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THE GROWTH OF ENTIRE SOLUTIONS
OF DIFFERENTIAL EQUATIONS
OF FINITE AND INFINITE ORDER

by Lawrence GRUMAN

Let \( f(z) \) be an entire function (of one or several variables) of finite order \( \rho \). A proximate order \( \rho(r) \) is a function which satisfies the conditions

\[
\lim_{r \to \infty} \rho(r) = \rho \quad \text{and} \quad \lim_{r \to \infty} r \rho'(r) \ln r = 0. \tag{1}
\]

The function \( L(r) = r^\rho - \rho \) satisfies

\[
\lim_{r \to \infty} \frac{L(kr)}{L(r)} = 1 \quad \text{uniformly for} \quad 0 < a < k < b < \infty. \tag{2}
\]

We assume in addition that \( \lim_{r \to \infty} L(r) \) exists (perhaps infinite). For every entire function of order \( \rho \), there exists a proximate order \( \rho(r) \) with respect to which \( f(z) \) has normal type [5].

For a given proximate order \( \rho(r) \), we define the functions

\[
h_r^*(z) = \lim_{z' \to z} \left( \lim_{r \to \infty} \frac{\ln |f(rz')|}{r^{\rho(r)}} \right), \quad r > 0
\]

(resp. \( h_c^*(z) = \lim_{u \to 0} \lim_{|z'| \to \infty} \frac{\ln |f(uz')|}{|u|^{\rho(u)}} \), \( u \in \mathbb{C} \)).

If \( f(z) \) is of normal type with respect to the proximate order \( \rho(r) \), it follows from (2) that these functions are pluri-subharmonic and real positive homogeneous (resp. complex homogeneous) of order \( \rho \) [4]. The function \( h_r^*(z) \) (resp. \( h_c^*(z) \)) is called the radial (resp. circular) indicator of growth function of \( f(z) \).

A convex homogeneous function \( g(z) \) is one which satisfies \( g(z_1 + z_2) \leq g(z_1) + g(z_2) \) and \( g(tz) = tg(z), t \geq 0 \). To every convex
homogeneous function $g(z)$, we associate the compact convex set $K_g = \{w : \text{Re } < w, z > \leq g(z) \text{ } \forall z \in \mathbb{C}^n\}$, and to every compact convex set $K$, we associate the convex homogeneous function

$$g_K(z) = \sup_{w \in K} \text{Re } < w, z > ,$$

which is called the support function of $K$. If $\rho \equiv 1$, we define $h_K(z)$, the convex indicator of growth function of $f(z)$, to be the least convex homogeneous majorant of $h^*(z)$. It is evidently the support function of the closed convex hull of the set

$$\{w : \text{Re } < w, z > \leq h^*(z) \text{ } \forall z \in \mathbb{C}^n\} .$$

If the dimension $n = 1$, these two functions are the same [5].

In § 1, we investigate for the case $n = 1$ the relationship between the growth of the function $f(z)$ and that of solutions $u(z)$ of the differential equation $P(D)u = f$ (where $D = \frac{\partial}{\partial z}$ and $P(D)$ is a differential polynomial).

Let $p(z)$ be a complex norm (i.e. $p(\lambda z) = |\lambda| p(z), \lambda \in \mathbb{C}$), $B^p_A$ the space of functions which satisfy a majoration

$$|f(z)| \leq C_A \exp \{(A p(z))^\rho\}$$

and $E^\rho_R = \bigcap_{\lambda > R} B^p_A$. In [8], A. Martineau introduced the notion of a constant coefficient differential operator as a convolution operator on the dual space $(E^\rho_R)'$ of continuous linear functionals defined on $E^\rho_R$. We will take as our definition of such an operator the transpose, which is a linear operator on the space $E^\rho_R$ into itself. This category includes the usual constant coefficient differential operator as a special case. For $\rho \geq 1$, Martineau showed that for every such operator $\hat{\mu}$ on $E^\rho_R$ and every $f \in E^\rho_R$, there exists a solution $g \in E^\rho_R$ of the equation $\hat{\mu}(g) = f$.

In § 2, we extend this notion and result to the case of $p(z)$ a pseudo-norm and $\rho(r)$ a proximate order ($\rho \neq 1$), including the important case of $\rho < 1$. In § 3, we extend this notion and result to the case $\rho = 1$ and $p(z)$ an arbitrary convex homogeneous function. In § 4, we extend this notion and result to those functions which satisfy a majoration of the type $\exp \{k (\ln r)^\rho\}$ for $\rho > 1$. 
Remark. — The case of proximate orders for \( \rho = 1 \) is rendered much more difficult by the special role played by the exponentials. We do not treat this case.

1. Ordinary differential equations.

Let \( f(z) \) be an entire function of a single variable and \( h^*_r(z) \) its indicator function with respect to a proximate order \( \rho(r) \). We will henceforth in this section use the notation \( k_f(\theta) = h^*_r(e^{i\theta}) \), which is the standard notation for \( n = 1 \). If \( u(z) \) is a solution of the constant coefficient differential equation \( P(D) u = f \), then it is an easy consequence of Cauchy's theorem that \( k_f(\theta) \leq k_u(\theta) \). We are interested in seeing if we can choose a solution such that equality holds (at least locally). We will need

**Lemma 1.** — The number of disjoint open intervals on which \( k_f(\theta) \) can be negative is at most \( \sup \{2\alpha\rho\} \) (where \( \lfloor \cdot \rfloor \) means "greatest integer in").

**Proof.** — For \( \theta_1 < \theta_2 < \theta_3 \) and \( \theta_3 - \theta_1 < \pi/\rho \), we have [5, p. 70]

\[
k_f(\theta_1) \sin \rho(\theta_2 - \theta_3) + k_f(\theta_2) \sin \rho(\theta_3 - \theta_1) + k_f(\theta_3) \sin \rho(\theta_1 - \theta_2) \leq 0 .
\]

Thus, any two disjoint intervals on which \( k_f(\theta) \) is negative are separated by an interval of length at least \( \pi/\rho \) on which \( k_f(\theta) \) is non-negative.

**Theorem 1.** — Let \( f(z) \) be an entire function with indicator \( k_f(\theta) \) with respect to the proximate order \( \rho(r) \). Then there exists a solution \( u(z) \) of the differential equation \( P(D) u = f \) such that

i) \( k_u(\theta) = k_f(\theta) \) for \( \rho \leq 1 \).

ii) \( k_f(\theta) \leq k_u(\theta) \leq k^*_f(\theta) = \max (k_f(\theta), 0) \) for \( \rho > 1 \) and for any specific interval \((\theta_1, \theta_2)\) on which \( k_f(\theta) \) is negative, there exists a unique solution \( u \) with this property such that \( k_u(\theta) = k_f(\theta) \) for \( \theta_1 \leq \theta \leq \theta_2 \).
Proof. — It is enough to consider solutions of the equation 
\((D - a) u = f\) and then iterate the result. All such solutions are given by 
\[ u(z) = e^{az} \int_0^z f(\xi) e^{-a\xi} d\xi + Ce^{az} \quad (3) \]
If for some open interval of \(\theta\), the function \(f(z) e^{-az}\) has negative 
indicator (with respect to any proximate order), then 
\[ C = \int_0^\infty f(t\xi) e^{-at\xi} \xi dt \quad , \quad \xi = e^{i\theta} \]
defines a constant for all \(\theta\) in this interval. If there is no such region, 
we choose \(C = 0\). By Lemma 1, for \(\rho < 1\), there is at most one such 
interval, but for \(\rho > 1\) there may be more than one such interval and 
we may only be able to choose \(C\) to satisfy this relation in one of 
the intervals. (This explains the difference between i) and ii) above).

