ANNALES DE L'INSTITUT FOURIER

MANUEL VALDIVIA

Some examples on quasi-barrelled spaces

Annales de l'institut Fourier, tome 22, nº 2 (1972), p. 21-26

<http://www.numdam.org/item?id=AIF_1972__22_2_21_0>

© Annales de l'institut Fourier, 1972, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
22, 2 (1972), 21-26.

SOME EXAMPLES ON QUASI-BARRELLED SPACES (1)

by Manuel VALDIVIA

J. Dieudonné has proved in [2] the following theorem:

a) Let E *be a bornological space. If* F *is a subspace of* E, *of finite codimension, then* F *is bornological.*

We have given in [6] and [7], respectively, the following results:

b) Let E *be a quasi-barrelled space. If* F *is a subspace of* E, *of finite codimension, then* F *is quasi-barrelled.*

c) Let E *be an ultrabornological space. If* F *is a subspace of* E, *of infinite countable codimension, then* F *is bornological.*

The results *a*), *b*) and *c*) lead to the question if the results *a*) and *b*) will be true in the case of being F a subspace of infinite countable codimension. In this paper we give an example of a bornological space E, which has a subspace F, of infinite countable codimension, such that F is not quasi-barrelled.

In [8] we have proved the two following theorems:

d) Let E *be a* $\mathfrak{D}\mathfrak{F}$*-space. If* G *is a subspace of* E, *of finite codimension, then* G *is a* $\mathfrak{D}\mathfrak{F}$*-space.*

e) Let E *be a sequentially complete* $\mathfrak{D}\mathfrak{F}$*-space. If* G *is a subspace of* E, *of infinite countable codimension, then* G *is a* $\mathfrak{D}\mathfrak{F}$*-space.*

Another question is if the result *d*) is also true for subspaces of infinite countable codimension. Here we give an example of a quasi-barrelled $\mathfrak{D}\mathfrak{F}$-space, which has a subspace G, of infinite countable codimension, which is not a $\mathfrak{D}\mathfrak{F}$-space.

(1) Supported in part by the « Patronato para el Fomento de la Investigación en la Universidad ».

N. Bourbaki, [1, p. 35], notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have proved that if E is the topological product of an infinite family of bornological barrelled space, of non-zero dimension, there exists an infinite number of bornological barrelled subspaces of E, which are not ultrabornological. In this paper we give an example of a bornological barrelled space, which is not inductive limit of Baire spaces.

We use here vector spaces on the field K of real or complex numbers. The topologies on these spaces are separated.

In [10] we have proved the following result:

f) Let E *be a barrelled space. If* $\{E_n\}_{n=1}^{\infty}$ *is an increasing sequence of subspaces of* E, *such that* $\bigcup_{n=1}^{\infty} E_n = E$, *then* E *is the inductive limit of* $\{E_n\}_{n=1}^{\infty}$.

THEOREM 1. — *Let* E *be the strict inductive limit of an increasing sequence* $\{E_n\}_{n=1}^{\infty}$ *of metrizable locally convex spaces. Let* F *be a sequentially dense subspace of* E. *If* E *is barrelled, then* F *is bornological.*

Proof. — Let $\overline{E}_n$, $n = 1, 2, \ldots$, be the closure of E_n in E. Obviously E is the strict inductive limit of the sequence $\{\overline{E}_n\}_{n=1}^{\infty}$. Let F_n be the closure in E of $F \cap \overline{E}_n$, $n = 1, 2, \ldots$ If $x \in E$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of F, which converges to x. Since the set of points of this sequence is bounded, there exists a positive integer n_0 such that $x_n \in \overline{E}_{n_0}$, $n = 1, 2, \ldots$, and, therefore, $x \in F_{n_0}$. Hence $E = \bigcup_{n=1}^{\infty} F_n$. Since E is barrelled, aplying the result $f)$, we obtain that E is the strict inductive limit of the sequence $\{F_n\}_{n=1}^{\infty}$.

Given any Banach space L and a linear and locally bounded mapping u from F into L, we must to prove that u is continuous. Let u_n be the restriction of u to $F \cap \overline{E}_n$. Since $F \cap \overline{E}_n$ is a metrizable space and u_n is locally bounded, u_n is continuous. Let v_n be the continuous extension of u_n to F_n. Let v be the linear mapping from E into L, which coincides with v_n in F_n, $n = 1, 2, \ldots$ Since v_n is equal to φ_{n+1} on $F \cap \overline{E}_n$, then they are equal on F_n and, therefore, φ is well defined. Since E is the inductive limit of $\{F_n\}_{n=1}^{\infty}$ and since the restriction of φ to F_n is continuous, $n = 1, 2, \ldots$, then φ is continuous. On other hand u is the restriction of φ to F and, therefore, u is continuous. Q.E.D.

