barrelled spaces $E_i$, $i \in I$, where $I$ is an uncountable set, and there exists a $i_0 \in I$, such that $E_{i_0}$ is not barrelled, then there
exists a infinite family $\mathcal{F}$ of quasi-barrelled dense subspaces of $E$, which are not bornological nor barreled.

**Proof.** — It is enough to prove if in the Theorem 3, $F \in \mathcal{F}$, then $F$ is not barreled. Indeed, if $F$ is barreled, then its closure in $E$, which is equal to $E$, is a barreled space. In contradiction with the fact that $E^*$ is not barreled. Q.E.D.

**BIBLIOGRAPHY**


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