From (1), we have that 
\[(r^\rho(r))' = \rho(r) r^{\rho(r)-1} + r^\rho(r) \rho'(r) \ln r \to \rho(r) r^{\rho(r)-1} \quad (4)\]
Let us consider the case \(\rho < 1\). For a given \(\xi = e^{i\theta}\), let \(b = k_f(\theta)\) 
and \(s = \text{Re} a\xi\). Then given \(\varepsilon > 0\), we have \(|f(t\xi)| \leq K \exp(b + \varepsilon) t^{\rho(t)}\).

i) If \(s < 0\) and \(b < 0\) and if \(\varepsilon < -\frac{b}{2}\), then 
\[ |u(r\xi)| \leq Ke^{sr} \int_0^r e^{(b+\varepsilon) t^{\rho(t)} - st} dt + |C| e^{sr} \]
\[ \leq K_1' e^{sr} \int_0^r \left[ (b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right] e^{(b+\varepsilon) t^{\rho(t)} - st} dt + |C| e^{sr} \]
where \(q_0\) is chosen so large that \(\left[ (b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right]\) is bounded 
below and \(K_1'\) depends on \(q_0\).
\[ |u(r\xi)| \leq K_1' \left[ e^{(b+\varepsilon) r^{\rho(r)}} - e^{sr} \cdot K_{q_0} \right] + |C| e^{sr} \]
\[ \leq K_1'' e^{(b+\varepsilon) r^{\rho(r)}} \quad . \]
ii) If \(s > 0\) and \(b < 0\), then by the choice of \(C\), we have
\[ |u(r\xi)| \leq K e^{sr} \int_{r}^{\infty} e^{(b+\varepsilon)t^{\rho(t)}} e^{-st} dt \]
\[ \leq K \int_{r}^{\infty} e^{(b+\varepsilon)t^{\rho(t)}} dt \leq K e^{(b+\varepsilon)r^{\rho(r)}} \int_{r}^{\infty} e^{-t^{\rho(t)}} dt \]
\[ \leq K_2 e^{(b+2\varepsilon)r^{\rho(r)}}, \]

since by (4), \( r^{\rho(r)} \) is increasing for sufficiently large \( r \).

iii) If \( s > 0 \) and \( b \geq 0 \), then
\[ |u(r\xi)| \leq K e^{sr} \int_{r}^{\infty} e^{(b+\varepsilon)t^{\rho(t)}} e^{-st} dt \]
\[ \leq K_3 e^{sr} \int_{r}^{\infty} \left[ (b+\varepsilon) \frac{d}{dt} (t^{\rho(t)}) - s \right] e^{(b+\varepsilon)t^{\rho(t)}} - st dt \]
\[ \leq K_3' e^{(b+\varepsilon)r^{\rho(r)}}. \]

iv) If \( s \leq 0 \) and \( b \geq 0 \), then
\[ |u(r\xi)| \leq K e^{sr} \int_{0}^{r} e^{(b+\varepsilon)t^{\rho(t)}} - st dt + |C| e^{sr} \]
\[ \leq K_4' e^{(b+\varepsilon)r^{\rho(r)}}. \]

The case \( \rho \geq 1 \) is treated similarly (for \( \rho = 1 \), we must make use of the assumption that \( \lim_{r \to \infty} r^{\rho(r)} - \rho \) exists). For \( \rho > 1 \), if for some \( \theta \), \( k_f(\theta) \neq k_u(\theta) \), then \( u(z) = w(z) + Ce^{az} \), where \( k_f(\theta) = k_w(\theta) < 0 \), so \( k_u(\theta) = 0 \). Q.E.D.

Remark. — It follows from Theorem 6 below that if \( P(D) \) has a non-zero constant term, then for \( \rho < 1 \), the solution \( u(z) \) in i) is unique.

The following example shows that it is not always possible to find a solution \( u \) of \( P(D) u = f \) with the same indicator as \( f \). Let \( f(z) = e^{az} \) and let \( u \) be a solution of \( Du = f \). The function \( f(z) \) has two intervals on which its indicator is negative. If we integrate \( f(z) \) along the positive imaginary axis, we obtain a constant different from that which we obtain by integrating along the negative imaginary axis.

There is even a more intimate connection between the growth of the function \( f(z) \) and the solution \( u(z) \) of \( P(D) u = f \). If \( f(z) \) grows regularly in a given direction, then so will \( u(z) \). We introduce our criterion for regularity of growth.
Let $E$ be a measurable set of positive real numbers and let $E' = E \cap [0, r]$. A set is said to have upper relative measure $U$ if
\[ \lim_{r \to \infty} \frac{\text{meas}(E')}{r} = U. \] If $U = 0$, $E$ is an $E^0$-set.

**Definition [5].** Let $f(z)$ be an entire function with indicator $k_f(\theta)$ with respect to a given proximate order $\rho(r)$; $f(z)$ is said to be of completely regular growth along the ray $re^{i\theta}$ if
\[ \lim_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{\rho(r)} = k_f(\theta), \]
where $r$ takes on all values except perhaps for some $E^0$-set.

**Remark.** The property of being of completely regular growth is not invariant with respect to a change in proximate orders.

**Theorem 2.** If $u(z)$ is a solution of $P(D)u = f$ for an entire function $f(z)$ and if $\rho(r)$ is a proximate order with respect to which both $k_f(\theta)$ and $k_u(\theta)$ are bounded, then if $f(z)$ is of completely regular growth along the ray $re^{i\theta}$, so is $u(z)$.

**Proof.** We consider a solution of $(D - a) u = f$. By Theorem 1, for given $\theta$, there is an interval $(\theta_1, \theta_2)$ containing $\theta$ such that $u = w + C e^{az}$ and $w$ has the same indicator as $f$ in the interval $(\theta_1, \theta_2)$. Thus, if $k_u(\theta) \neq k_f(\theta)$, we have that $\lim_{r \to \infty} \frac{\ln |u(re^{i\theta})|}{\rho(r)}$ exists with no exceptional set. Hence, in the following, we assume that $k_u(\theta) = k_f(\theta)$. We assume without loss of generality that $\theta = 0$.

Let $\varepsilon$ and $\eta$ be given positive numbers. Then there exists a set $E_1$ of upper relative measure less than $\eta/4$ such that if $r \notin E_1$, the family of functions $k_{u,r}(\phi) = \frac{\ln |u(re^{i\phi})|}{\rho(r)}$ is equicontinuous [5, p. 96]. Thus, there is a $\delta > 0$ such that for $|\phi| < \delta$,
\[ |k_{u,r}(\phi) - k_{u,r}(0)| < \frac{\varepsilon}{4} \text{ and } |k_u(\phi) - k_u(0)| < \frac{\varepsilon}{4} \text{ for } r \notin E_1. \]

Since $f$ is of completely regular growth along the positive real axis, given $\gamma > 0$ (depending eventually on $\eta$ and $\varepsilon$), for $r$ not in some $E^0$-set $E_2$. 

We choose \( r \) so large that \( \text{meas}(E_2') < \frac{\eta}{4} r \) and \( \frac{\ln |u(re^{i\phi})|}{r^\rho(r)} \leq k_u(\phi) + \frac{\gamma}{4} \) [5, p. 71]. By Cauchy's formula,

\[
f(r) e^{-ar} = \frac{1}{2\pi i} \int_{|\xi| = 1} \frac{u(\xi + r)}{\xi^2} e^{-a(\xi + r)} d\xi.
\]

So by (5) for \( r \not\in E_2' \) and \( r \) sufficiently large, there exists \( w \) with \( |w - r| = 1 \) such that, noting \( \phi_w = \arg w \),

\[
|a| + \ln |u(w)| > \left\{ k_f(0) - \frac{\gamma}{4} \right\} r^\rho(r) \geq \left\{ k_f(\phi_w) - \frac{\gamma}{2} \right\} |w|^\rho(|w|)
\]

Let \( R_m = \left( \frac{1 + \eta}{1 - \eta} \right)^m \). Then, as in the proof of Theorem 31 [5, p. 73], we can choose \( \gamma \) so small (depending on \( \varepsilon \) and \( \eta \) but independent of \( w \) since \( k_u(\theta) \) is bounded) such that

\[
\frac{\ln |u(r \phi_w)|}{r^\rho(r)} > k_u(\phi_w) - \frac{\varepsilon}{4}
\]

except perhaps on a set of measure at most \( \frac{\eta^2}{4} R_m \) for

\[
(1 - 2\eta) R_m \leq r \leq (1 + 2\eta) R_m
\]

(for \( m \geq m_0 \) so large that the above inequalities hold). Let

\[
E_3 = [0, R_{m_0}] \cup \left( \bigcup_{m \geq m_0} E_m \right).
\]

Then

\[
\frac{\text{meas}(E_3')}{r} \leq \frac{R_{m_0} + \sum_{i = m_0}^{m} \frac{\eta^2}{4} \frac{(R_{m_0} - R_m)}{1 - \left(1 + \eta\right)} (1 - \eta)}{R_m \left(1 - \eta\right)} \leq 0(1) + \frac{\eta}{2} \left(1 - \frac{R_{m_0}}{R_m}\right) < \frac{\eta}{4}
\]
for \( m \) sufficiently large. Let \( E_\eta = E_1 \cup E_2 \cup E_3 \). Then

\[
\lim_{r \to \infty} \frac{\text{meas}(E'_n)}{r} < \eta,
\]

and gathering together our inequalities, we have \( |k_{u,r}(0) - k_u(0)| < \varepsilon \) for \( r \notin E \). To see that this implies the theorem, we refer the reader to Theorem 1, part 3 [5, p. 141]. Q.E.D.

