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I. An obstruction to constructing PL-manifolds
from homology manifolds**

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**CONSTRUCTING MANIFOLDS
BY HOMOTOPY EQUIVALENCES I.
AN OBSTRUCTION
TO CONSTRUCTING PL-MANIFOLDS
FROM HOMOLOGY MANIFOLDS**

by Hajime SATO

0. Introduction.

A homology manifold can be given a canonical cell complex structure, where each cell is a contractible homology manifold. In this paper, given a homology manifold M , we aim at constructing a PL-manifold with a cell complex structure, where each cell is an acyclic PL-manifold, which is cellularly equivalent to the canonical cell complex structure of M . We obtain a theorem that, if the dimension n of M is greater than 4 and if the boundary ∂M is a PL-manifold or empty, there is a unique obstruction element in $H_{n-4}(M; \mathcal{H}^3)$, where \mathcal{H}^3 is the group of 3-dimensional PL-homology spheres modulo those which are the boundary of an acyclic PL-manifold. If the manifold is compact, the constructed PL-manifold is simple homotopy equivalent to M .

I have heard that similar results have been obtained independently and previously by M. Cohen and D. Sullivan, refer [1] and [9].

I would like to thank Professors V. Poénaru and F. Laudenbach for their kind support.

1. Definition of homology manifold with boundary⁽¹⁾.

Let K be a locally finite simplicial complex and let σ be a simplex of K . We define the subcomplexes of K as follows.

⁽¹⁾ We can refer the chapter 5 of the book : C.R.F. Maunder, "Algebraic topology", Van Nostrand, London (1970).

$$\begin{aligned} St(\sigma, K) &= St(\sigma) = \{\tau \in K, \exists \alpha > \tau, \alpha > \sigma\} \\ \partial St(\sigma, K) &= \partial St(\sigma) = \{\tau \in St(\sigma), \tau \triangleright \sigma\} \\ Lk(\sigma, K) &= Lk(\sigma) = \{\tau \in St(\sigma), \tau \cap \sigma = \emptyset\} \end{aligned}$$

We write by K' , K'' , the first and the second barycentric subdivisions of K .

Let M be a locally finite full simplicial complex of dimension n . We say that M is a homology manifold of dimension n if the following equivalent condition holds :

LEMMA 1. — *The followings are equivalent :*

i) *for any simplex σ of dimension p ,*

$$\tilde{H}_i(Lk(\sigma, M)) = \tilde{H}_i(S^{n-p-1}) \quad \text{or} \quad 0.$$

ii) *for any simplex σ of dimension p ,*

$$\tilde{H}_i(St(\sigma, M)/\partial St(\sigma, M)) = \tilde{H}_i(S^n) \quad \text{or} \quad 0.$$

iii) *for any point x of $|M|$, where $|M|$ denotes the underlying topological space of M ,*

$$H_i(|M|, |M| - x) = \tilde{H}_i(S^n) \quad \text{or} \quad 0.$$

The definition is invariant by the PL-homeomorphism in the category of simplicial complexes.

LEMMA 2. — *For any p -simplex σ of M , $Lk(\sigma, M)$ is a compact $(n - p - 1)$ -dimensional homology manifold.*

Proof. — It is compact because M is locally finite. Let τ be a q -simplex of $Lk(\sigma, M)$. We have

$$Lk(\tau, Lk(\sigma, M)) = Lk(\tau\sigma, M).$$

Hence $\tilde{H}_i(Lk(\tau, Lk(\sigma, M))) = \tilde{H}_i(S^{n-p-q-1})$ or 0, which completes the proof.

Let us define the subset ∂M of M by

$$\partial M = \{\sigma \in M \mid \tilde{H}_i(\sigma, M) = 0\}$$

We call it as the boundary of M . If $\partial M = \emptyset$, the manifold is classical

and the following Poincaré duality is well known (see for example [7 ; (7,4)]).

LEMMA 3. — *Let M be an orientable compact n-dimensional homology manifold without boundary. Let $A_1 \supset A_2$ be subcomplexes of M. Then we have the isomorphism*

$$H^i(A_1, A_2) = H_{n-i}(|M| - |A_2|, |M| - |A_1|).$$

Using this we will prove the followings. By lemma 2, for p-simplex σ , $Lk(\sigma, M)$ is a homology manifold and we can define $\partial Lk(\sigma, M)$.

LEMMA 4. — *If $\partial Lk(\sigma, M) \neq \emptyset$, $Lk(\sigma, M)$ is acyclic and $\partial Lk(\sigma, M)$ is an $(n - p - 2)$ -dimensional homology manifold such that*

$$\tilde{H}_i(\partial Lk(\sigma, M)) = \tilde{H}_i(S^{n-p-2}).$$

PROPOSITION 5. — *If $\partial M \neq \emptyset$, ∂M is a subcomplex and is an $(n - 1)$ -dimensional homology manifold without boundary.*

We prove that lemma 4 for $n = k$ implies proposition 5 for $n = k$ and proposition 5 for $n \leq k$ implies lemma 4 for $n = k + 1$. Since lemma 4 holds for $n = 1$, we can continue by induction.

