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INFINITELY DIVISIBLE PROCESSES AND THEIR POTENTIAL THEORY, (II. Part) [1]

by S. C. PORT and C. J. STONE

16. Some Fourier Analysis.

The characteristic function $\hat{\mu}^t(\theta)$, $\theta \in \hat{\mathfrak{G}}$, is defined for $t \geqslant 0$ by

$$\hat{\mu}^t(\theta) = \int_{\mathfrak{G}} \langle \theta, x \rangle \mu^t\,(dx) = \int_{\mathfrak{G}} \langle \theta, x \rangle \mathrm{P}^t(0, dx).$$

It is jointly continuous in t and θ and satisfies the equation

$$(16.1) \quad \hat{\mu}^{s+t}(\theta) = \hat{\mu}^s(\theta)\hat{\mu}^t(\theta), \quad s, t \geqslant 0 \quad \text{and} \quad \theta \in \hat{\mathfrak{G}}.$$

For convenience we set $\hat{\mu}(\theta) = \hat{\mu}^1(\theta)$.

PROPOSITION 16.1. — *There is a uniquely defined function* $\log \hat{\mu}(\theta)$, $\theta \in \hat{\mathfrak{G}}$, *such that*

$$\hat{\mu}^t(\theta) = e^{t \log \hat{\mu}(\theta)}, \qquad t \geqslant 0 \qquad \textit{and} \qquad \theta \in \hat{\mathfrak{G}}.$$

This function is continuous, vanishes at $\theta = 0$ *and nowhere else, and has non-positive real part.*

Proof. — By (16.1) there is a uniquely defined number $\varphi(\theta)$ such that

$$\hat{\mu}^t(\theta) = e^{t\varphi(\theta)}, \qquad t \geqslant 0 \qquad \text{and} \qquad \theta \in \hat{\mathfrak{G}}.$$

Since $\hat{\mu}^t(0) = 1$ for $t \geqslant 0$, it follows that $\varphi(0) = 1$. Suppose $\varphi(\theta_0) = 1$. Then $\hat{\mu}^t(\theta_0) = 1$ for $t \geqslant 0$. This implies that

[1] The fist part was published in Ann. Inst. Fourier, tome 21, 2.

each S_t is contained in the subgroup

$$\{x \in \mathfrak{G} \mid \langle x, \theta_0 \rangle = 1\}.$$

Therefore this subgroup equals the whole group and hence $\theta_0 = 0$.

We will now show that φ is continuous. Let $\log z$ be defined continuously for $|z - 1| < 1$ so that $\log 1 = 0$. Let N be a relatively compact open subset of $\hat{\mathfrak{G}}$. There is a $t_0 > 0$ such that

$$|\hat{\mu}^t(\theta) - 1| < 1, \qquad 0 \leqslant t \leqslant t_0 \qquad \text{and} \qquad \theta \in N.$$

Thus $\log \hat{\mu}^t(\theta)$ is well defined and continuous in t and θ for $0 \leqslant t \leqslant t_0$ and $\theta \in N$. If $s \geqslant 0$, $t \geqslant 0$, $s + t \leqslant t_0$, and $\theta \in N$, then

$$\log \hat{\mu}^t(\theta) = \log \hat{\mu}^s(\theta) + \log \hat{\mu}^t(\theta).$$

Consequently there is a continuous function $\psi(\theta)$, $\theta \in N$, such that

$$\log \hat{\mu}^t(\theta) = t\psi(\theta), \qquad 0 \leqslant t \leqslant t_0 \qquad \text{and} \qquad \theta \in N.$$

It follows that

$$\hat{\mu}^t(\theta) = e^{t\psi(\theta)}, \qquad 0 \leqslant t \leqslant t_0 \qquad \text{and} \qquad \theta \in N.$$

From (16.1) we see that

$$\hat{\mu}^t(\theta) = e^{t\psi(\theta)}, \qquad 0 \leqslant t < \infty \qquad \text{and} \qquad \theta \in N,$$

and consequently that ψ agrees with φ on N. Since N is any relatively compact open subset of $\hat{\mathfrak{G}}$, we see that φ is a continuous function on $\mathfrak{G}$. Since $\varphi(\theta)$ is a logarithm of $\hat{\mu}(\theta)$, the proof of the proposition is complete.

THEOREM 16.1. — *The function*

$$\frac{1 - \Re\langle x, \theta\rangle}{\Re \log \hat{\mu}(\theta)}$$

is bounded for x in compact subsets of $\mathfrak{G}$ and θ in compact subsets of the complement of the origin of $\hat{\mathfrak{G}}$.

The proof of this result is an obvious modification of the proof of Theorem 3.1 of [7] and will be omitted.