**Remark.** The fact that a function is of completely regular growth in an interval has important consequences for the distribution of its zeros. This is fully discussed in [5].

2. Differential operators with constant coefficients.

Let \( p_n(z) \) be a decreasing sequence of real valued functions and \( b_n \) the space of entire functions such that \(| f(z) \exp(-p_n(z)) | \) goes to zero at infinity. This is a Banach space with norm

\[
\| f \|_n = \sup_z | f(z) \exp(-p_n(z)) |.
\]

We then set

\[
E = \bigcap_n B_n,
\]

which is a Fréchet space when we equip it with the projective limit topology. If \( B'_n \) is the dual space of \( B_n \), \( E' \) that of \( E \), then \( E' = \bigcup_n B'_n \).

Let \( \rho(z) \) be a complex pseudo-norm and \( \rho(r) \) a proximate order. The space \( E_{\rho(r)} \) will designate the space we get in (6) by setting

\[
p_n(z) = p(z) + \frac{1}{n} \| z \|^{\rho(r)}
\]

(where \( r = \| z \| \), and we use the Euclidean norm). The space \( E^0 \) will be the space we get in (6) by setting

\[
p_n(z) = \| z \|^{1/n}
\]

(the space of entire functions of zero order).

For a given proximate order \( \rho(r) \), we have by (4) that \( \rho(r) \) is increasing for sufficiently large \( r \). For a given integer \( q \), we define \( \phi(q) = r_q \) to be the largest solution of \( q = r^{\rho(r)} \). Then the type with respect to \( \rho(r) \) of an entire function of one variable with coefficients \( c_q \) (in its Taylor series expansion at the origin) is given by the formula
\( (a_\rho e)^{1/\rho} = \lim_{q \to +\infty} (\phi(q) | c_q |^{1/q}) \quad [5, \text{p. 42}] . \)

If \( f \in E^p_{o}(r) \), we expand \( f \) at the origin in homogeneous polynomials \( f(z) = \sum_q P_q(z) \). Let \( A_q = \left( \frac{\phi(q)^{q/p}}{e^p} \right) \). If we set
\[
f_t(z) = \sum_q A_q P_q(z),
\]
then \( f_t(z) \) is a holomorphic function in the open set \( D = \{ z : p(z) < 1 \} \), and when we equip the space \( \mathcal{H}(D) \) of holomorphic functions defined on \( D \) with the topology of uniform convergence on compact subsets, the mapping \( f \to f_t \) becomes an isomorphism of \( E^p_{o}(r) \) onto \( \mathcal{H}(D) \) (cf. [8], Prop. 4, p. 116 and [4]).

For \( \mu \in (E^p_{o}(r))' \), we define the linear functional \( \mu_t \) on \( \mathcal{H}(D) \) by \( (\mu_t, \mu_q) = (f_t, \mu) \). This is an isomorphism of \( (E^p_{o}(r))' \) onto \( \mathcal{H}'(D) \), the space of continuous linear functionals on \( \mathcal{H}(D) \). We say that a linear functional \( \mu_t \) is carried by the compact convex set \( K \) if for every open neighborhood \( \Omega \) of \( K \), there exists a constant \( C_\Omega \) such that \( |\mu_t(f_t)| \leq C_\Omega \sup_{\Omega} |f_t| \). Every \( \mu_t \in \mathcal{H}'(D) \) is carried by one of the sets \( K_n = \{ z : p(z) + \frac{1}{n} \| z \| \leq 1 \} \).

We define the Fourier-Borel transform of the functional \( \mu_t \) to be the entire function \( \tilde{\mu}_t(u) = \mu_t(\exp <z,u>) \). Then we have [3], [7].

**Proposition 1.** — The functional \( \mu_t \) is carried by the compact convex set \( K \) if and only if
\[
\tilde{\mu}_t(u) \leq C_8 \exp(H_K(u) + \delta \| u \|) \quad \text{for all} \quad \delta > 0 ,
\]
where \( H_K(u) \) is the support function of \( K \).

Let \( p'_n(u) = \sup_{z \in K_n} \Re <z,u> \). Then \( p'_n(u) \) is a family of increasing complex norms, and since each \( \mu_t \in \mathcal{H}'(D) \) is carried by some \( K_n \), we have
\[
\tilde{\mu}_t(u) \leq C_n \exp H_{K_n}(u) \quad \text{for} \quad n \text{ sufficiently large}.
\]

Let \( \alpha \) be a multi-index of positive numbers, \( |\alpha| = \Sigma \alpha_i \) and
$z^\alpha = z^{\alpha_1} \ldots z^{\alpha_n}$. Since the polynomials converge to $\exp < z, u >$ in $\mathcal{S}(\mathbb{D})$, we have

$$\mu_t(\exp < z, u >) = \mu_t \sum_q \sum_{|\alpha| = q} z^\alpha u^\alpha \frac{1}{\alpha!} = \sum_q \sum_{|\alpha| = q} \mu_t(z^\alpha) \frac{u^\alpha}{\alpha!} = \sum_q P^\mu_t(u)$$

and from (7) and Proposition 1, we have

$$\lim_{q \to \infty} \left\{ \frac{q}{e} \left| P^\mu_t(u) \right|^{1/q} \right\} \leq p'_n(u)$$

for $n$ sufficiently large. From the relation $\mu_t(z^\alpha) = \frac{1}{A_{|\alpha|}} \mu(z^\alpha)$, we see that $\mu \in (E^p(r))'$ (resp. $(E^0)^'$) if and only if

$$\lim_{q \to \infty} \left\{ \frac{q}{e} \left| \frac{1}{A_q} \sum_{|\alpha| = q} \mu(z^\alpha) \frac{u^\alpha}{\alpha!} \right|^{1/q} \right\} \leq p'_n(u) \quad (8)$$

for $n$ sufficiently large (resp. for $p$ sufficiently small).

For $\mu \in (E^p(r))'$ (resp. $(E^0)^'$), we define its Fourier-Borel transform to be the formal power series

$$\widehat{\mu}(u) = \mu(\exp < z, u >) = \sum_q \sum_{|\alpha| = q} \mu(z^\alpha) \frac{u^\alpha}{\alpha!} = \sum_q P^\mu(u).$$

If $\rho > 1$, we assume that the proximate order $\rho(r)$ satisfies:

i) $\rho(r) > 1$ for all $r$

ii) $\frac{d}{dr} (r^{\rho(r)-1}) > 0$ for all $r$.

By (1), these properties hold eventually, so this is an inessential assumption. Then the equation $r = t^{\rho(t)-1}$ has a unique solution for all $r$. We define

$$\rho^*(r) = \frac{\rho(t)}{\rho(t) - 1}, \text{ where } t \text{ is this unique solution.}$$

It is an easy calculation to show that $\rho^*(r)$ satisfies the conditions (1) and so is a proximate order. For $\rho > 1$, we designate
where \( A = \frac{(\rho - 1)^{\rho}}{\rho} \)

**Theorem 3.** — The mapping \( \mu \mapsto \tilde{\mu}(u) \) is a one-to-one linear mapping of \((E_p^\rho)'\) (resp. \((E^0)'\)) onto

i) \( F_{\Lambda p}^\rho(r) \) for \( \rho > 1 \)

ii) the set \( Q_p^\rho(r) \) of formal power series at the origin which satisfy (8) for some \( n \) for \( \rho < 1 \)

iii) the set \( Q_0 \) of formal power series at the origin which satisfy (8) for some \( \rho > 0 \) for \((E^0)'\).