Lemma $4_{n=k} \Rightarrow$ Proposition $5_{n=k}$. Let σ be a p-simplex of ∂M and let $\sigma_0 < \sigma$. Then we can write $\sigma = \sigma_0 \sigma_1$. We have

$$\tilde{H}_*(Lk(\sigma_1, Lk(\sigma_0, M))) = \tilde{H}_*(Lk(\sigma, M)) = 0,$$

which shows that $\sigma_1 \in \partial Lk(\sigma_0, M)$ and so $\partial Lk(\sigma_0, M) \neq \emptyset$. By the lemma 4, $Lk(\sigma_0, M)$ is acyclic and it follows that $\sigma_0 \in \partial M$. Hence ∂M is a well-defined subcomplex of M. A q-simplex τ of $Lk(\sigma, M)$ is in $Lk(\sigma, \partial M)$ if and only if $\tilde{H}_i(Lk(\tau\sigma, M)) = 0$. Since

$$Lk(\tau\sigma, M) = Lk(\tau, Lk(\sigma, M)),$$

it is equivalent to that τ belongs to $\partial Lk(\sigma, M)$. Hence the complex $Lk(\sigma, \partial M)$ coincides with $\partial Lk(\sigma, M)$. By lemma 4_k , we have $\tilde{H}_i(\partial Lk(\sigma, M)) = \tilde{H}_i(S^{k-p-2})$, which shows that ∂M is a $(k - 1)$ -dimensional homology manifold without boundary.

Proposition $5_{n \leq k} \Rightarrow$ Lemma $4_{n=k+1}$. Let M be a homology manifold of dimension $k + 1$. Let σ be a p-simplex of M. By lemma 2,

$Lk(\sigma, M)$ is a homology manifold of dimension $k - p$. By proposition 5 for $n = k - p$, $\partial Lk(\sigma, M)$ is a $(k - p - 1)$ -dimensional homology manifold without boundary if it is not empty. Let $2Lk(\sigma, M)$ be the double of $Lk(\sigma, M)$, i.e.,

$$2Lk(\sigma, M) = Lk(\sigma, M) \cup_{\partial Lk(\sigma, M)} Lk(\sigma, M) .$$

Let τ be a q -simplex of $2Lk(\sigma, M)$. If τ is not a simplex of $\partial Lk(\sigma, M)$, clearly,

$$\tilde{H}_i(Lk(\tau, 2Lk(\sigma, M))) = \tilde{H}_i(Lk(\tau, Lk(\sigma, M))) = \tilde{H}_i(S^{k-p-q-1}) .$$

If τ is a simplex of $\partial Lk(\sigma, M)$, we have

$$Lk(\tau, 2Lk(\sigma, M))$$

$$= Lk(\tau, Lk(\sigma, M)) \cup_{Lk(\tau, \partial Lk(\sigma, M))} Lk(\tau, Lk(\sigma, M)) .$$

By definition $\tilde{H}_i(Lk(\tau, Lk(\sigma, M))) = 0$ and by the proposition 5 for $n = k - p - 1$, we have

$$\tilde{H}_i(Lk(\tau, \partial Lk(\sigma, M))) = \tilde{H}_i(S^{k-p-q-2}) .$$

Hence in any case $\tilde{H}_i(Lk(\tau, 2Lk(\sigma, M))) = \tilde{H}_i(S^{k-p-q-1})$, which shows that $2Lk(\sigma, M)$ is a $(k - p)$ -dimensional homology manifold without boundary. Applying lemma 3, we have

$$H^i(Lk(\sigma, M), \partial Lk(\sigma, M)) = H_{k-p-i}(|Lk(\sigma, M)| - |\partial Lk(\sigma, M)|) .$$

Notice that for any homology manifold M , $H_i(|M| - |\partial M|) = H_i(M)$. Hence $H^i(Lk(\sigma, M), \partial Lk(\sigma, M)) = H_{k-p-i}(S^{k-p})$ or $H_{k-p-i}(pt.)$. But if it is isomorphic to $H_{k-p-i}(S^{k-p})$, we have

$$H^0(Lk(\sigma, M), \partial Lk(\sigma, M)) = Z ,$$

which contradicts to the definition that $\tilde{H}_0(Lk(\sigma, M)) = 0$. Hence $Lk(\sigma, M)$ is acyclic and consequently $\tilde{H}_i(\partial Lk(\sigma, M)) = \tilde{H}_i(S^{k-p-1})$, which completes the proof.