Proposition 16.2. — *There is a relatively compact open neighborhood* Q *of the origin of* $\hat{\mathfrak{G}}$ *such that*

$$\left|\frac{1}{\lambda - \log \hat{\mu}(\theta)} - \frac{1}{\lambda + 1 - \hat{\mu}(\theta)}\right| \leqslant 2, \qquad \lambda > 0 \qquad \text{and} \qquad \theta \in Q.$$

Proof. — There is a relatively compact open neighborhood Q of the origin of $\hat{\mathfrak{G}}$ such that

$$|\hat{\mu}(\theta) - 1| \geqslant \frac{|\log \hat{\mu}(\theta)|}{2} \qquad \theta \in Q,$$

and

$$|\hat{\mu}(\theta) - 1 - \log \hat{\mu}(\theta)| \leqslant |\log \hat{\mu}(\theta)|^2, \quad \theta \in Q.$$

It follows that

$$|\lambda + 1 - \hat{\mu}(\theta)| \geqslant \frac{|\log \hat{\mu}(\theta)|}{2}, \qquad \lambda > 0 \qquad \text{and} \qquad \theta \in Q.$$

We also have that

$$|\lambda - \log \hat{\mu}(\theta)| \geqslant |\log \hat{\mu}(\theta)|, \qquad \lambda > 0 \qquad \text{and} \qquad \theta \in \hat{\mathfrak{G}}.$$

Thus for $\lambda > 0$ and $\theta \in Q$

$$\left|\frac{1}{\lambda - \log \hat{\mu}(\theta)} - \frac{1}{\lambda + 1 - \hat{\mu}(\theta)}\right| = \frac{|\hat{\mu}(\theta) - 1 - \log \hat{\mu}(\theta)|}{|\lambda + 1 - \hat{\mu}(\theta)||\lambda - \log \hat{\mu}(\theta)|} \leqslant 2,$$

as desired.

Theorem 16.2. — *Let* Q *be a relatively compact open neighborhood of the origin of* $\hat{\mathfrak{G}}$. *The process is transient or recurrent according as*

$$\int_Q \Re\left(\frac{1}{\log \hat{\mu}(\theta)}\right) d\theta$$

converges or diverges.

Proof. — Suppose the process is transient. Choose $f(x)$, $x \in \mathfrak{G}$, such that $f \in C_c^+$, $J(f) > 0$, and $\hat{f}$ is non-negative and integrable. Then

$$G^\lambda f(0) = \int_{\hat{\mathfrak{G}}} \frac{\hat{f}(\theta)}{\lambda - \log \hat{\mu}(\theta)} d\theta.$$

By letting $\lambda \downarrow 0$ and using Fatou's Lemma, we find that

$$\int_{\mathfrak{G}} \hat{f}(\theta)\Re\left(\frac{-1}{\log \hat{\mu}(\theta)}\right) d\theta \leqslant Gf(0) < \infty.$$

Since $\hat{f}(0) = J(f) > 0$ and $\log \hat{\mu}(\theta) \neq 0$ for $\theta \neq 0$, we see that

$$\int_{Q} \Re\left(\frac{1}{\log \hat{\mu}(\theta)}\right) d\theta$$

converges.

Suppose instead that the process is recurrent. Then by Proposition 5.3 S_t generates $\mathfrak{G}$ for some $t \supset 0$. Without loss of generality we can assume that S_1 generates $\mathfrak{G}$. Then the random walk obtained by looking at the process at integer times is a recurrent random walk on $\mathfrak{G}$. By Theorem 5.1 of [7]

$$\int_{Q} \Re\left(\frac{1}{1 - \hat{\mu}(\theta)}\right) d\theta$$

diverges. By Proposition 16.2

$$\int_{Q} \left(\Re\left(\frac{1}{\log \hat{\mu}(\theta)}\right) + \Re\left(\frac{1}{1 - \hat{\mu}(\theta)}\right)\right) d\theta$$

converges. Thus

$$\int_{Q} \Re\left(\frac{1}{\log \hat{\mu}(\theta)}\right) d\theta$$

diverges as desired.

17. The Recurrent Potential Operator.

Throughout this section it will be assumed that the process is recurrent. Let $\mathscr{F}^{+}$ denote the collection of functions $f(x)$, $x \in \mathfrak{G}$, such that

(i) f is a continuous, non-negative and integrable function;

(ii) f is supported by a compactly generated subgroup of $\mathfrak{G}$;

(iii) $\hat{f}$, the Fourier transform of f, has compact support; and

(iv) there is a compact subset C of $\mathfrak{G}$, a constant c such that $0 < c < \infty$, and an open neighborhood Q of the origin of $\hat{\mathfrak{G}}$ such that

$$\mathrm{J}(f) - \mathfrak{R}\hat{f}(\theta) \leqslant c \max_{x \in \mathrm{C}} (1 - \mathfrak{R}\langle x, \theta\rangle), \qquad \theta \in \mathrm{Q}.$$

Properties of this family of functions were developed in [7]. Let $\mathfrak{F}$ denote the collection of differences of elements of $\mathfrak{F}^+$. Recall that

$$\mathrm{G}^\lambda f = \int_0^\infty e^{-\lambda t} \mathrm{P}^t\, f dt.$$

For $\lambda > 0$ and $f \in \mathfrak{F}$ we have

$$\mathrm{G}^\lambda f(x) = \int_{\hat{\mathfrak{G}}} \frac{\langle x, \theta\rangle \hat{f}(-\theta)}{\lambda - \hat{\mu}(\theta)}\, d\theta.$$