**Proof.** — We have that (8) holds for some \( n_0 \). Since

\[
A_{1/q}^{1/q} = \frac{\phi(q)}{(e\rho)^{1/\rho}}, \quad q \frac{1}{e} A_{1/q}^{1/q} = \frac{A r_q^{\rho(r_q)^{-1}}}{(e\rho^*)^{1/\rho^*}}
\]

(where \( r_q = \phi(q) \)). Let \( r'_q = r_q^{\rho(r_q)^{-1}} \). Then

\[
(r'_q)^{\rho^*} (r'_q) = (r_q^{\rho(r_q)^{-1}})^{\rho^*} (r_q^{\rho(r_q)^{-1}}) = (r_q^{\rho(r_q)^{-1}})^{\rho(r_q)^{-1}} = r_q^{\rho(r_q)} = q
\]

so if \( \phi'(q) \) is the unique solution of \((r'_q)^{\rho^*} (r'_q) = q\), we have that

\[
q \frac{1}{e} A_{1/q}^{1/q} = A \frac{\phi'(q)}{(e\rho^*)^{1/\rho^*}}
\]

so the mapping is into. Since the calculations are all reversible, the mapping is also onto. This proves case i). Cases ii) and iii) follow directly from (8). Q.E.D.

Let \( \mu \in (E_p^\rho(r))' \). Then for any other element \( \nu \), we define the convolution of \( \nu \) with \( \mu \), \( \mu * \nu = \tau \) by \((f(z), \mu * \nu) = (\mu_w f(z + w), \nu)\). This is defined at least on the polynomials, which are dense in \( E_p^\rho(r) \). For \( \rho > 1 \), it is also defined on the exponentials [8]. We then have the relationship (for \( \rho \neq 1 \)) \( \tau(u) = \tilde{\mu}(u) * \tilde{\nu}(u) \), which, for the case \( \rho < 1 \), follows from...
LEMMA 2. – For \( \tilde{\mu}(u), \tilde{\nu}(u) \in Q_0^{\rho(r)} \) (resp. \( Q_0 \)), we have \( \tilde{\tau}(u) = \tilde{\mu}(u) \tilde{\nu}(u) \in Q_0^{\rho(r)} \) (resp. \( Q_0 \)) for \( \rho < 1 \) (i.e. these spaces are algebras).

Proof. – We choose \( n_0 \) so large so that for \( n \geq n_0 \), (8) holds for both \( \mu \) and \( \nu \). Consider such an \( n \) and let \( \varepsilon > 0 \) be given. Then there exist constants \( C_\varepsilon^\mu \) and \( C_\varepsilon^\nu \) such that

\[
|P_\mu^\nu(u)| \leq C_\varepsilon^\mu [p_n'(u) + \varepsilon \|u\|]^q \left( \frac{\Phi(q)^\rho}{e_p} \right)^{q/\rho} \left( \frac{e}{q} \right)^q
\]

and

\[
|P_\nu^\mu(u)| \leq C_\varepsilon^\nu [p_n'(u) + \varepsilon \|u\|]^q \left( \frac{\Phi(q)^\rho}{e_p} \right)^{q/\rho} \left( \frac{e}{q} \right)^q.
\]

Then

\[
|P_q^\mu(u)| = \sum_{m+n-q} |P_m^\nu(u) P_n^\mu(u)| \leq C_\varepsilon^\mu C_\varepsilon^\nu [p'(u) + \varepsilon \|u\|]^q \left( \frac{\Phi(q)^\rho}{e_p} \right)^{q/\rho} \left( \frac{e}{q} \right)^q \sum_{m+n-q} \left[ \frac{\Phi(m)^m \Phi(n)^n}{\Phi(m+n)^{m+n}} \right] (m+n)^{m+n} / m^m n^n.
\]

Let \( r_q = \Phi(q) \). Then \( \frac{q}{\Phi(q)} = r_q^{\rho(r_q)^{-1}} \), and hence, since by (1), \( r^{\rho(r)^{-1}} \) is decreasing for \( r \) sufficiently large

\[
\sum_{m+n-q} \frac{[r_m^{\rho(r_m+n)^{-1}}]^{m+n}}{[r_m^{\rho(r_m)^{-1}}]^m [r_n^{\rho(r_n)^{-1}}]^n} \leq K q \text{ for some constant } K.
\]

Thus \( |P_q^\mu(u)| \) satisfies (8). For \( Q_0 \), we choose \( \rho_0 \) so small that (8) holds for both \( \mu \) and \( \nu \) for \( \rho < \rho_0 \). The result then follows from the above calculations. Q.E.D.

Thus, by Theorem 3, for \( \rho < 1 \), the mapping \( \nu \to \mu \ast \nu \) is a map of \( (E_0^\rho)' \) (resp. \( (E_0^0)' \)) into \( (E_0^\rho)' \) (resp. \( (E_0)' \)). If \( \rho > 1 \), this is only the case if \( \tilde{\mu}(u) \) is of minimal type with respect to the proximate order \( \rho^*(r) \). Assuming \( \mu \) to satisfy these conditions, we define \( \tilde{\mu} \) to be the transpose of \( \mu \), \( (\tilde{\mu}(f), \nu) = (f, \mu \ast \nu) \). We are interested in proving that the mapping \( \tilde{\mu}(E_0^\rho) \) (resp. \( E_0^0 \)) is onto (i.e. that there always exists a solution \( g \) such that \( \tilde{\mu}(g) = f \)). We will make use of [cf. 9, p. 85].
PROPOSITION 2. — Let \( E, F \) be two Fréchet spaces, \( \alpha \) a continuous linear map of \( E \) into \( F \). The two following are equivalent

i) \( \alpha \) is onto

ii) \( '\alpha : F' \rightarrow E' \) (the transpose map) is one-to-one and its image \( t_{\alpha(F')} \) is weakly closed in \( E' \).

We shall prove the closure of \( \mu \ast \nu \) in the equivalent spaces as determined by Theorem 3, but first we must equip these spaces with topologies. For \( \rho > 1 \), we equip the space \( F^\rho_p(r) \) with the topology of pointwise convergence. For \( \rho < 1 \), we equip \( Q^\rho_p(r) \) (resp. \( Q_0 \)) with the topology of convergence of Taylor’s series coefficients. Each of these topologies is at least as weak as the weak topology.

We define a differential operator with constant coefficients (with respect to a given proximate order \( \rho(r) \)) to be

i) \( \tilde{\mu} \) for \( \mu \in (E^\rho_p(r))' \) for \( \rho < 1 \)

ii) \( \tilde{\mu} \) for \( \mu \in (E^0)' \)

iii) \( \tilde{\mu} \) for \( \mu \in (E^\rho_p(r))' \) such that \( \tilde{\mu}(u) \) is of minimal type with respect to \( \rho^*(r) \) for \( \rho > 1 \).

For \( \rho > 1 \), the mapping \( \nabla(u) \rightarrow \nabla(u) \nabla(u) \) is closed in the topology we have chosen (the proof is carried out in [8]; the modifications necessary to treat the case of proximate orders are obvious). Thus, we limit ourselves to the case \( \rho < 1 \) and \( E^0 \).

LEMMA 3. — Let \( A_n(u) = \frac{B_{n+m}(u)}{C_m(u)} \) be a homogeneous polynomial which is the ratio of two homogeneous polynomials. Furthermore, assume that for some complex norm \( p_0(u) \) that

\[ \| B_{n+m}(u) \| \leq C[p_0(u)]^{n+m} . \]

Then given \( \delta > 0 \), there is a constant \( K_\delta \) (depending only on \( C_m(u) \) and \( \delta \)) such that \( | A_n(u) | \leq C K_\delta [p_0(u)]^n (1 + \delta)^{n+m} \).

Proof. — Let \( \Omega = \{ u : 1 - \delta \leq p_0(u) \leq 1 + \delta \} \). For every point \( u \) in \( \Omega \) we find a polydisc (by making a non-singular linear change of variable if necessary) \( \Delta(u ; r^u) \) centered at \( u \) and lying in \( \Omega \) such that \( C_m(u_1', \ldots , u_{n-1}', \xi_n) \neq 0 \) for \( | \xi_n - u_n | = r_n^u \) and
Let $n' = \{u : p_0(u) = 1\}$. We now consider the polydisc $\Delta'_u = \Delta \left( u ; \frac{r^u}{2} \right)$.