2. Cell decomposition of a homology manifold.

We mean by a homology cell (resp. pseudo homology cell) of dimension n or homology n -cell (resp. pseudo homology n -cell) a

compact contractible (resp. acyclic) homology manifold of dimension n with a boundary, the boundary being a homology sphere but not necessarily simply connected. A (pseudo) homology cell complex is a complex K with a locally finite family of (pseudo) homology cells $C = \{C_\alpha\}$, such that :

- i) $K = \cup C_\alpha$
- ii) $C_\alpha, C_\beta \in C$ implies $\partial C_\alpha, C_\alpha \cap C_\beta$ are unions of cells in C
- iii) If $\alpha \neq \beta$, then $\text{Int } C_\alpha \cap \text{Int } C_\beta = \emptyset$.

If a homology manifold M has a (pseudo) homology cell complex structure, we call it a (pseudo) cellular decomposition of M . Two (pseudo) homology cell complexes $K = \cup C_\alpha, K' = \cup C'_\alpha$ are isomorphic if there exists a bijection $k : C \rightarrow C'$ such that both k and k^{-1} are incidence preserving. In such a case we say that they are cellularly equivalent.

Now we have the following :

PROPOSITION 1. — *If two finite homology cell complexes K, K' are cellularly equivalent, then they are simple homotopy equivalent.*

We can define a simplicial map $f : K \rightarrow K'$ inductively by the dimension of the cells. Hence it is sufficient to prove the following lemma.

LEMMA 2. — *Let $A_j^i (j = 1, 2, \dots, r)$ be subcomplex of simplicial complexes B^i for $i = 1, 2$ respectively such that $B^i = \cup_j A_j^i$, and let $f : B^1 \rightarrow B^2$ be a simplicial map. For any subset s of $\{1, 2, \dots, r\}$, let $A_s^i = \cap_{j \in s} A_j^i$ and let f_s be the restriction of f on A_s^1 . If f_s is a mapping from A_s^1 to A_s^2 which is a simple homotopy equivalence for any s , then f itself is a simple homotopy equivalence.*

Proof. — First suppose that $r = 2$. We have the exact sequence

$$0 \rightarrow C_*(A_1^i) \rightarrow C_*(B^i) \rightarrow C_*(A_1^i \cap A_2^i) \rightarrow 0$$

of the chain complexes. Let $g : A_2^1/(A_1^1 \cap A_2^1) \rightarrow A_2^2/(A_1^2 \cap A_2^2)$ be the map induced by f and let us denote by $w()$ the Whitehead torsion. Then by theorem 10 of [8], we have

$$w(f) = w(f_{\{1\}}) + w(g) .$$

Remark here that f and g can easily be seen to be homotopy equivalences. Further we have the exact sequence

$$0 \rightarrow C_*(A_1^i \cap A_2^i) \rightarrow C_*(A_2^i) \rightarrow C_*(A_2^i/(A_1^i \cap A_2^i)) \rightarrow 0$$

which shows that

$$w(f_{\{2\}}) = w(f_{\{1,2\}}) + w(g) .$$

Since $w(f_{\{1\}}) = w(f_{\{2\}}) = w(f_{\{1,2\}}) = 0$, we have $w(f) = 0$. If $r \geq 3$, we can repeat this argument, which shows that f is a simple homotopy equivalence for any r .

Now let σ be a simplex of a locally finite simplicial complex K . We denote by $b_\sigma \in K'$ its barycenter. We define dual complex $D(\sigma)$ and its subcomplex $\delta D(\sigma)$ which are subcomplexes of K' by

$$D(\sigma) = D(\sigma, K) = \{b_{\sigma_0} \dots b_{\sigma_r} \mid \sigma < \sigma_0 < \dots < \sigma_r \in K\}$$

$$\delta D(\sigma) = \delta D(\sigma, K) = \{b_{\sigma_0} \dots b_{\sigma_r} \mid \sigma \not\leq \sigma_0 < \dots < \sigma_r \in K\}$$

The followings are easy to see.

- i) if $\sigma < \sigma' \Rightarrow D(\sigma) \supset D(\sigma')$
- ii) $D(\sigma) = b_\sigma * \delta D(\sigma)$
- iii) $\delta D(\sigma) = \bigcup_{\tau} D(\tau)$ where $\tau > \sigma$ and $\tau \neq \sigma$
- iv) $\delta D(\sigma)$ is isomorphic to $Lk(\sigma, K)'$.