Choose $g \in \mathfrak{F}^+$ such that g is symmetric and $\mathrm{J}(g) = 1$. For $\lambda > 0$ set $c^\lambda = \mathrm{G}^\lambda g(0)$ and define A^λ for $f \in \mathfrak{F}$ and $x \in \mathfrak{G}$ by

$$\mathrm{A}^\lambda f(x) = c^\lambda \mathrm{J}(f) - \mathrm{G}^\lambda f(x).$$

Then

$$\mathrm{A}^\lambda f(x) = \int_{\hat{\mathfrak{G}}} \frac{\hat{g}(-\theta)\mathrm{J}(f) - \langle x, \theta\rangle \hat{f}(-\theta)}{\lambda - \log \hat{\mu}(\theta)}\, d\theta.$$

Also

$$\mathrm{A}^\lambda f_y(x) - \mathrm{A}^\lambda f_y(0) = \int_{\hat{\mathfrak{G}}} \frac{\langle -y, \theta\rangle (1 - \langle x, \theta\rangle) \hat{f}(-\theta)}{\lambda - \log \hat{\mu}(\theta)}\, d\theta.$$

In the non-singular case it is convenient to modify the definition of c^λ. Let $\mu^{(t)}$ be the distribution of X_t when $\mathrm{X}_0 = 0$ and define

$$\mu^\lambda = \int_0^\infty e^{-\lambda t} \mu^{(t)}\, dt.$$

Then we can write

$$\mu^\lambda = \mu_1^\lambda + \varphi * \mu_2^\lambda * \mu^\lambda,$$

where, as $\lambda \downarrow 0$, μ_1^λ and μ_2^λ increase to finite measures μ_1 and μ_2 and φ is a probability measure having compact support and a continuous density and is such that $\hat{\varphi}$ is absolutely integrable. This decomposition is obtained as in Port and Stone [8].

The measure $\varphi * \mu_2^\lambda * \mu^\lambda$ is absolutely continuous and has a continuous density p^λ given by

$$p^\lambda(y) = \int_{\hat{\mathfrak{G}}} \frac{\langle -y, \theta\rangle \hat{\varphi}(\theta) \hat{\mu}_2^\lambda(\theta)}{\lambda - \log \hat{\mu}(\theta)}\, d\theta.$$

Set $c^\lambda = p^\lambda(0)$ and $a^\lambda(y) = c^\lambda - p^\lambda(y)$. Then

$$a^\lambda(y) = \int_{\hat{\mathfrak{G}}} \frac{(1 - \langle -y, \theta\rangle)\hat{\varphi}(\theta)\hat{\mu}_{\frac{1}{2}}^{\lambda}(\theta)}{\lambda - \log \hat{\mu}(\theta)}\, d\theta$$

and

$$a^\lambda(y - x) - a^\lambda(y) = \int_{\hat{\mathfrak{G}}} \frac{\langle -y, \theta\rangle(1 - \langle x, \theta\rangle)\hat{\varphi}(\theta)\hat{\mu}_{\frac{1}{2}}^{\lambda}(\theta)}{\lambda - \log \hat{\mu}(\theta)}\, d\theta.$$

We now define

$$A^\lambda f(x) = c^\lambda J(f) - G^\lambda f(x)$$

then

$$A^\lambda f(x) = \int_{\mathfrak{G}} a^\lambda(y - x) f(y)\, dy - \int_{\mathfrak{G}} \mu_{\frac{1}{2}}^{\lambda}\, (dy) f(s + y)$$

and

$$A^\lambda f_y(x) - A^\lambda f_y(0) = \int_{\mathfrak{G}} (a^\lambda(z + y - x) - a^\lambda(z + y)) f(z)\, dz - \int_{\mathfrak{G}} \mu_{\frac{1}{2}}^{\lambda}\, (dz)(f(z + x - y) - f(z - y)).$$

The process is said to be type II recurrent if $\mathfrak{G} \cong R \oplus H$ or $\mathfrak{G} \cong Z \oplus H$, where H is compact, and the induced process on R or Z has mean zero and finite variance. Otherwise the process is said to be type I recurrent. In the type II case we can assume that $\mathfrak{G} = R \oplus H$ or $Z \oplus H$ and Haar measure on $\mathfrak{G}$ is the direct product of Lebesgue measure on R or counting measure on Z and normalized Haar measure on H. Let ψ be the projection from $\mathfrak{G}$ onto R or Z and let σ^2 denote the variance of $\psi(X_1)$. Then $\psi(X_t)$ has mean 0 and variance $\sigma^2 t$. We set

$$\mathfrak{G}^+ = \{x \in \mathfrak{G} \mid \psi(x) \geqslant 0\} \quad \text{and} \quad \mathfrak{G}^- = \{x \in \mathfrak{G} \mid \psi(x) < 0\}.$$

By $x \to +\infty$ or $x \to -\infty$ we mean $x \to \infty$ and $x \in \mathfrak{G}^+$ or $x \in \mathfrak{G}^+$.