Since $\Omega'$ is compact, it can be covered by a finite number of $\Delta'_{u_j}$, $j = 1, \ldots, N$. The function $\frac{1}{C_m(u)}$ is bounded, say by $\frac{K_\delta}{2}$, on the compact set

$$K = \bigcup_i \{u' : u' \in \Delta_{u_i}, |u'_i - u_i| \leq r^u_i, i = 1, \ldots, n-1, |u_n' - u_n| = r^u_n \}.$$

Let the function $A_n$ take its maximum on $\Omega'$ at the point $u^0$. Then $u^0 \in \Delta'_{u_j}$ for some $j$. By Cauchy's formula

$$|A_n(u^0)| = \left| \frac{1}{2\pi i} \int_{|\xi_n - u_n'| = r^u_n} \frac{B_{n+m}(u_1^0, \ldots, u_{n-1}^0, \xi_n) d\xi_n}{C_m(u_1^0, \ldots, u_{n-1}^0, \xi_n)(\xi_n - u_n^0)} \right|$$

$$= K_\delta C p_0(u) (1 + \delta)^{n+m}.$$ 

Q.E.D.

**Theorem 4 (Division Theorem).** Let $H(u), F(u) \in \mathbb{O}_p^{(r)}$ for $p < 1$ (resp. $Q_0$) with $H(u) = F(u) G(u)$, where $G(u)$ is a formal power series at the origin. Then $G(u) \in \mathbb{O}_p^{(r)}$ (resp. $Q_0$).

**Proof.** Let $\varepsilon > 0$ be given and let

$$G(u) = \sum_q R_q(u), \quad H(u) = \sum_q P_q(u), \quad \text{and} \quad F(u) = \sum_q T_q(u),$$

with $s$ the smallest integer such that $T_s(u) \neq 0$. We choose $n_0$ so large that (8) holds for both $H(u)$ and $F(u)$ for $n \geq n_0$. Thus, there exist constants $C_1$ and $C_2$ such that

$$|P_q(u)| \leq C_1 [p'_n(u) + \varepsilon \|u\|] q (\frac{\phi(q)}{e^p})^{q/p} \left( \frac{e}{q} \right)^q$$

and

$$|T_q(u)| \leq C_2 [p'_n(u) + \varepsilon \|u\|] q (\frac{\phi(q)}{e^p})^{q/p} \left( \frac{e}{q} \right)^q.$$

We have
\[ P_{q+s}(u) = \sum_{m+k=q} R_m(u) T_{k+s}(u) \]

or

\[ P_{q+s}(u) - \sum_{m+k=q} R_m(u) T_{k+s}(u) = \frac{R_q(u)}{T_s(u)} \]

We now show by induction that there exist constants \( K_q \) (with \( K_{q-1} \leq K_q \)) such that

\[ |R_q(u)| \leq K_q \left[ p_n'(u) + \varepsilon \|u\| \right]^q (1 + \delta)^q \left( \frac{\phi(q+s)^p}{ep} \right)^{q+s} \left( \frac{e}{q+s} \right)^{q+s}, \]

where \( K_q = K_{q-1} \) for \( q \) sufficiently large.

For \( q = 0 \), it follows from Lemma 3. We assume it true for \( q \leq q_0 - 1 \).

\[ |P_{q_0+s}(u)| + \sum_{m+k=q_0} |R_m(u) T_{k+s}(u)| \]

\[ |R_{q_0}(u)| \leq \frac{1}{|T_s(u)|} \]

\[ \leq K_s (1 + \delta)^p \left[ p_n'(u) + \varepsilon \|u\| \right]^q (1 + \delta)^q \left( \frac{\phi(q+s)^p}{ep} \right)^{q+s} \left( \frac{e}{q+s} \right)^{q+s} \times \]

\[ \left\{ C_1 + \sum_{m+k=q_0} K_{q-1} C_2 m \left[ \frac{\phi(m)^m \phi(k+s)^{k+s}}{\phi(k+m+s)^m} \right] \right\} \]

\[ \leq \max \{ K_0 (1 + \delta)^p C_1, K_{q-1} C_2 \} \left[ p_n'(u) + \varepsilon \|u\| \right]^q (1 + \delta)^q \left( \frac{\phi(q+s)^p}{ep} \right)^{q+s} \left( \frac{e}{q+s} \right)^{q+s} \times \]

\[ \left\{ 1 + \sum_{m+k=q_0} K_{q_0} (1 + \delta)^p m \left[ \frac{\phi(m)^m \phi(k+s)^{k+s}}{\phi(k+m+s)^m} \right] \right\}. \]

We assume that the function \( r^{1 - \rho(p)} \) is increasing. By (1), this holds eventually, so this is an inessential assumption.
Let us assume for the moment that $k + s \leq \frac{3}{4} (q_0 + s)$. Then

$$\begin{bmatrix}
\frac{1-\rho(r_{q_0+s})}{r_{q_0+s}} \\
\frac{1-\rho(r_{k+s})}{r_{k+s}}
\end{bmatrix}^{k+s} = \begin{bmatrix}
\frac{1-\rho(r_{q_0+s})}{r_{q_0+s}} \\
\frac{1-\rho(r_{k+s})}{r_{k+s}}
\end{bmatrix}^{k+s} \cdot \begin{bmatrix}
\frac{1-\rho(r_{q_0+s})}{r_{q_0+s}} \\
\frac{1-\rho(r_{k+s})}{r_{k+s}}
\end{bmatrix}^{k+s} \cdot \frac{1}{\left(\frac{3}{4}\right)^{k+s}}$$

Let $\psi(r) = r^{1-\rho}$. Then

$$\psi(r_{q_0+s}) - \psi(r_{k+s}) = \int_{r_{k+s}}^{r_{q_0+s}} \frac{d\psi(r)}{dr} \, dr \geq \int_{\frac{3}{4}(r_{q_0+s})}^{r_{q_0+s}} \frac{d\psi(r)}{dr} \, dr \geq \int_{\frac{3}{4}(r_{q_0+s})}^{r_{q_0+s}} \frac{d\rho(r_{q_0+s})}{dr} \, dr$$

for $q_0 + s$ sufficiently large, by (1). Thus

$$\psi(r_{q_0+s}) - \psi(r_{k+s}) \geq r_{q_0+s}^{\frac{4}{3} - 1} \left[ 1 - \frac{3}{4} \right] = T(q_0 + s)^{1/4}.$$ 

For $(k + s) \geq 12 \frac{\rho}{2} \frac{1}{1 - \rho} = \alpha$, we have

$$\begin{bmatrix}
\frac{1-\rho(r_{q_0+s})}{r_{q_0+s}} \\
\frac{1-\rho(r_{k+s})}{r_{k+s}}
\end{bmatrix}^{k+s} = \begin{bmatrix}
\frac{1-\rho(r_{q_0+s})}{r_{q_0+s}} \\
\frac{1-\rho(r_{k+s})}{r_{k+s}}
\end{bmatrix}^{k+s} \cdot \begin{bmatrix}
\frac{1-\rho(r_{q_0+s})}{r_{q_0+s}} \\
\frac{1-\rho(r_{k+s})}{r_{k+s}}
\end{bmatrix}^{k+s} \cdot \frac{1}{\left(\frac{3}{4}\right)^{k+s}}$$

$$\geq \left[ 1 + \frac{T(q_0 + s)^{1/4}}{1 - \rho(r_{k+s})} \right]^{k+s} \cdot \frac{2}{\rho} (1 - \rho)$$

$$+ \cdots + K T^{(q_0 + s)^{1/4} + \cdots} \cdot \frac{2}{\rho} (1 - \rho),$$
where \( \gamma \geq 3 \frac{\rho}{2} \) \( 1 - \frac{1}{\rho} \left( \frac{1 - \rho(r_q + s)}{r_q + s} \right) \cdot \frac{\rho}{2} = 0(k + s)^{1/2} \)

\[ \geq T'(q_0 + s)^3. \]

For \( (k + s) \leq \alpha + 1 \)

\[ \left[ \frac{1 - \rho(r_q + s)}{r_q + s} \right]^{k+s} \]

\[ \geq \left[ \frac{1 - \rho(r_q + s)}{r_q + s} \right]^{k+s} \]

\[ \geq (\alpha + 1)^2 K_6 (1 + \delta)^s 3, \]

(\( \beta = \max_{(k + s) \leq \alpha} \frac{1 - \rho(r_q + s)}{r_q + s} \)) for \( q_0 \) sufficiently large. By symmetry, similar inequalities exist if we replace \( (k + s) \) by \( m \). We choose \( q_0 \) so large that \( \frac{K_6 (1 + \delta)^s}{T'(q_0 + s)^2} \leq \frac{1}{q_0} \). Thus

\[ \left\{ 1 + \sum_{m+k=q_0} K_6 (1 + \delta)^s m \left[ \frac{\phi(m)^m}{\phi(m+k+s)^{m+k+s}} \right] \frac{(m+k+s)^{m+k+s}}{m^m(k+s)^{k+s}} \right\} \]

\[ \leq 1 + \frac{(q_0 - 1)}{3} + 2 \leq q_0 \]

for \( q_0 \) sufficiently large, which establishes the induction.