Let M be a homology manifold. For each simplex

$$\sigma = b_{\sigma_0} b_{\sigma_1} \dots b_{\sigma_r}$$

of M' , where $\sigma_0^{n_0} < \sigma_1^{n_1} < \dots < \sigma_r^{n_r}$ are a set of simplexes of M , we have the dual cell $D(\sigma, M')$. It is a compact homology manifold by lemma 2 of § 1. Further we have

$$\begin{aligned} \delta D(\sigma, M') &\cong Lk(\sigma, M) \\ &\cong Lk(\sigma, \sigma_r) * Lk(\sigma_r, M) \\ &\cong S^{n_r - r - 1} * Lk(\sigma_r, M) \\ &\cong Lk(\sigma_r, M) \times D^{n_r - r} \cup (Lk(\sigma_r, M) * (pt.)) \times S^{n_r - r - 1} , \end{aligned}$$

where \cong denotes that both sides are PL-homeomorphic and let

$$d_\sigma : \delta D(\sigma, M') \rightarrow Lk(\sigma_r, M) \times D^{n_r-r} \cup (Lk(\sigma_r, M) * (pt.)) \times S^{n_r-r-1}$$

be the PL-homeomorphism, which we call the trivialization of $\delta D(\sigma, M')$. If σ is not in ∂M , $\delta D(\sigma, M')$ is a homology manifold whose homology groups are isomorphic to those of S^{n-1} , boundary being empty. If $\sigma \in \partial M$, $\delta D(\sigma, M')$ is an acyclic homology manifold with the boundary $Lk(\sigma, \partial M'')$ which is PL-homeomorphic to $\partial Lk(\sigma_r, M) \times D^{n_r-r} \cup (\partial Lk(\sigma_r, M) * (pt.)) \times S^{n_r-r-1}$. The union $St(\sigma, \partial M'') \cup \delta(\sigma, M') = \partial D(\sigma, M')$ is a homology manifold without boundary whose homology groups are isomorphic to those of S^{n-1} . Hence in any case $D(\sigma, M')$ is a homology cell. The union $\cup D(\sigma, M')$, σ moving all simplexes of M' , gives the cellular decomposition of M , which we call the canonical one.

We define the handle M_i of index i by the disjoint union

$$M_i = \cup D(b_{\sigma_{n-i}})$$

where σ changes all $(n - i)$ -simplexes of M . We have $\delta D(b_\sigma) = \cup D(\tau)$, where $\sigma < \tau \in M'$, and it gives a cellular decomposition of M_i . We can divide the boundary as $\delta D(b_\sigma) = LD(b_\sigma) \cup HD(b_\sigma)$, which consists of unions of celles attached to the handles of lower indexes and higher indexes. We define them as

$$\begin{aligned} LD(b_\sigma) &= \delta D(b_\sigma) \cap \left(\bigcup_{j < i} M_j \right) \\ HD(b_\sigma) &= \delta D(b_\sigma) \cap \left(\bigcup_{j > i} M_j \right). \end{aligned}$$

Let $\tau = b_{\tau_0} b_{\tau_1} \dots b_{\tau_r} \neq \sigma$ be a simplex of M' , where

$$\tau_0^{m_0} < \tau_1^{m_1} \dots < \tau_r^{m_r} \in M.$$

Then $D(\tau) \in LD(b_\sigma)$ if and only if $\tau_r > \sigma$ and $D(\tau) \in HD(b_\sigma)$ if and only if $\tau_0 < \sigma$. It is easy to see that

$$\begin{aligned} LD(b_\sigma) &\cong Lk(\sigma, M) \times D^{n-i} \\ HD(b_\sigma) &\cong (Lk(\sigma, M) * (pt.)) \times S^{n-i-1}, \end{aligned}$$

and these isomorphism together give the trivialization d_{b_σ} of $\delta D(b_\sigma)$.

Let Δ^{n-i} be the standard $(n-i)$ -simplex and let

$$\partial\Delta^{n-i} = S^{n-i-1} = \bigcup_{\alpha} C_{\alpha}$$

be the cell decomposition defined as above, which we call the standard decomposition of S^{n-i-1} . The decomposition

$$HD(b_{\sigma}) = \bigcup D(\tau) .$$

is equal to the standard product decomposition

$$\{Lk(\sigma, M) * (pt.)\} \times \left(\bigcup_{\alpha} C_{\alpha}\right) .$$

All the cells of $HD(b_{\sigma})$ which is not contained in $LD(b_{\sigma}) \cap HD(b_{\sigma})$ is written as

$$(Lk(\sigma, M) * (pt.)) \times C_{\alpha} .$$

Finally we define $M_{(i)}$ the subcomplex of M composed of handles whose indexes are inferior or equal to i , that is,

$$M_{(i)} = \bigcup_{j \leq i} M_j \subset M .$$

Then we have

$$M_{(i)} = M_{(i-1)} \cup M_i$$

attached on $\bigcup_{\sigma} LD(b_{\sigma})$, σ being $(n-i)$ -simplexes.