Example 1. — A. Grothendieck, [3], has given an example of a space E, which is strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of separable Frechet spaces, so that there exists in E a non-closed subspace G, such that $G \cap E_n$ in closed, $n = 1, 2, \ldots$ In this example let A_n. be a countable set of E_n, dense in E_n. Let P be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let F be the linear hull of $P \cup G$. Since P is sequentially dense in E, applying Theorem 1, it results that F is a bornological space. Applying theorem *f*) it results that G is not barrelled and since G is quasi-complete, then G is not quasi-barrelled. Since P has a countable basis, G is a subspace of F, of countable codimension, and by *a*) the codimension of G is infinite. Therefore, F is a bornological space, which has a subspace G, of infinite countable codimension, so that G is not quasi-barrelled.

Example 2. — G. Kothe, [4, p. 433-434] gives an example of a Montel $\mathfrak{DF}$-space, which has a closed subspace L, which is not a $\mathfrak{DF}$-space. In this example, let $\{B_n\}_{n=1}^{\infty}$ be a fundamental sequence of bounded sets. Since E is a Montel $\mathfrak{DF}$-space, then B_n is separable, $n = 1, 2, \ldots$ Let A_n be a countable subset of B_n, dense in B_n, $n = 1, 2, \ldots$ Let Q be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let M be the linear hull of $Q \cup L$. Now, we shall prove that M is quasi-barrelled. Indeed, given a closed, absolutely convex and bornivorous set U in M, let $\overline{U}$ be its closure in E. If $x \in E$, there exists a positive integer n_0, such that $x \in B_{n_0}$ and, therefore, x is in the closure of A_{n_0}. Hence, there exists a $\lambda \in K, \lambda > 0$, such that $\lambda \times \chi \in \overline{U}$, i.e. $\overline{U}$ is a barrel in E, and therefore,

$U = \dot{U} \cap M$ is a neighborhood of the origin in M. Since Q has a countable basis, L is a subspace of M, of countable codimension, and by *d*), the codimension of L is infinite. The space M is, therefore, an example of quasi-barrelled $\mathfrak{DF}$-space which has a subspace L, of infinite countable codimension, so that L is not a $\mathfrak{DF}$-space.

We say that a subspace E of F is locally dense if, for every $x \in F$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E, which converges to x in the Mackey sense. In [9] we have proved the following result:

g) Let F *be a locally convex space. If* E *is a bornological locally dense subspace of* F, *then* F *is bornological.*

THEOREM 2. — *Let* E *be a bornological barrelled space which has a family* $\{E_n\}_{n=1}^{\infty}$ *of subspaces, which satisfy the following conditions:*

I. $\bigcup_{n=1}^{\infty} E_n = E.$

II. *For every positive integer* n, *there exists a topology* $\mathfrak{T}_n$ *on* E_n, *finer than the initial one, so that* $E_n[\mathfrak{T}_n]$ *is a Frechet space.*

III. — *There exists in* E *a bounded set* A, *such that* $A \not\subset E_n$, $n = 1, 2, \ldots$

Then there exists a bornological barrelled space F, *which is not inductive limit of Baire spaces, so that* E *is a hyperplane of* F.

Proof. — Let B be the closed, absolutely convex hull of A and let u be the canonical injection of E_B in E. If E_B is a Banach space, there exists, according to a theorem of Grothendieck, [4] or [5, p. 225], a positive integer n_1, such that $u(E_B) = E_B \subset E_{n_1}$, hence $A \subset E_{n_1}$, which is in contradiction with the condition III. We take in E_B a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ which is not convergent. Let $\hat{B}$ be the closure of B in the completion $\hat{E}$ of E. Since the topology of the Banach space $\hat{E}_{\hat{B}}$ induces in E_B a topology coarser than the initial one, $\{x_n\}_{n=1}^{\infty}$ converges in $\hat{E}_{\hat{B}}$ to an element

x. Since the set $M = \{x_1, x_2, \dots\}$ is bounded in E_B, there exists a $\lambda \in K$, such that $M \subset \lambda B$ and, therefore, if $x \in E$, then $x \in \lambda B \subset E_B$, hence, according to a result of N. Bourbaki, [5, p. 210-211], $\{x_n\}_{n=1}^{\infty}$ converges to x in E_B. This is a contradiction and, therefore, $x \notin E$. Let F be the space generated by $E \cup \{x\}$, equipped with the topology induced by $\hat{E}$. Obviously F is a barrelled space and, according to result *g*), F is bornological.