Furthermore,

\[ \left[ \frac{\phi(q + s)}{q + s} \right]^{q+s} = \left[ \frac{1 - \rho(r_q + s)}{r_q + s} \right]^{q+s} = \left[ \frac{\phi(q)}{q} \right]^{q+s} \left[ \frac{1 - \rho(r_q)}{r_q} \right]^{q+s} \]

\[ \leq (1 + \delta)^{q+s} \left[ \frac{\phi(q)}{q} \right]^{q+s+1} \]

for arbitrary \( \delta > 0 \) when \( q \) is sufficiently large. Thus

\[ \lim_{q \to \infty} \left\{ \frac{q}{e} \left| \frac{1}{A_q} R_q(u) \right|^{1/q} \right\} \leq p'_n(u), \]

which proves the theorem. Q.E.D.
COROLLARY. — Let \( F(u) = \sum_q T_q(u), \) \( H(u) = \sum_q P_q(u) \) be in \( Q_{p(r)}^p \) (resp. \( Q_0 \)) and assume \( T_0 \neq 0. \) Then there exists a unique \( G(u) \in Q_{p(r)}^p \) (resp. \( Q_0 \)) such that \( F(u) G(u) = H(u) \).

Proof. — It is well known that the set of formal power series with non-zero constant term forms a group under multiplication. By Theorem 4, \( G(u) \in Q_{p(r)}^p \) (resp. \( Q_0 \)). Q.E.D.

Combining Theorem 4 with Proposition 2, we obtain the following

THEOREM 5. — Let \( \hat{\mu} \) be a differential operator with constant coefficients for some space \( E_{p(r)}^p \) for a complex pseudo-norm \( p(z) \) and a proximate order \( \rho(r) \) (\( \rho \neq 1 \)) (resp. \( E^0 \)). Then for \( f \in E_{p(r)}^p \) (resp. \( E^0 \)), there always exists \( g \in E_{p(r)}^p \) (resp. \( E^0 \)) such that \( \hat{\mu}(g) = f. \)

For \( \rho < 1 \) (resp. \( E^0 \)), if \( \hat{\mu}(1) \neq 0, \) the solution \( g \) is unique.

Proof. — As a result of Theorem 4, the mapping \( \nu \rightarrow \mu \ast \nu \) is one-to-one and closed. If \( \hat{\mu}(u) \) has a non-zero constant term, then by the corollary to Theorem 4, this mapping is also onto, so its transpose \( \hat{\mu} \) is one-to-one. Q.E.D.

We now show that for \( \rho < 1, \) the uniqueness of the solution has important consequences for the circular indicator function. Instead of a complex pseudo-norm, we let \( p_0(z) \) be any positive upper semi-continuous complex homogeneous function (i.e. \( p_0(\lambda z) = |\lambda| p_0(z) \)). We construct the space \( E_{p_0}^p \) as in (6).

LEMMA 4. — Let \( p_0(z) \) be a positive upper semi-continuous complex homogeneous function, \( \mathfrak{S} = \{ p(z) : p(z) \) a complex norm, \( p(z) \geq p_0(z) \} \). Then \( p_0(z) = \inf_{p(z) \in \mathfrak{S}} \{ p(z) \} \).

Proof. — Let \( D = \{ z : p_0(z) < 1 \}, D_\varepsilon = \{ z : p_0(z) + \varepsilon \| z \| < 1 \}, \) which are open. Consider a complex line \( (\lambda z_0), \lambda \in \mathbb{C} \) (which we assume to be \( (\lambda(z_1, 0, \ldots, 0)) \)), and let

\[
D^{r_0} = D \cap (\lambda z_0), \quad D_\varepsilon^{r_0} = D_\varepsilon \cap (\lambda z_0).
\]

This determines two concentric circles in the \( (\lambda z_0) \) line. We choose a radius \( r_{z_0} < \infty \) between the radii of these two concentric circles and \( \varepsilon_{z_0} \) so small that the convex set
We define $p_{z_0}(z) = \inf_{t \in K_{z_0}} t$, which is a complex norm. Since $D_\varepsilon$ is a compact set, it can be covered by a finite number of the open sets $K_{z_j}$, $j = 1, \ldots, N$. Then $p_0(z) \leq \inf_{j} p_{z_j}(z) \leq p_0(t) + \varepsilon \|t\|$. Q.E.D.

**Theorem 6.** — Let $\rho < 1$ and let $f$ have circular indicator $h^*_e(z)$ with respect to $\rho(r)$. Let $\mu \in \bigcap_{\lambda > 0} (E_\lambda^{\rho(r)})'$ such that $\mu(1) \neq 0$. Then there is a unique solution $g$ of the equation $\ddot{y}(x) = f$ such that, if $k_e^*(z)$ is the circular indicator of $g$ with respect to $\rho(r)$, $k_e^*(z) \leq h^*_e(z)$.

**Proof.** — Let $p_\alpha(z)$ be a family of norms such that

$$h^*_e(z)^{1/\rho} = \inf_\alpha p_\alpha(z).$$

Then $\mu \in (E_\lambda^{\rho(r)})'$ for every $\alpha$, and by Theorem 5, there exists a unique solution $g$ to the equation $\ddot{y}(g) = f$. We clearly have

$$k_e^*(z) \leq h^*_e(z).$$

Q.E.D.

In particular, if $P(D)$ is a differential polynomial with constant coefficients and non-zero constant term, then for $\rho < 1$, there is a unique solution $g$ of the differential equation $P(D)g = f$ where $g$ has the same circular indicator as $f$.

3. The case of $\rho = 1$ and convex functions.

Let $h_k$ be a convex function, $K$ the associated convex compact set. We make the space $E_{h_k}$ of entire functions $F(u)$ whose convex indicator functions are less than or equal to $h_k$ into a Frechet space as in (6) by choosing $p_n(z) = h_k(z) + \frac{1}{n} \|z\|; (E_{h_k})'$ is its dual space. We have the following characterization of $(E_{h_k})'$ [8].
PROPOSITION 3. — The space \( (E_n^g)^\prime \) is just the set of measures \( m \) for which there exists an \( \varepsilon > 0 \) such that \( m \cdot e^{n^{-1} g(z) + \varepsilon \| z \|} \) is a bounded measure.

We recall some of the basic notions that A. Martineau [8] used in defining the projective Laplace transformation of a function \( f(z) \) of exponential type. Let \( V \) be an \( n \)-dimensional linear vector space, \( V' \) its dual. Let \( P(V) \) be the projective space obtained from \( V \) by adding the points at infinity, \( P(V') \) that obtained from \( V' \) by adding the points at infinity. We write the coordinates of \( P(V) \) as \((\xi_0, \xi)\), those of \( P(V') \) as \((\xi_0', \xi)\), and we let \( \xi \) be the hyperplane

\[ \xi_0 \cdot \xi_0 + z, \xi = 0. \]

We introduce the differential forms \( \pi(z) = dz_1 \wedge \ldots \wedge dz_n \),

\[ \theta(\xi) = \sum_{j=1}^{n} (-1)^{j+1} \xi_j \ dx_1 \wedge \ldots \wedge \hat{dx}_j \wedge \ldots \wedge dx_n \]

\((d\xi_j \text{ omitted})\) and \( \omega(\xi, z) = \theta(\xi) \wedge \pi(z) \), which is defined in \( V \times P(V') \).

Let \( \Gamma \) be the boundary of a strictly convex open set \( \Omega \) and assume \( \Gamma \) regular and oriented by Stokes' formula \( \int_{\partial \Omega} \pi = \int_{\Omega} d\pi \). To each point \( z \in \Gamma \), we have the associated hyperplane \( \xi(z) \) through \( z \) tangent to \( \Gamma \). This defines a manifold \( \Sigma(\Gamma) \) in \( V \times P(V') \).

For a compact convex set \( K \), we designate by \( \hat{K} \) the open subset of \( P(V') \) formed of hyperplanes \( \xi \) such that \( \xi \cap K = \{0\} \).