Proposition 17.1. — *Let Q be a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Then*

$$\lim_{\lambda \downarrow 0} \int_Q \frac{1 - \langle x, \theta\rangle}{\lambda - \log \hat{\mu}(\theta)}\, d\theta$$

exists and is finite. In the type I case

$$\lim_{y \to \infty} \lim_{\lambda \downarrow 0} \int_Q \frac{\langle -y, \theta\rangle(1 - \langle x, \theta\rangle)}{\lambda - \log \hat{\mu}(\theta)}\, d\theta = 0.$$

In the type II *case*

$$\lim_{y \to +\infty} \lim_{\lambda \downarrow 0} \int_{\mathrm{Q}} \frac{\langle -y, \theta \rangle (1 - \langle x, \theta \rangle)}{\lambda - \log \hat{\mu}(\theta)} \, d\theta = \mp \sigma^{-2} \psi(x).$$

Proof. — We can assume that the random walk obtained by looking at the process at integer times is a random walk on $\mathfrak{G}$. Then by Proposition 16.2, the result reduces immediately to Theorem 5.2 of [7].

THEOREM 17.1. — *In the non-singular case there is a continuous function* $a(x)$, $x \in \mathfrak{G}$, *such that*

$$\lim_{\lambda \downarrow 0} a^{\lambda}(x) = a(x), \qquad x \in \mathfrak{G}.$$

In the type I *case*

$$\lim_{y \to \infty} (a(y - x) - a(y)) = 0$$

and in the type II *case*

$$\lim_{y \to \pm\infty} (a(y - x) - a(y)) = \mp \sigma^{-2} \psi(x).$$

The convergence in these limits is uniform for x *in compacts.*

Proof. — Let Q denote a relatively compact open neighborhood of the origin of $\hat{\mathfrak{G}}$. Set

$$a^{\lambda}_{\mathrm{Q}^c}(y) = \int_{\mathrm{Q}^c} \frac{(1 - \langle -y, \theta \rangle) \hat{\varphi}(\theta) \hat{\mu}^{\lambda}_{2}(\theta)}{\lambda - \log \hat{\mu}(\theta)} \, d\theta.$$

Then as $\lambda \downarrow 0$, $a^{\lambda}_{\mathrm{Q}^c}$ converges uniformly on compacts to

$$a_{\mathrm{Q}^c}(y) = \int_{\mathrm{Q}^c} \frac{(1 - \langle -y, \theta \rangle) \hat{\varphi}(\theta) \hat{\mu}_2(\theta)}{- \log \hat{\mu}(\theta)} \, d\theta.$$

By the Riemann-Lebesgue Lemma

$$\lim_{y \to \infty} (\hbar)^{\circ}\eth v = \int_{\mathrm{Q}^c} \frac{\hat{\varphi}(\theta) \hat{\mu}_2(\theta)}{- \log \hat{u}(\theta)} \, d\theta$$

and hence

$$\lim_{y \to \infty} (a_{\mathrm{Q}_c}(y - x) - a_{\mathrm{Q}_c}(y)) = 0$$

uniformly for x in compacts. Set

$$ {}_1a_Q^\lambda(y) = \int_Q \frac{(1 - \langle -y, \theta\rangle)(\hat\varphi(\theta)\hat\mu_2^\lambda(\theta) - 1)}{\lambda - \log \hat\mu(\theta)}\, d\theta $$
$$ = - \int_Q (1 - \langle -y, \theta\rangle)\hat\mu_1^\lambda(\theta)\, d\theta. $$

Thus, as $\lambda \downarrow 0$, ${}_1a_Q^\lambda(y)$ converges uniformly on compacts to

$$ {}_1a_Q(y) = - \int_Q (1 - \langle -y, \theta\rangle)\hat\mu_1(\theta)\, d\theta. $$

Again by the Riemann-Lebesgue Lemma $\lim\limits_{y\to\infty} {}_1a_Q(y)$ exists and is finite and hence

$$ \lim_{y\to\infty} ({}_1a_Q(y-x) - {}_1a_Q(y)) = 0 $$

uniformly for x in compacts.

Finally, set

$$ {}_2a_Q^\lambda(y) = \int_Q \frac{1 - \langle -y, \theta\rangle}{\lambda - \log \hat\mu(\theta)}\, d\theta. $$

By Proposition 17.1, as $\lambda \downarrow 0$, ${}_2a_Q^\lambda$ converges uniformly on compacts to

$$ {}_2a_Q^\lambda(y) = \lim_{\lambda\downarrow 0} \int_Q \frac{1 - \langle -y, \theta\rangle}{\lambda - \log \hat\mu(\theta)}\, d\theta. $$

From the same theorem it follows that, uniformly for x in compacts, in the type I case

$$ \lim_{y\to\infty} ({}_2a_Q(y-x) - {}_2a_Q(y)) = 0 $$

and in the type II case

$$ \lim_{y\to\pm\infty} ({}_2a_Q(y-x) - {}_2a_Q(y)) = \mp\, \sigma^{-2}\psi(x). $$

This completes the proof of the theorem.

In order, to state results involving Φ in the non-singular case and $\mathcal{F}$ otherwise, we let $\mathcal{F}^* = \Phi$ in the non-singular case and $\mathcal{F}$ otherwise.