3. PL-homology spheres.

We call an n -dimensional homology manifold whose homology groups are isomorphic to those of S^n a homology n -sphere or homology sphere of dimension n . If it is a PL-manifold, it is called a PL-homology n -sphere.

If dimension is smaller than 3, a homology sphere is the natural sphere. And so any 3-dimensional homology manifold is a PL-manifold. In order to study higher dimensional cases we define the group $\mathcal{H}e^3$.

Let X^3 be the set of oriented 3-dimensional PL-homology spheres. Note that any homology sphere is orientable. We say that $H_1^3 \in X^3$ is equivalent to $H_2^3 \in X^3$ if $H_1^3 \# (-H_2^3)$ is the boundary of an acyclic PL-manifold, where $\#$ denotes the connected sum and

– H_2^3 is H_2^3 with the orientation inversed. Let $\mathcal{H}e^3 = X^3/\sim$ be the set of equivalence classes. By the connected sum operation, $\mathcal{H}e^3$ is an abelian group. Let G be the binary dodecahedral group. The quotient space S^3/G is a PL-homology sphere whose class in $\mathcal{H}e^3$ is non trivial.

On the contrary, for higher dimensions the following is known [2] [6] [4].

PROPOSITION 1 (Hsiang-Hsiang, Tamura, Kervaire). – *Any PL-homology sphere is the boundary of a contractible PL-manifold, if the dimension is greater than 3.*

We will prove the followings, where x is a point in S^i , $i \geq 1$.

PROPOSITION 2. – *Let $H^3 \in X^3$, then $H^3 \times S^1$ is the boundary of a PL-manifold K^5 such that $H_*(K) \cong H_*(S^1)$ and the inclusion*

$$j : S^1 \hookrightarrow \{x\} \times S^1 \hookrightarrow H^3 \times S^1 \hookrightarrow K$$

induce an isomorphism of the fundamental groups.

PROPOSITION 3. – *Let $H^3 \in X^3$ and let $i \geq 2$. Then $H^3 \times S^i$ is the boundary of a PL-manifold K^{4+i} such that the inclusion*

$$j : S^i \hookrightarrow \{x\} \times S^i \hookrightarrow H^3 \times S^i \hookrightarrow K$$

induces a homotopy equivalence.

Proof of Proposition 2. – Since any orientable closed 3-dimensional PL-manifold is a boundary of a 4-dimensional parallelizable PL-manifold (See by example [3]), we have a parallelizable PL-manifold L^4 such that $\partial L = H$. By doing surgery we can assume that $\pi_1(L) = 0$. By the Poincaré duality theorem, $H_2(L)$ is free abelian. Let $p : L \times S^1 \rightarrow S^1$ be the projection. Then it induces an isomorphism of the fundamental groups. Remark that if we have a manifold K with boundary $H^3 \times S^1$ such that $H_2(K) \cong 0$ and the inclusion $j : S^1 \hookrightarrow K$ induces the isomorphism of the fundamental groups, then, by the Poincaré duality, we have $H_i(K) = 0$ for $i \geq 2$. Hence it is sufficient to kill $H_2(L \times S^1)$. Since $H_2(L)$ is free, so is $H_2(L \times S^1)$. We can follow the method of lemma 5.7 of Kervaire-Milnor [5]. Since $\pi_1(L) = 0$, the Hurewicz map of L , $\pi_2(L) \rightarrow H_2(L)$, is isomorphic,

and so is the Hurewicz map of $L \times S^1$

$$h : \pi_2(L \times S^1) \rightarrow H_2(L \times S^1) .$$

Hence we can represent any element of $H_2(L \times S^1)$ by an embedded sphere. In our case the boundary $\partial(L \times S^1)$ is $H^3 \times S^1$ and it does not satisfy the hypothesis of that lemma. But since we have

$$H_2(\partial(L \times S^1)) = 0 ,$$

the result is the same.

Proof of Proposition 3. — Let K^5 be the 5-dimensional PL-manifold of proposition 2. Attach K with $H^3 \times D^2$ by the identity map on $H^3 \times S^1$. The constructed manifold W^5 is a simply connected PL-homology sphere, and by the generalized Poincaré conjecture, it is the natural sphere S^5 . It shows that we can embed H^3 in S^5 with a trivial normal bundle. By composing with the natural embedding $S^5 \hookrightarrow S^{4+i}$, we have an embedding of H^3 in S^{4+i} with the trivial normal bundle. The manifold N which is the complement of the open regular neighbourhood of H^3 in S^{4+i} has $H^3 \times S^i$ as the boundary and the inclusion $j : S^i \hookrightarrow N$ induces an isomorphism of homology groups, hence homotopy equivalence, which completes the proof.

