

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 22, n° 2 (1972), p. 21-26

http://www.numdam.org/item?id=AIF_1972__22_2_21_0

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SOME EXAMPLES ON QUASI-BARRELLED SPACES ⁽¹⁾

by Manuel VALDIVIA

J. Dieudonné has proved in [2] the following theorem :

a) *Let E be a bornological space. If F is a subspace of E , of finite codimension, then F is bornological.*

We have given in [6] and [7], respectively, the following results :

b) *Let E be a quasi-barrelled space. If F is a subspace of E , of finite codimension, then F is quasi-barrelled.*

c) *Let E be an ultrabornological space. If F is a subspace of E , of infinite countable codimension, then F is bornological.*

The results a), b) and c) lead to the question if the results a) and b) will be true in the case of being F a subspace of infinite countable codimension. In this paper we give an example of a bornological space E , which has a subspace F , of infinite countable codimension, such that F is not quasi-barrelled.

In [8] we have proved the two following theorems :

d) *Let E be a \mathcal{DF} -space. If G is a subspace of E , of finite codimension, then G is a \mathcal{DF} -space.*

e) *Let E be a sequentially complete \mathcal{DF} -space. If G is a subspace of E , of infinite countable codimension, then G is a \mathcal{DF} -space.*

Another question is if the result d) is also true for subspaces of infinite countable codimension. Here we give an example of a quasi-barrelled \mathcal{DF} -space, which has a subspace G , of infinite countable codimension, which is not a \mathcal{DF} -space.

⁽¹⁾ Supported in part by the « Patronato para el Fomento de la Investigación en la Universidad ».

N. Bourbaki, [1, p. 35], notices that it is not known if every bornological barrelled space is ultrabornological. In [9] we have proved that if E is the topological product of an infinite family of bornological barrelled space, of non-zero dimension, there exists an infinite number of bornological barrelled subspaces of E , which are not ultrabornological. In this paper we give an example of a bornological barrelled space, which is not inductive limit of Baire spaces.

We use here vector spaces on the field K of real or complex numbers. The topologies on these spaces are separated.

In [10] we have proved the following result:

f) Let E be a barrelled space. If $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of subspaces of E , such that $\bigcup_{n=1}^{\infty} E_n = E$, then E is the inductive limit of $\{E_n\}_{n=1}^{\infty}$.

THEOREM 1. — *Let E be the strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of metrizable locally convex spaces. Let F be a sequentially dense subspace of E . If E is barrelled, then F is bornological.*

Proof. — Let \bar{E}_n , $n = 1, 2, \dots$, be the closure of E_n in E . Obviously E is the strict inductive limit of the sequence $\{\bar{E}_n\}_{n=1}^{\infty}$. Let F_n be the closure in E of $F \cap \bar{E}_n$, $n = 1, 2, \dots$. If $x \in E$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of F , which converges to x . Since the set of points of this sequence is bounded, there exists a positive integer n_0 such that $x_n \in \bar{E}_{n_0}$, $n = 1, 2, \dots$, and, therefore, $x \in F_{n_0}$. Hence $E = \bigcup_{n=1}^{\infty} F_n$. Since E is barrelled, applying the result *f)*, we obtain that E is the strict inductive limit of the sequence $\{F_n\}_{n=1}^{\infty}$.

Given any Banach space L and a linear and locally bounded mapping u from F into L , we must to prove that u is continuous. Let u_n be the restriction of u to $F \cap \bar{E}_n$. Since $F \cap \bar{E}_n$ is a metrizable space and u_n is locally bounded, u_n is continuous. Let φ_n be the continuous extension of u_n to F_n . Let φ be the linear mapping from E into L , which coincides with φ_n in F_n , $n = 1, 2, \dots$. Since φ_n is

equal to φ_{n+1} on $F \cap \overline{E}_n$, then they are equal on F_n and, therefore, φ is well defined. Since E is the inductive limit of $\{F_n\}_{n=1}^{\infty}$ and since the restriction of φ to F_n is continuous, $n = 1, 2, \dots$, then φ is continuous. On other hand u is the restriction of φ to F and, therefore, u is continuous. Q.E.D.