THEOREM 17.2. — *Let* $f \in \mathcal{F}^*$. *Then for* $x \in \mathfrak{G}$

$$ \lim_{\lambda\downarrow 0} A^\lambda f(x) = Af(x) $$

exists and is finite. In the type I *case*

$$\lim_{y\to\infty} (Af_y(x) - Af_y(0)) = 0$$

and in the type II *case*

$$\lim_{y\to\pm\infty} (Af_y(x) - Af_y(0)) = \mp\ \psi(x)\sigma^{-2}J(f).$$

These limits are all uniform for x *in compacts.*

Proof. — In the non-singular case, this result follows from Theorem 17.1. In the singular case it follows from Theorems 16.1 and 16.2, and Proposition 17.1.

In the type II recurrent case define

$$K(f) = \int_{\mathfrak{G}} \psi(x)f(x)\, dx.$$

THEOREM 17.3. — *Let* $f \in \mathcal{F}^*$ *with* $J(f) = 0$. *Then*

$$\lim_{\lambda\downarrow 0} G^\lambda f(x) = Gf(x)$$

exists and is finite and the convergence is uniform for x *in compacts. In the type* I *case*

$$\lim_{x\to\infty} Gf(x) = 0$$

and in the type II *case*

$$\lim_{x\to\pm\infty} Gf(x) = \pm\ \sigma^{-2}K(f).$$

Proof. — In the non-singular case the result follows from Theorem 17.1 and the formula

$$G^\lambda f(x) = \int_{\mathfrak{G}} (a^\lambda(-x) - a^\lambda(y-x))f(y)\,dy + \int_{\mathfrak{G}} \mu^\lambda_2\,(dy)f(x+y).$$

In the general case it follows by the same argument used in proving the corresponding result in discrete time, Theorem 5.8 of [7].

COROLLARY 17.1. — *Let* $f, f_1 \in \mathcal{F}^*$. *In the type* I *case*

$$\lim_{x\to\infty} (J(f_1)Af(x) - J(f)Af_1(x)) = 0$$

and in the type II *case*

$$\lim_{x \to \pm\infty} (J(f_1)Af(x) - J(f)Af_1(x)) = \mp \sigma^{-2}(J(f_1)K(f) - K(f_1)J(f)).$$

PROPOSITION 17.2. — *Let* C *be a compact subset of* $\mathfrak{G}$ *and* Q *a relatively compact open neighborhood of the origin of* $\hat{\mathfrak{G}}$. *Then there is an* M *such that* $0 < M < \infty$ *and*

$$\left| \int_Q \frac{\langle y, \theta \rangle (\langle x, \theta \rangle - 1)}{\lambda - \log \hat{\mu}(\theta)} \, d\theta \right| \leqslant M$$

for $y \in \mathfrak{G}$, $x \in C$, *and* $\lambda > 0$.

Proof. — By Proposition 16.2 this reduces immediately to the corresponding result in discrete time, Theorem 5.13 of [7].

In stating the next several results it is convenient to set $a^0 = a$ and $A^0 = A$.

THEOREM 17.4. — *In the non-singular case for any compact subset* C *of* $\mathfrak{G}$ *there is an* M *such that* $0 < M < \infty$ *and*

$$|a^\lambda(y - x) - a^\lambda(y)| \leqslant M, \quad y \in \mathfrak{G}, \quad x \in C, \quad \textit{and} \quad \lambda \geqslant 0.$$

Proof. — This result follows immediately from Proposition 17.2 and the definition of a^λ.

THEOREM 17.5. — *Let* $f \in \mathcal{F}^*$ *and let* C *be a compact subset of* $\mathfrak{G}$. *Then there is an* M *such that* $0 < M < \infty$ *and*

$$|A^\lambda f_y(x) - A^\lambda f_y(0)| \leqslant M, \quad y \in \mathfrak{G}, \quad x \in C, \quad \textit{and} \quad \lambda \geqslant 0.$$

Proof. — In the non-singular case this result follows from Theorem 17.4. In the singular case it follows from Proposition 17.2.

THEOREM 17.6. — *In the non-singular case there is an* M *such that* $0 < M < \infty$ *and*

$$a^\lambda(x) \geqslant -M, \qquad x \in \mathfrak{G} \qquad \textit{and} \qquad \lambda \geqslant 0.$$

THEOREM 17.7. — *Let* $f \in \mathcal{F}^*$ *with* $J(f) \geqslant 0$. *Then there is an* M *such that* $0 < M < \infty$ *and*

$$A^\lambda f(x) \geqslant -M, \qquad x \in \hat{\mathfrak{G}} \qquad \textit{and} \qquad \lambda \geqslant 0.$$

Proof of Theorems 17.6 *and* 17.7. Theorem 17.6 follows from Proposition 16.2 and the arguments used in proving the corresponding result in discrete time, Theorem 7.8 of [7]. Theorem 17.7 follows from Theorem 17.6 in the non-singular case. Otherwise it follows from Proposition 16.2 and the arguments used in proving the corresponding result in discrete time, Theorem 5.15 of [7].