4. An obstruction to constructing PL-manifold.

Let M be a homology manifold of dimension greater than 4. We assume that the boundary ∂M is a PL-manifold if it is not empty. As in § 2, it has the handle decomposition

$$M = M_{(n)} = \bigcup_{0 \leq i \leq n} M_i$$

which has also the canonical homology cell complex structure. We want to construct a PL-manifold with a pseudo homology cell complex structure which is cellularly equivalent to M . Since $M_{(3)}$ is a PL-manifold, a problem first arises when we attach handles of index 4.

Let σ be an $(n - 4)$ -simplex in the interior of M . Then $Lk(\sigma, M)$ is a 3-dimensional PL-homology sphere. Connecting σ by a path from

a fixed base point of M , we can give the orientation for the neighbourhood of σ , and hence for $Lk(\sigma, M)$.

Let $Lk(\sigma, M)$ be the class in the group \mathfrak{H}^3 . To each $(n - 4)$ -simplex σ of M , we define a function $\lambda(M) : \{(n - 4)\text{-simplex}\} \rightarrow \mathfrak{H}^3$ by

$$\lambda(M)(\sigma) = \begin{cases} \{Lk(\sigma, M)\} & \text{if } \sigma \in \text{Int. } M \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda(M)$ is an element of the chain group $C_{n-4}(M, \mathfrak{H}^3)$. The coefficient may be twisted if the manifold is not orientable.

LEMMA 1. — $\lambda(M)$ is a cycle.

Proof. — Let μ be an $(n - 5)$ -simplex. In the homology 4-sphere $Lk(\mu)$, the complex $\cup Lk(\sigma_i) * (x_i)$, where x_i denotes the barycenter of the 1-simplex $b_\mu b_{\sigma_i}$ and the sum extends to all the $(n - 4)$ -simplexes such that $\sigma_i > \mu$, is a subcomplex whose complement in $Lk(\mu)$ is a PL-manifold. So the connected-summed PL-manifold $\Sigma Lk(\sigma_i)$ bounds an acyclic PL-manifold.

Hence $\lambda(M)$ represents an element $\{\lambda(M)\}$ of $H_{n-4}(M, \mathfrak{H}^3)$. Now we have the theorem :

THEOREM. — Let M^n be a homology manifold with the dimension $n > 4$. Assume that ∂M is a PL-manifold if $\partial M \neq \emptyset$. If the obstruction class

$$\{\lambda(M)\} \in H_{n-4}(M, \mathfrak{H}^3)$$

is zero, then there exists a PL-manifold N with a pseudo homology cell decomposition which is cellularly equivalent to M .

Proof. — Since $\{\lambda(M)\} = 0$, there exists a correspondance

$$g : \{(n - 3)\text{-simplex}\} \rightarrow \mathfrak{H}^3$$

such that

$$\sum_{\tau_i > \sigma} g(\tau_i) = \{Lk(\sigma, M)\} \in \mathfrak{H}^3 .$$

We will inductively construct PL-manifolds N_p and $N_{(p)} = \bigcup_{q \leq p} N_q$ with a pseudo homology cell decomposition $N_p = \cup E_\alpha$ where all

pseudo cells are PL-manifolds such that $N_{(p)}$ is cellularly equivalent to $M_{(p)}$.

(a) $p \leq 2$. In this case, the manifolds N_p , $N_{(p)}$ and their cells are just equal to M_p , $M_{(p)}$ and their cells. That is, for any j -simplex σ , $j \geq n - 2$, we define the PL-manifolds as

$$E(b_\sigma) = D(b_\sigma)$$

$$N_p = \cup \{E(b_\sigma) \mid \dim \sigma = n - p\} = \cup \{D(b_\sigma) \mid \dim \sigma = n - p\} = M_p$$

For any simplex $\mu \in M'$ such that $\mu > b_\sigma$, we put

$$E(\mu) = D(\mu).$$

Hence $\partial E(b_\sigma) = \partial D(b_\sigma) = \cup D(\mu) = \cup E(\mu)$, and $N_{(p)} = M_{(p)}$.

(b) $p = 3$. Let τ_i be an $(n - 3)$ -simplex. Let H_i^3 be the 3-dimensional PL-homology sphere which represents $g(\tau_i)$ and let K_i be the PL-manifold whose boundary is $H_i^3 \times S^{n-4}$ such that the inclusion $j : S^{n-4} \hookrightarrow K_i$ induces the isomorphisms of the fundamental groups and the homology groups, whose existence is shown by propositions 2 and 3 of § 3. Let $D^3 \subset H_i^3$ be a disc. Then $D^3 \times S^{n-4} \subset \partial K_i$. We have the PL-homeomorphism $\partial D(b_{\tau_i}) = S^2 \times D^{n-3} \cup D^3 \times S^{n-4}$. We define the PL-manifolds $E(b_{\tau_i})$ and N_3 by