Finally we need to prove that F is not inductive limit of Baire spaces. Suppose that there exists in F a family $\{F_i : i \in I\}$ of subspaces, which union is F, so that for every $i \in I$, there exists a topology $\mathfrak{U}_i$ on F_i, such that $F_i[\mathfrak{U}_i]$ is a Baire space and F is the locally convex hull of $\{F_i[\mathfrak{U}_i] : i \in I\}$. Since E is a dense hyperplane of F, there exists an index $i_0 \in I$, such that $E \cap F_{i_0}$ is a dense hyperplane of $F_{i_0}[\mathfrak{U}_{i_0}]$. Let G be the vector space $E \cap F_{i_0}$ with the topology induced by $\mathfrak{U}_{i_0}$ and let x_0 be an element of F_{i_0}, which is not in G. If φ is the canonical injection of G in E, φ is continuous. Let G_n and H_n be the spaces $G \cap \varphi^{-1}(E_n)$ and that generated by $(G \cap \varphi^{-1}(E_n)) \cup \{x_0\}$, respectively, equipped with the topologies induced by $\mathfrak{U}_{i_0}$. Obviously $F_{i_0} = \bigcup_{n=1}^{\infty} H_n$ and, therefore, there exists a positive integer n_0 such that H_{n_0} is of the second category in $F_{i_0}[\mathfrak{U}_{i_0}]$. If φ_{n_0} is the restriction of φ to G_{n_0}, the graph of φ_{n_0} is closed in $G_{n_0} \times E_{n_0}[\mathfrak{T}_{n_0}]$ and, since G_{n_0} is barrelled and $E_{n_0}[\mathfrak{T}_{n_0}]$ is a Frechet space, φ_{n_0} is continuous from G_{n_0} into $E_{n_0}[\mathfrak{T}_{n_0}]$. If $\{y_m : m \in D\}$ is a net of elements of G_{n_0}, which converges to $y \in F_{i_0}[\mathfrak{U}_{i_0}]$, then $\{\varphi_{n_0}(y_m) = y_m : m \in D\}$ is a Cauchy net in the Frechet space $E_{n_0}[\mathfrak{T}_{n_0}]$, which converges to z, hence $y = z$ and G_{n_0} is closed in $F_{i_0}[\mathfrak{U}_{i_0}]$. Also H_{n_0} is closed in $F_{i_0}[\mathfrak{U}_{i_0}]$ and since H_{n_0} is of the second category in $F_{i_0}[\mathfrak{U}_{i_0}]$, then $H_{n_0} = F_{i_0}[\mathfrak{U}_{i_0}]$ and, therefore, $G_{n_0} = G$.

Finally, taking the net $\{y_m : m \in D\}$ converging to x_0, it results that $x_0 \in E$, which is not true. Hence F is not inductive limit of Baire spaces. Q.E.D.

Example 3. — G. Kothe has given an example of a non-complete (LB)-space, which is defined by a sequence $\{E_n\}_{n=1}^{\infty}$

of Banach spaces, so that there exists a bounded set A in E, which is not subset of E_n, $n = 1, 2, \ldots$ This example, and our Theorem 2, assure the existence of bornological barrelled spaces which are not inductive limits of Baire spaces.

BIBLIOGRAPHY

[1] N. Bourbaki, Éléments de Mathématiques, Livre V : Espaces vectoriels topologiques, (ch. iii, ch. iv, ch. v), Paris (1964).

[2] J. Dieudonné, Sur les propriétés de permanence de certains espaces vectoriels topologiques, *Ann. Soc. Polon. Math.*, 25, 50-55 (1952).

[3] A. Grothendieck, Sur les espaces (F) et (DF), *Summa Brasil. Math.*, 3, 57-123 (1954).

[4] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires *Mem. Math. Soc.*, 16 (1955).

[5] G. Kothe, Topological Vector Spaces I, Berlin-Heidelberg-New York, Springer (1969).

[6] M. Valdivia, A hereditary property in locally convex spaces, *Ann. Inst. Fourier*, 21, 1-2 (1971).

[7] M. Valdivia, On final topologies, *J. Reine angew. Math.*, 251, 193-199 (1971).

[8] M. Valdivia, On $\mathfrak{D}\mathfrak{F}$ spaces, *Math. Ann.*, 191, 38-43 (1971).

[9] M. Valdivia, A class of bornological barrelled spaces which are not ultrabornological, *Math. Ann.* 194, 43-51 (1971).

[10] M. Valdivia, Absolutely convex sets in barrelled spaces, *Ann. Inst. Fourier*, 21, 3-13 (1971).

Manuscrit reçu le 22 juin 1971.
accepté par J. Dieudonné

Manuel Valdivia,
Facultad de Ciencias,
Paseo Valencia al Mar, 13,
Valencia (Espagne).