PROPOSITION 4 [8]. — Suppose \( K \) convex and compact. Let \( \psi \) be a function defined in \( \hat{K}, \) holomorphic there, and zero at the points at infinity \((\xi_0 = 0)\). Let \( \tilde{f} \in \mathcal{H}(K) \) (functions holomorphic in a neighborhood of \( K \)) and \( f \) a representative of \( \tilde{f} \) in an open neighborhood \( \Omega \) of \( K \). Let \( \omega \) be a strictly convex neighborhood of \( K \) with regular boundary included in \( \Omega \). Posing

\[ T_{\psi}(\tilde{f}) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} f(z) \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left( \frac{1}{\xi_0} \psi(\xi) \right) \omega(z, \xi) \tag{9} \]

we define a continuous linear functional on \( \mathcal{H}(K) \) which is independent of the choice of the representative \( f \) and of \( \omega \).
Let $F(u)$ be an arbitrary element of $E_{h_k}$. We define the function

$$\mathcal{P}_F(\xi) = \xi_0 \int_0^\infty F(-\xi t) e^{-\xi_0 t} dt.$$ 

This defines a function in $\mathcal{C}K$ which is zero at the points at infinity $\xi_0 = 0$. The function $\mathcal{P}_F$ is called the projective Fourier-Borel transform of $F$. We then have

**Proposition 5 [8].** Let $F(u) \in E_{h_k}$.

Then

$$F(u) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} \exp <z, u> \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left( \frac{\mathcal{P}_F(\xi)}{\xi_0} \right) \omega(z, \xi),$$

where $\omega$ is any strictly convex neighborhood of $K$ with regular boundary.

Let $\mu \in (E_{h_k}')$. We define the Fourier-Borel transform of $\mu$ to be $f_\mu(z) = \mu(\exp <z, u>)$, which, by Proposition 3, defines a function holomorphic in a neighborhood of $K$. For $\nu \in (E_{h_k}')$, we define the convolution of $\mu$ with $\nu$ as $(\nu * \mu)(F(u)) = \mu(\nu, F(u + \nu))$. We refer the reader again to [8] to see that the convolution is well defined. We then have the relationship that $f_{\nu * \mu}(z) = f_\nu(z) \cdot f_\mu(z)$ where these functions are defined.

On the other hand, let $g(z)$ be a function holomorphic in a neighborhood of $K$. Then $g$ defines a continuous linear operator $S_g$ from $E_{h_k}$ into $E_{h_k}$ by

$$S_g(F(u)) = \frac{1}{(2\pi i)^n} \int_{\Sigma(\omega)} g(z) \exp <z, u> \frac{\partial^{n-1}}{\partial \xi_0^{n-1}} \left( \frac{\mathcal{P}_F(\xi)}{\xi_0} \right) \omega(z, \xi),$$

where $\omega$ is a suitably small strictly convex regular neighborhood of $K$.

**Lemma 5.** Let $\psi_{z_0} = \mathcal{P}_{\exp <z_0, u>}$ for $z_0 \in K$. Then the linear functional on $\mathcal{H}(K)$ determined by $\psi_{z_0}$, $T_{\psi_{z_0}} = \delta(z_0)$, the Dirac measure.

**Proof.** Let $f$ be a representative of $\overline{f} \in \mathcal{H}(K)$ defined in some convex neighborhood $\omega$ of $K$. Since $\omega$ is a Runge domain, $f$ can be
uniformly approximated by polynomials in an open neighborhood of 
K, and since $z_i = \lim_{|\lambda| \to 0} \frac{e^{z_i \lambda} - 1}{\lambda}, \lambda \in \mathbb{C}$, f can be uniformly approxi-
mated by exponentials. But by (10), we have that $T_{\psi z_0}$ is just $f(z_0)$
for the exponentials. It now follows from the uniform convergence in
a neighborhood of K that $T_{\psi z_0}(f) = f(z_0)$.

LEMMA 6. – Let $\nu \in (E^\ast_{h_k})'$. If $f_\nu$ is its Fourier-Borel transform,
then the linear operator $Q_{f_\nu} : E_{h_k} \to E_{h_k}$ is just the transpose of the
convolution $\nu * \mu$ (i.e. $(Q_{f_\nu}(F), \mu) = (F, \nu * \mu)$).

Proof. – By Proposition 3, we can represent $\mu$ by a measure $m_\mu$
such that $m_\mu e^{h_k(u) + \varepsilon \|u\|}$ is a bounded measure for $\varepsilon$
sufficiently small. Then

$$
\mu(F(u)) = \frac{1}{(2\pi)^n} \int_{\Sigma(\omega)} \mu(\exp < z, u >) \frac{\partial^{n-1}}{\partial \xi^{n-1}} \left( \frac{F(\xi)}{\xi_0} \right) \bar{\omega}(z, \xi)
$$

follows from Fubuni’s theorem for $\omega$ a sufficiently small, strictly
convex neighborhood of K. Thus, $\mu$ is completely determined by its
values on a set of exponentials $\exp < z, u >$ defined for $z$ in a
neighborhood of K. We choose $\omega$ so small that $f_\nu$ is defined and
bounded in $\omega$. Then for $z_0 \in \omega$,

$$(Q_h(\exp < z_0, u >), \mu) =$$

$$= \mu \left( \frac{1}{(2\pi)^n} \int_{\Sigma(\omega)} \exp < z, u > f_\nu(z) \frac{\partial^{n-1}}{\partial \xi^{n-1}} \left( \frac{\psi_{z_0}(\xi)}{\xi_0} \right) \bar{\omega}(z, \xi) \right) =$$

$$= f_\nu(z_0) \mu(\exp < z_0, u >) = f_\nu(z_0) f_\mu(z_0),$$

from which the lemma follows. Q.E.D.

For $\nu \in (E^\ast_{h_k})'$, we define the differential operator with constant
coefficients $\vec{\nu}$ on $E_{h_k}$ to be the transpose of the convolution operation
$\mu \to \nu * \mu$ on $(E^\ast_{h_k})'$. 
THEOREM 7. - Let \( \mathcal{D} \) be a differential operator with constant coefficients on \( E_{\mathbb{H}} \). Then

(a) for \( F \in E_{\mathbb{H}} \), there always exists \( G \in E_{\mathbb{H}} \) such that \( \mathcal{D} G = F \),

(b) if \( f_\nu \) has no zeros in \( K \), then \( G \) is unique

(c) the polynomial exponential solutions of \( \mathcal{D} x = 0 \) are dense in the space of all solutions of this equation.

Proof. - (a) The mapping \( \mu \to f_\mu \) is a one-to-one linear mapping of \((E_{\mathbb{H}})\)' onto \( \mathfrak{E}(K) \). We topologize \( \mathfrak{E}(K) \) with the topology of convergence of the Taylor series coefficients at each point of \( K \). This is at least as weak as the equivalent on \( \mathfrak{E}(K) \) of the weak topology on \((E_{\mathbb{H}})\)' , since, for a multi-index \( \alpha \),

\[
\mu(u^a \exp < z_0 , u >) = \frac{\partial^{\mid \alpha \mid} f_\mu(z_0)}{\partial z^\alpha}.
\]

If \( f_\nu \cdot f_\mu \gamma \) is a filter converging to \( g \in \mathfrak{E}(K) \), then we must have \( g = f_\gamma \cdot f_\mu \), since the Taylor series of \( g \) is divisible by that of \( f_\nu \) at each point of \( K \). Thus the mapping \( f_\mu \to f_\nu \cdot f_\mu \) is one-to-one and closed, so \( \mu \to \nu \ast \mu \) is also one-to-one and closed. By Proposition 2, its transpose is onto.

(b) If \( f_\nu \) has no zeros in \( K \), then \( f_\nu \to f_\nu \cdot f_\mu \) is onto so \( \mu \to \nu \ast \mu \) is onto and hence its transpose is one-to-one.

(c) See [8] and [6]. Q.E.D.

The following example, due to C.O. Kiselman, shows that in some sense the results of § 2 and § 3 are sharp. Let \( P(D) = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \) and let \( f(z) = \cos \sqrt{z_1 z_2} \), which is of exponential type. Let \( u \) be a solution of exponential type of \( P(D) u = f \). Then

\[
u(0,r) - \nu(-r,0) = \int_0^1 \frac{d}{dt} \nu(-r(1-t), tr) \, dt = \]

\[
= r \int_0^1 \cos r \sqrt{1-t(1-t)} \, dt = r \int_0^1 \left( e^{r \sqrt{1-t}} + e^{-r \sqrt{1-t}} \right) \, dt \\
\geq \frac{r}{2} \int_0^1 e^{r \sqrt{1-t}} \, dt = \frac{r}{2 \sqrt{2}} e^{r \sqrt{2}}.
\]
But $h^*(z)$ the circular indicator of $f(z)$, is zero in both the complex line $(\lambda(0, z_2))$ and $(\lambda(z_1, 0))$, so that the circular indicator (and hence the radial indicator) of $u$ is strictly greater than that of $f$.