$$E(b_{\tau_i}) = D(b_{\tau_i}) \cup_{D^3 \times S^{n-4}} K_i$$

$$N_3 = \cup_i E(b_{\tau_i})$$

where $D(b_{\tau_i})$ is attached to K_i by the identity map on $D^3 \times S^{n-4}$. It is easy to see that $E(b_{\tau_i})$ is a homology cell. We will give the pseudo cell decomposition for $\partial E(b_{\tau_i})$. First we divide $\partial E(b_{\tau_i})$ as the union $\partial E(b_{\tau_i}) = LE(b_{\tau_i}) \cup HE(b_{\tau_i})$, where

$$LE(b_{\tau_i}) = \partial D(b_{\tau_i}) - D^3 \times D^{n-3}$$

$$HE(b_{\tau_i}) = \partial K_i - D^3 \times S^{n-4} = (H_i^3 - D^3) \times S^{n-4}.$$

Since $LE(b_{\tau_i}) = LD(b_{\tau_i})$, we give the cell decomposition by that of $LD(b_{\tau_i})$. We give the pseudo cell decomposition in the interior of $HE(b_{\tau_i})$ as

$$(H_i^3 - D^3) \times S^{n-4} = (H_i^3 - D^3) \times \left(\bigcup_{\alpha} C_{\alpha} \right) = \bigcup_{\alpha} (H_i^3 - D^3) \times C_{\alpha} ,$$

where $S^{n-4} = \bigcup C_{\alpha}$ is the standard decomposition. These decompositions of $LE(b_{\tau_i})$ and $HE(b_{\tau_i})$ fit together on their intersection and give the decomposition of $\partial E(b_{\tau_i})$, which is clearly cellular equivalent to that of $\partial D(b_{\tau_i})$. For each simplex $\mu > b_{\tau_i}$, $\mu \in M'$, we denote by $E(\mu)$ the pseudo cell of $\partial E(b_{\tau_i})$ which corresponds by the equivalence to $D(\mu) \in \partial D(b_{\tau_i})$. We have $\partial E(b_{\tau_i}) = \bigcup E(\mu)$. We define $N_{(3)}$ by

$$N_{(3)} = N_{(2)} \cup N_3$$

attached by the identity on $LE(b_{\tau_i})$. $N_{(3)}$ is cellularly equivalent to $M_{(3)}$.

(c) $p = 4$. Let σ be a $(n - 4)$ -simplex. Let $\bigcup E(\mu) \subset \partial N_{(3)}$ be the union of pseudo cells such that $b_{\sigma} < \mu \in M'$, $\mu \neq b_{\sigma}$. Then by the definition, it is PL-homeomorphic to the PL-manifold

$$(Lk(\sigma) \# \Sigma(-H_i^3)) \times D^{n-4}$$

where H_i^3 represents $g(\tau_i)$ and the sum extends to all $\tau_i > \sigma$.

Since $\{Lk(\sigma)\} = \Sigma g(\tau_i)$ in \mathcal{H}^3 , the PL-homology 3-sphere

$$H_{\sigma}^3 = Lk(\sigma) \# \Sigma(-H_i^3)$$

is the boundary of an acyclic PL-manifold W_{σ}^4 . The union

$$W_{\sigma}^4 \times S^{n-5} \cup H_{\sigma}^3 \times D^{n-4}$$

is a PL-homology $(n - 1)$ -sphere. By the proposition 1 of § 3, it is the boundary of a contractible PL-manifold Y_{σ} . We define the PL-manifolds $E(b_{\sigma})$ and N_4 as

$$\begin{aligned} E(b_{\sigma}) &= Y_{\sigma} \\ N_4 &= \bigcup E(b_{\sigma}) . \end{aligned}$$

Further we define $LE(b_{\sigma})$ and $HE(b_{\sigma})$ by

$$\begin{aligned} LE(b_{\sigma}) &= H_{\sigma}^3 \times D^{n-4} \\ HE(b_{\sigma}) &= W_{\sigma}^4 \times S^{n-5} . \end{aligned}$$

The pseudo cellular decomposition for $LE(b_{\sigma})$ is already defined and we give for $HE(b_{\sigma})$ by the product with the standard decomposition

of S^{n-5} . They give a pseudo cellular decomposition of

$$\partial E(b_\sigma) = LE(b_\sigma) \cup HE(b_\sigma),$$

which is cellularly equivalent to that of $\partial D(b_\sigma)$. For each simplex $\mu > b_\sigma, \mu \in M'$, we define $E(\mu)$ by the pseudo cell which corresponds to $D(\mu)$ by this equivalence. We define $N_{(4)}$ by $N_{(3)} \cup N_4$ attached by the identity of $LE(b_\sigma)$, which is cellularly equivalent to $M_{(4)}$.