THEOREM 17.8. — *Let* $f \in \mathcal{F}^*$ *with* $J(f) = 0$. *Then there is an* M *such that* $0 < M < \infty$ *and*

$$|G^\lambda f(x)| \leqslant M, \qquad x \in \mathfrak{G} \qquad \textit{and} \qquad \lambda \geqslant 0.$$

Proof. — This result is an immediate corollary of the above theorem.

THEOREM 17.9. — *Let the process be non-singular. If* $\mathfrak{G} \cong R \oplus H$ *or* $\mathfrak{G} \cong Z \oplus H$, *where* H *is compact, there is an* L *such that* $0 \leqslant L \leqslant \infty$ *and either*

$$\lim_{x \to -\infty} a(x) = L \qquad \textit{and} \qquad \lim_{x \to +\infty} a(x) = \infty$$

or

$$\lim_{x \to -\infty} a(x) = \infty \qquad \text{and} \qquad \lim_{x \to +\infty} a(x) = L.$$

If $\mathfrak{G}$ *is not of the above type, then*

$$\lim_{x \to \infty} a(x) = \infty.$$

THEOREM 17.10. — *If* $\mathfrak{G} \cong R \oplus H$ *or* $\mathfrak{G} \cong Z \oplus H$ *where* H *is compact there is an* L *such that* $0 \leqslant L \leqslant \infty$ *and either*

$$\lim_{x \to +\infty} Af(x) = LJ(f) \qquad \textit{and} \qquad \lim_{x \to -\infty} Af(x) = \infty$$

for all $f \in \mathcal{F}^*$ *with* $J(f) > 0$ *or*

$$\lim_{x \to +\infty} Af(x) = \infty \qquad \textit{and} \qquad \lim_{x \to -\infty} Af(x) = LJ(f)$$

for all $f \in \mathcal{F}^*$ *with* $J(f) > 0$. *If* $\mathfrak{G}$ *is not of the above type, then for all* $f \in \mathcal{F}^*$ *with* $J(f) > 0$

$$\lim_{x \to \infty} Af(x) = \infty.$$

Proof of Theorems 17.9 *and* 17.10. — By using Proposition 16.2, we can extend Theorem 9.4 of [7] to continuous time. From this point on, the proof of these two theorems is similar to the arguments used in proving the corresponding discrete time results in Section 9 of [7].

In the type I case let $C_0(\mathfrak{G})$ be the collection of continuous functions on $\mathfrak{G}$ which vanish at ∞. In the type II case let $C_0(\mathfrak{G})$ denote the collection of continuous functions f on $\mathfrak{G}$ having finite limits $f(+\infty)$ and $f(-\infty)$ such that $f(+\infty)+f(-\infty)=0$ then $C_0(\mathfrak{G})$ with sup norm is a Banach space.

PROPOSITION 17.3. — *Let* t *be such that* S_t *generates* $\mathfrak{G}$. *Then*

$$\int_0^t P^s \, ds$$

maps $C_0(\mathfrak{G})$ *onto a dense subset of* $C_0(\mathfrak{G})$.

Proof. — Consider first the type I case. Let γ be a bounded signed measure such that

$$\left(\gamma, \int_0^t P^s f \, ds\right) = 0, \qquad f \in C_0(\mathfrak{G}).$$

Let μ^s denote the distribution of X_s when $X_0 = 0$. Then

$$\left(\gamma * \int_0^t \mu^s \, ds, f\right) = 0, \qquad f \in C_0(\mathfrak{G}).$$

Consequently

$$\gamma * \int_0^t \mu^s \, ds = 0$$

and by taking Fourier transforms we see that

$$\hat{\gamma}(\theta) \int_0^t \hat{\mu}^s(\theta) \, ds = 0.$$

Thus $\hat{\gamma}(0) = 0$. For each $\theta \neq 0$ there is a $c \neq 0$ such that

$$\hat{\mu}^s(\theta) = e^{cs}.$$

Then

$$\int_0^t \hat{\mu}^s(\theta) \, d\theta = \frac{e^{ct} - 1}{c} = \frac{\hat{\mu}^t(\theta) - 1}{c}.$$

Since S_t generates $\mathfrak{G}$ it follows that $\hat{\mu}^t(\theta) \neq 1$ for $\theta \neq 0$. Thus $\hat{\gamma}(\theta) = 0$ for all θ and hence γ is the trivial measure. This shows that the collection

$$\int_0^t P^s f \, ds, \qquad f \in C_0(\mathfrak{G})$$

is dense in $C_0(\mathfrak{G})$.

Consider next the type II case. Let α be a finite constant and γ a bounded signed measure such that

$$\left(\gamma, \int_0^t P^s f \, ds\right) + \alpha f(+\infty) = 0, \qquad f \in C_0(\mathfrak{G}).$$

Then

$$\left(\gamma, \int_0^t P^s f \, ds\right) = 0, \qquad f \in C_c$$

and arguing as in the type I case we conclude that γ is the zero measure. This implies that $\alpha = 0$. Thus the conclusion in the type II case is also established.

THEOREM 17.11. — *Suppose the process is non-singular. Then* $\{Af \mid f \in C_c$ *and* $J(f) = 0\}$ *is dense in* $C_0(\mathfrak{G})$.