4. Functions of slow growth.

In this section, we extend the notion of a differential operator with constant coefficients to entire functions which satisfy a majoration of the form

$$|f(z)| \leq C_k \exp(\ln[p(z)])^k$$

asymptotically for some $k > 1$ and some norm $p(z)$. These functions are known to have very even growth [1].

We define the logarithmic order $\rho$ of such a function to be the infimum of all $k$ for which (11) holds. We define the logarithmic type $\sigma$ of $f$ (with respect to a logarithmic order $\rho$) to be the infimum of all $b$ such that

$$|f(z)| \leq C_b \exp b(\ln[p(z)])^\rho .$$

These values are clearly independent of the norm used to define them.

**Theorem 8.** — Let $m$ be a multi-index of positive numbers $m = (m_1, \ldots, m_n)$, $|m| = \sum m_i$. Then the logarithmic order and logarithmic type of a function $f$ are given by

$$\frac{\rho}{\rho - 1} = \lim_{|m| \to \infty} \frac{\ln \ln^+ \frac{1}{|c_m|}}{\ln n} \text{ and } \left(\frac{\rho - 1}{\rho}\right)^{\frac{1}{\rho - 1}} = \lim_{|m| \to \infty} \frac{\ln \frac{1}{|c_m|}}{\frac{\rho}{n^{\rho - 1}}}$$

where $f(z) = \sum_m c_m z^m$ and $\ln^+ a = \sup (0, \ln a)$.

**Remark.** — We interpret this to mean $\rho = 1$ if the limit in (12) is infinite. In this case, if we have $\sigma < +\infty$, we have a polynomial. We do not consider this case but rather assume that if $\rho = 1$ that $\sigma = +\infty$. 
Proof. — Let \( b > 0 \) and \( k > 1 \) be numbers such that

\[
|f(z)| \leq C \exp b(\ln r)^k.
\]

We assume without loss of generality that \( r = \|z\|_1 \), where \( \|z\|_1 = \max_i |z_i| \). By applying Cauchy's formula to the distinguished boundary of the polydisc of radius \( r \), we get

\[
|c_n| \leq C \exp \{ b(\ln r)^k - |m| \ln r \}.
\]

This function takes on its maximum (for \( k > 1 \)) when \( \ln r = \frac{1}{kb} \)

and equals \( \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) m^{\frac{k}{k-1}} \right\} \), which establishes the theorem in one direction.

On the other hand, if \( |c_m| \leq K \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) m^{\frac{k}{k-1}} \right\} \),

\[
|f(z)| \leq \sum_m K |m|^n \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) m^{\frac{k}{k-1}} + |m| \ln r \right\}
\]

on the distinguished boundary of the polydisc of radius \( r \). The function

\[
\left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) x^{\frac{k}{k-1}} + x \ln r
\]

takes on its maximum for

\[
x = \{(kb)^{\frac{1}{k-1}} \ln r\}
\]

and equals \( \exp b(\ln r)^k \).

Let \( M_0 = \{(kb)^{\frac{1}{k-1}} \ln r\}^{k-1} \) and

\[
M'_0 = \left\{ \left( \frac{k}{2(k-1)} (kb)^{\frac{1}{k-1}} \ln r \right)^{k-1} \right\}
\]

("greatest integer in"). Then

\[
|f(z)| \leq K'(\ln r)^{2n(k-1)} \exp b(\ln r)^k + \\
+ \sum_{|m|=M_0+1}^\infty r^{|m|} \exp \left\{ \left( \frac{1}{kb} \right)^{\frac{1}{k-1}} \left( \frac{1}{k} - 1 \right) m^{\frac{k}{k-1}} \right\}.
\]
But
\[ \sum_{|m| > M_0 + 1} r^{|m|} \exp \left\{ \frac{1}{k} \left( \frac{1}{k^2} \right) \left( \frac{1}{k} - 1 \right) |m|^k \right\} \leq \]
\[ \sum_{|m| > M_0 + 1} \exp \left\{ \frac{1}{k} \left( \frac{1}{k^2} \right) \left( \frac{1}{k} - 1 \right) |m|^k + |m| (M_0 + 1)^{k-1} \right\} \]
and this last series is bounded independently of \( M_0 \) since
\[ \left( \frac{1}{k} - 1 \right) |m|^{k-1} + |m| (M_0 + 1)^{k-1} = \]
\[ = |m| \left( \frac{1}{k} - 1 \right) \left( |m|^{k-1} - \frac{(k-1)}{k} (M_0 + 1)^{k-1} \right) < |m| \left( \frac{1}{k} - 1 \right) T \]
for some \( T > 0 \). Q.E.D.

We let \( \mathcal{E}_{\sigma, \rho} \) be the Fréchet space that we get by taking
\[ p_n = \left( \sigma + \frac{1}{n} \right) (\ln r)^\rho \]
in (6), \( \mathcal{E}_1 \) that which we get by taking \( p_n = (\ln r)^{1 + \frac{1}{n}} \), and we de-
signate their duals by \( (\mathcal{E}_{\sigma, \rho})' \) and \( (\mathcal{E}_1)' \).

**Lemma 7.** — A linear functional \( \mu \) on \( \mathcal{E}_{\sigma, \rho} \) (resp. \( \mathcal{E}_1 \)) is in \( (\mathcal{E}_{\sigma, \rho})' \) (resp. \( (\mathcal{E}_1)' \)) if and only if
\[ |\mu(z^m)| \leq K_\varepsilon \exp \left\{ \frac{1}{(\sigma + \varepsilon) \rho} \right\} |m|^{\rho-\frac{1}{\rho}} \tag{13} \]
(resp.
\[ |\mu(z^m)| \leq K_\varepsilon \exp \left\{ \frac{1}{1 + \varepsilon} \right\} \left( \frac{\varepsilon}{1 + \varepsilon} \right)^{\frac{1+\varepsilon}{\varepsilon}} |m|^{\frac{1+\varepsilon}{\varepsilon}} \tag{14} \]
for some \( \varepsilon > 0 \).

**Proof.** — It follows from the proof of Theorem 8 that the Taylor series of an element in \( \mathcal{E}_{\sigma, \rho} \) (resp. \( \mathcal{E}_1 \)) converges to the function in this space (cf. [8]). Thus, if \( \mu \) is a continuous linear functional, it follows that (13) (resp. (14)) holds.
On the other hand, if (13) (resp. (14)) holds, it follows from the estimates of Theorem 8 that $\mu$ is a continuous linear functional on $E_{a,\rho}$ (resp. $E_1$).

For $\mu \in (E_{a,\rho})'$ (resp. $(E_1)'$), we define its Fourier-Borel transform

$$\tilde{\mu}(u) = \mu(\exp <z, u>) = \sum \frac{\mu(z^n)}{n!} u^n,$$

in the sense of a formal power series at the origin. We topologize this space with the topology of convergence of coefficients. Let $Q_{a,\rho}$ (resp. $Q_1$) be the space of formal power series whose coefficients satisfy (13) (resp. (14)) above.

For $\nu, \mu \in (E_{r,\rho})'$ (resp. $(E_1)'$), we define the convolution of $\mu$ with $\nu$, $\nu \ast \mu$ to be

$$\nu \ast \mu(f(u)) = \mu(\nu(f(u) + v)) .$$

A differential operator with constant coefficients on $E_{a,\rho}$ (resp. $E_1$) is defined as the transpose of this convolution operation. We then have the following

**Theorem 9.** — Let $\tilde{\nu}$ be a differential operator with constant coefficients on the space $E_{a,\rho}$ (resp. $E_1$). Then for $f \in E_{a,\rho}$ (resp. $E_1$) there is always a solution $g \in E_{a,\rho}$ (resp. $E_1$) of the equation $\tilde{\nu}(g) = f$. If $\tilde{\nu}(1) = 0$, then $g$ is unique.

The proof is the same as that of Theorem 6, with some alterations in the calculations of Theorem 5 to prove that the operation of convolution is closed. The details are left to the interested reader.

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