(d) $p \geq 5$. Let σ be a j -simplex $j \leq n - 5$. Let $\cup E(\mu) \subset \partial N_{(n-j-1)}$ be the union of pseudo cells such that $\mu > b_\sigma, \mu \neq b_\sigma$. Then by our definition, it is a PL-manifold

$$H_\sigma^{p-1} \times D^{n-p}$$

where H_σ^{p-1} is a PL-homology $(p-1)$ -sphere, where $p = n - j$. By the proposition 1 of § 3, H^{p-1} is the boundary of a contractible PL-manifold W_σ^p . We define $E(b_\sigma)$ by

$$E(b_\sigma) = W_\sigma^p \times D^{n-p}.$$

The other definitions are just similar to the case when $p = 4$.

Continuing this process, we obtain a PL-manifold $N = N_{(n)}$ which is cellularly equivalent to $M = M_{(n)}$. Q.E.D.

5. Simple homotopy equivalence.

By the theorem of § 4, for the same M , if the obstruction class is 0, we can construct a PL-manifold N . In this section, we prove the following.

THEOREM. — *If M is compact, the constructed manifold N is simple homotopy equivalent to M .*

Let $M^{(k)}$ denote the k -skelton of M . Let L be a subcomplex of $M^{(k)}$, we define the PL-submanifold $N^{(L)}$ of N by

$$N^{(L)} = \cup \{E(b_\sigma) \mid \sigma \in L\}.$$

We put

$$N^{(k)} = N^{(M^{(k)})} = \cup \{E(b) \mid \sigma \in M^{(k)}\}.$$

By the induction of k , we prove the stronger

LEMMA 1. — *There exists a simple homotopy equivalence*

$$f : M^{(k)} \rightarrow N^{(k)}$$

such that, for any $(k + 1)$ -simplex μ , $f(\partial\mu) \subset N^{(\partial\mu)}$ and

$$f|_{\partial\mu} : \partial\mu \rightarrow N^{(\partial\mu)}$$

is a simple homotopy equivalence.

Proof. — If $k = 0$, it holds obviously. Now we will prove the lemma for $k + 1$ assuming the lemma for k . Let μ be a $(k + 1)$ -simplex. Since the collar of $\partial\mu$ is PL-homeomorphic to $S^k \times I$, we can write

$$\mu = S^k \times I \cup S^k * (b_\mu)$$

where $S_0^k = S^k \times \{0\} = \partial\mu$ and $S_1^k = S^k \times \{1\} = S^k \times I \cap S^k * (b_\mu)$. Recall that

$$N^{(M^{(k)} \cup \mu)} = N^{(k)} \cup E(b_\mu)$$

$$N^{(k)} \cap E(b_\mu) = N^{(\partial\mu)} \cap E(b_\mu) = HE(b_\mu) = W_\mu^{n-k-1} \times S^k$$

where W_μ^{n-k-1} is an acyclic (or contractible) PL-manifold. Let x be a point in the interior of W_μ and let $d : S^k \rightarrow W_\mu \times S^k$ be the embedding defined by $d(S^k) = \{x\} \times S^k$. We define a map

$$\tilde{f} : S_0^k \cup S_1^k \rightarrow N^{(k)}$$

by

$$\tilde{f}|_{S_0^k} = f$$

$$\tilde{f}|_{S_1^k} = d.$$

Since $\tilde{f}|_{\partial M}$ gives a simple homotopy equivalence $\partial\mu \rightarrow N^{(\partial\mu)}$, $N^{(\partial\mu)}$ is homotopy equivalent to S^k , and so $\tilde{f}|_{S_0^k}$ and $\tilde{f}|_{S_1^k}$ are homotopic. Hence we can extend \tilde{f} on $S^k \times I$. Further since $E(b_\mu)$ is contractible, we can extend \tilde{f} to a map from $\mu = S^k \times I \cup S^k * (b_\mu)$ to $N^{(M^{(k)} \cup \mu)}$. By the definition, f and \tilde{f} coincide on $\partial\mu$, and so we have a map

$$g = f \cup \tilde{f} : M^{(k)} \cup \mu \rightarrow N^{(M^{(k)} \cup \mu)}.$$

Repeating this for all $(k + 1)$ -simplexes of M , we obtain a map $g : M^{(k+1)} \rightarrow N^{(k+1)}$. We have the exact sequences of chain groups,

$$0 \rightarrow C_*(M^{(k)}) \rightarrow C_*(M^{(k+1)}) \rightarrow \Sigma C_*(\mu/\partial\mu) \rightarrow 0$$

$$0 \rightarrow C_*(N^{(k)}) \rightarrow C_*(N^{(k+1)}) \rightarrow \Sigma C_*(E(b_\mu)/HE(b_\mu)) \rightarrow 0,$$

where we regard them as $Z\pi_1(M^{(k+1)}) = Z\pi_1(N^{(k+1)})$ -modules.