Proof. — Since the process is non-singular S_t generates $\mathfrak{G}$ for all $t > 0$ and in particular S_1 generates $\mathfrak{G}$. If $f \in C_c$ and $J(f) = 0$ then $Af = -Gf$. Let U denote the analog of G for the random walk obtained by looking at the process at integer times. Then

$$G = \int_0^1 P^s \, ds \, U.$$

By Theorem 12.1 of [7], $\{Uf, f \in C_c$ and $J(f) = 0\}$ is dense in $C_0(\mathfrak{G})$ and hence by Proposition 17.3

$$\left\{ Gf = \int_0^1 P^s \, ds \, Uf \mid f \in C_c \quad \text{and} \quad J(f) = 0 \right\}$$

is dense in $C_0(\mathfrak{G})$ as desired.

From Theorem 17.11 we obtain

COROLLARY 17.2. — *Let* $g \in C_c^+$, $J(g) = 1$, *and let* $\chi = C_0(\mathfrak{G}) \oplus \{Ag\}$. *In the non-singular case the linear manifold* $\{Af : f \in C_c\}$ *is dense in* χ.

18. The Behaviour of G_B.

Throughout this section the process is assumed to be recurrent. Recall that $\mathcal{B}$ denotes the collection of relatively compact Borel sets in $\mathfrak{G}$. We let $\mathcal{B}_1$ denote those sets $B \in \mathcal{B}$ such that $C^\lambda(B) > 0$. If $B \in \mathcal{B}_1$, then $P_x(T_B < \infty) = 1$ a.e. $x \in \mathfrak{G}$. In the non-singular case if $B \in \mathcal{B}_1$ then $P_x(T_B < \infty) = 1$ for all $x \in \mathfrak{G}$. We let $\mathcal{B}_2$ denote those sets $B \in \mathcal{B}$ such that $G_B(x, A)$ is integrable on compacts whenever A is compact. We let $\mathcal{B}_3$ denote those $B \in \mathcal{B}$ such that for every compact set C there exist finite positive constants t and δ such that

$$P_x(T_B \leqslant t) \geqslant \delta, \qquad x \in C.$$

It is clear that if $B \in \mathcal{B}$, then $G_B(x, A)$ is bounded in x whenever A is compact. Thus $\mathcal{B}_2 \supseteq \mathcal{B}_3$. If $B \in \mathcal{B}$ and B has a non-empty interior, then $B \in \mathcal{B}_3$. We let $\mathcal{B}_4$ be those sets in $\mathcal{B}$ having a non-empty interior and such that $P_x(T_{\bar{B}} = T_{\mathring{B}})$ a.e. $x \in \mathfrak{G}$. In general $\mathcal{B} \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3 \supseteq \mathcal{B}_4$. Set $\mathcal{B}_1^* = \mathcal{B}_1$ in the non-singular case and $\mathcal{B}_1^* = \mathcal{B}_4$ otherwise.

Proposition 18.1. — *In the non-singular case* $\mathcal{B}_1 = \mathcal{B}_3$.

Proof. — Let the process be non-singular and let $B \in \mathcal{B}_1$. To prove that $B \in \mathcal{B}_3$ it suffices to show that for some non-empty open set P and finite positive constants t and δ

$$(18.1) \qquad P_x(T_B \leqslant t) \geqslant \delta, \qquad x \in P.$$

Since $B \in \mathcal{B}_1$ we can find a set $D \in \mathcal{B}$ having positive measure and finite positive constants s and α such that

$$P_y(T_B \leqslant s) \geqslant \alpha, \qquad y \in D.$$

Let $y_0 \in \mathfrak{G}$ be such that if Q is an open neighborhood of y_0, then $|Q \cap B_1| > 0$ (such a y_0 clearly exists). We can find non-empty open sets P and Q such that $y_0 \in Q$ and for some $c > 0$ and $0 < r < \infty$

$$P^r(x, dy) \geqslant c\, dy, \qquad x \in P \quad \text{and} \quad y \in Q.$$

Set $t = r + s$ and $\delta = \alpha c|Q \cap B_1|$. Then for $x \in P$

$$P_x(T_B \leqslant t) \geqslant \int_{\mathfrak{G}} P^r(x, dy) P_y(T_B \leqslant s) \geqslant \int_{B_1 \cap Q} C\alpha \; dy = \delta,$$

so that (18.1) holds as desired.

Proposition 18.2. — *The collection $\mathcal{B}_2$ is the same for the dual process as for the original process.*

Proof. — Let A and C be compact and $B \in \mathcal{B}$. Then

$$\int_C \tilde{G}_B(x, A)\, dx = \int_A G_B(x, C)\, dx.$$

Proof. — We can assume that A and C are relatively compact non-empty open sets. Let A_1 and C_1 be compact sets such that

$$\int_{C_1} G_B(z, A_1)\, dz = \infty. \tag{18.3}$$

We will first prove that

$$\int_C G_B(z, A_1)\, dz = \infty. \tag{18.4}$$

Let C_2 be a compact subset of C having a non-empty interior. Clearly

$$G_B(z, A_1) \leqslant G_{C_2}(z, A_1) + \int_{C_2} H_{C_2}(z, dy) G_B(y, A_1).$$