Example 1. — A. Grothendieck, [3], has given an example of a space E , which is strict inductive limit of an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of separable Frechet spaces, so that there exists in E a non-closed subspace G , such that $G \cap E_n$ is closed, $n = 1, 2, \dots$. In this example let A_n be a countable set of E_n , dense in E_n . Let P be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let F be the linear hull of $P \cup G$. Since P is sequentially dense in E , applying Theorem 1, it results that F is a bornological space. Applying theorem f) it results that G is not barrelled and since G is quasi-complete, then G is not quasi-barrelled. Since P has a countable basis, G is a subspace of F , of countable codimension, and by a) the codimension of G is infinite. Therefore, F is a bornological space, which has a subspace G , of infinite countable codimension, so that G is not quasi-barrelled.

Example 2. — G. Kothe, [4, p. 433-434] gives an example of a Montel \mathcal{DF} -space, which has a closed subspace L , which is not a \mathcal{DF} -space. In this example, let $\{B_n\}_{n=1}^{\infty}$ be a fundamental sequence of bounded sets. Since E is a Montel \mathcal{DF} -space, then B_n is separable, $n = 1, 2, \dots$. Let A_n be a countable subset of B_n , dense in B_n , $n = 1, 2, \dots$. Let Q be the linear space generated by $\bigcup_{n=1}^{\infty} A_n$. Let M be the linear hull of $Q \cup L$. Now, we shall prove that M is quasi-barrelled. Indeed, given a closed, absolutely convex and bornivorous set U in M , let \overline{U} be its closure in E . If $x \in E$, there exists a positive integer n_0 , such that $x \in B_{n_0}$ and, therefore, x is in the closure of A_{n_0} . Hence, there exists a $\lambda \in K$, $\lambda > 0$, such that $\lambda \times \chi \in \overline{U}$, i.e. \overline{U} is a barrel in E , and therefore,

$U = \bar{U} \cap M$ is a neighborhood of the origin in M . Since Q has a countable basis, L is a subspace of M , of countable codimension, and by d), the codimension of L is infinite. The space M is, therefore, an example of quasi-barrelled \mathcal{DF} -space which has a subspace L , of infinite countable codimension, so that L is not a \mathcal{DF} -space.

We say that a subspace E of F is locally dense if, for every $x \in F$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E , which converges to x in the Mackey sense. In [9] we have proved the following result:

g) Let F be a locally convex space. If E is a bornological locally dense subspace of F , then F is bornological.

THEOREM 2. — *Let E be a bornological barrelled space which has a family $\{E_n\}_{n=1}^{\infty}$ of subspaces, which satisfy the following conditions:*

$$I. \bigcup_{n=1}^{\infty} E_n = E.$$

II. *For every positive integer n , there exists a topology \mathcal{C}_n on E_n , finer than the initial one, so that $E_n[\mathcal{C}_n]$ is a Frechet space.*

III. — *There exists in E a bounded set A , such that $A \not\subset E_n$, $n = 1, 2, \dots$*

Then there exists a bornological barrelled space F , which is not inductive limit of Baire spaces, so that E is a hyperplane of F .

Proof. — Let B be the closed, absolutely convex hull of A and let u be the canonical injection of E_B in E . If E_B is a Banach space, there exists, according to a theorem of Grothendieck, [4] or [5, p. 225], a positive integer n_1 , such that $u(E_B) = E_B \subset E_{n_1}$, hence $A \subset E_{n_1}$, which is in contradiction with the condition III. We take in E_B a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ which is not convergent. Let \hat{B} be the closure of B in the completion \hat{E} of E . Since the topology of the Banach space \hat{E}_B induces in E_B a topology coarser than the initial one, $\{x_n\}_{n=1}^{\infty}$ converges in \hat{E}_B to an element

x . Since the set $M = \{x_1, x_2, \dots\}$ is bounded in E_B , there exists a $\lambda \in K$, such that $M \subset \lambda B$ and, therefore, if $x \in E$, then $x \in \lambda B \subset E_B$, hence, according to a result of N. Bourbaki, [5, p. 210-211], $\{x_n\}_{n=1}^\infty$ converges to x in E_B . This is a contradiction and, therefore, $x \notin E$. Let F be the space generated by $E \cup \{x\}$, equipped with the topology induced by \hat{E} . Obviously F is a barrelled space and, according to result g), F is bornological.

Finally we need to prove that F is not inductive limit of Baire spaces. Suppose that there exists in F a family $\{F_i: i \in I\}$ of subspaces, which union is F , so that for every $i \in I$, there exists a topology \mathcal{U}_i on F_i , such that $F_i[\mathcal{U}_i]$ is a Baire space and F is the locally convex hull of $\{F_i[\mathcal{U}_i]: i \in I\}$. Since E is a dense hyperplane of F , there exists an index $i_0 \in I$, such that $E \cap F_{i_0}$ is a dense hyperplane of $F_{i_0}[\mathcal{U}_{i_0}]$. Let G be the vector space $E \cap F_{i_0}$ with the topology induced by \mathcal{U}_{i_0} and let x_0 be an element of F_{i_0} , which is not in G . If ν is the canonical injection of G in E , ν is continuous. Let G_n and H_n be the spaces $G \cap \nu^{-1}(E_n)$ and that generated by $(G \cap \nu^{-1}(E_n)) \cup \{x_0\}$, respectively, equipped with the topologies induced by \mathcal{U}_{i_0} .

Obviously $F_{i_0} = \bigcup_{n=1}^\infty H_n$ and, therefore, there exists a positive integer n_0 such that H_{n_0} is of the second category in $F_{i_0}[\mathcal{U}_{i_0}]$. If ν_{n_0} is the restriction of ν to G_{n_0} , the graph of ν_{n_0} is closed in $G_{n_0} \times E_{n_0}[\mathcal{C}_{n_0}]$ and, since G_{n_0} is barrelled and $E_{n_0}[\mathcal{C}_{n_0}]$ is a Frechet space, ν_{n_0} is continuous from G_{n_0} into $E_{n_0}[\mathcal{C}_{n_0}]$. If $\{y_m: m \in D\}$ is a net of elements of G_{n_0} , which converges to $y \in F_{i_0}[\mathcal{U}_{i_0}]$, then $\{\nu_{n_0}(y_m) = y_m: m \in D\}$ is a Cauchy net in the Frechet space $E_{n_0}[\mathcal{C}_{n_0}]$, which converges to z , hence $y = z$ and G_{n_0} is closed in $F_{i_0}[\mathcal{U}_{i_0}]$. Also H_{n_0} is closed in $F_{i_0}[\mathcal{U}_{i_0}]$ and since H_{n_0} is of the second category in $F_{i_0}[\mathcal{U}_{i_0}]$, then $H_{n_0} = F_{i_0}[\mathcal{U}_{i_0}]$ and, therefore, $G_{n_0} = G$.

Finally, taking the net $\{y_m: m \in D\}$ converging to x_0 , it results that $x_0 \in E$, which is not true. Hence F is not inductive limit of Baire spaces. Q.E.D.

Example 3. — G. Kothe has given an example of a non-complete (LB)-space, which is defined by a sequence $\{E_n\}_{n=1}^\infty$

of Banach spaces, so that there exists a bounded set A in E , which is not subset of E_n , $n = 1, 2, \dots$. This example, and our Theorem 2, assure the existence of bornological barrelled spaces which are not inductive limits of Baire spaces.

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Manuscrit reçu le 22 juin 1971.
accepté par J. Dieudonné

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