Since $G_{C_2}(z, A)$ is bounded in z it follows from (18.3) that

$$\int_{C_1} dz \int_{C_2} H_{C_2}(z, dy) G_B(y, A_1) = \infty.$$

Let D be a non-empty open set such that $D + C_2 \subseteq C$ Then

$$\int_D du \int_{C_1} dz \int_{C_2} H_{u+C_2}(z, dy) G_B(y, A_1) = \infty. \tag{18.5}$$

Equation (18.4) now follows from (18.5) and Proposition 18.3

For r a non-negative integer define.

$$V_A(r) = \min\left[t \geqslant 0 \,\middle|\, \int_0^t 1_A(X_s)\, ds \geqslant r\right].$$

Then $V_A(0) = 0$, $V_A(r) \to \infty$ as $r \to \infty$ and for all $x \in G$

with P_x probability one, $V_A(r) < \infty$ for all r. Thus

$$G_B(x, A_1) = E_x \int_0^{T_B} 1_{A_1}(X_t)\, dt$$
$$= \sum_{r=0}^{\infty} E_x \left[\int_{V_A(r)}^{V_A(r+1)} 1_{A_1}(X_t);\ T_B \geqslant V_A(r)\right].$$

There is an $M < \infty$ such that

$$E_x \left[\int_{V_A(r)}^{V_A(r+1)} 1_{A_1}(X_t);\ T_B \geqslant V_A(r)\right] \leqslant MP_x(T_B \geqslant V_A(r)).$$

Hence

$$G_B(x, A_1) \leqslant M \sum_{r=0}^{\infty} P_x(T_B \geqslant V_A(r)) \leqslant M(1 + G_B(x, A)).$$

The conclusion of the theorem now follows from (18.4).

THEOREM 18.2. — *There exist recurrent processes such that for some* $B \in \mathcal{B}$, $|B| > 0$ *but* $B \notin \mathcal{B}_2$.

Remark. — If $B \in \mathcal{B}$ and $|B| > 0$, then $B \in \mathcal{B}_1$, so the theorem shows that B_1 may be strictly larger than $\mathcal{B}_2$.

Proof. — Consider a symmetric random walk on the line which assigns mass 2^{-8n} to the numbers $\pm 2^{-n}$, $n \geqslant 1$, the remainder of the mass being assigned to the origin. Then the random walk has mean 0 and finite variance and hence is recurrent. We can convert the random walk into an infinitely divisible process on the real line by letting one unit of time for the random walk correspond to an exponential length of time with mean one for the infinitely divisible process.

By the local central limit theorem we can find a compact set A and a $c > 0$ such that

$$\int_0^t P_x(X_s \in A)\, ds \geqslant c\sqrt{t}, \quad t \geqslant 0 \quad \text{and} \quad 1 \leqslant x \leqslant 2.$$

Let $B \in \mathcal{B}$. Then for $t \geqslant 0$ and $1 \leqslant x \leqslant 2$

$$G_B(x, A) = E_x \int_0^{T_B} 1_A(X_s)\, dx$$
$$\geqslant E_x\left[\int_0^{T_B} 1_A(X_s)\, ds;\ T_B \geqslant t\right] \geqslant c\sqrt{t} - tP_x(T_B \leqslant t).$$

In order to construct the set B start out with $[0, 1]$ and delete all points which, for some $k \geqslant 2$, are within 2^{-2k} of some number of the form $j/2^k$. Then $B \in \mathcal{B}$ and $|B| > 0$.

Let $1 \leqslant x \leqslant 2$ and $k \geqslant 2$. If x is within 2^{-2k} of some number of the form $j/2^k$, then the process starting at x cannot hit B until at least one non-trivial jump occurs whose magnitude is strictly smaller than 2^{-k}. The probability that a jump of magnitude smaller than 2^{-k} occurs by time t is less than $t2^{-8k}$ and hence $P_x(T_B \leqslant t) \leqslant t2^{-8k}$. Consequently

$$G_B(x, A) \geqslant \sqrt{t} - t^2 2^{-8k}.$$

Choose $t = 2^{4k}$. Then

$$G_B(x, A) \geqslant c2^{2k} - 1.$$

The measure of such $x's$ is 2^{1-k}. Thus

$$\int_1^2 G_B(x, A)\, dx \geqslant 2^{k+1}c - 2^{1-k}.$$

Since k can be made arbitrarily large

$$\int_1^2 G_B(x, A)\, dx = \infty$$

and $B \notin \mathcal{B}_2$, as desired.

We now begin to study the main properties of sets in $\mathcal{B}_2$ and $\mathcal{B}_3$.

Proposition 18.4. — *Let* $B \in \mathcal{B}_2$ *and let* A *and* C *be compact. Then*

$$\int_{x+C} G_B(z, A)\, dz$$

is bounded in x.

Proof. — Let F be a compact set such that $A \subseteq \mathring{F}$ and $\overline{B} \subseteq \mathring{F}$. Let D be a relatively compact non-empty open set such $A - D \subseteq F$ and $\overline{B} - D \subseteq F$.

For $u \in D$

$$\int_{x+C} G_B(z, A)\, dz = \int_{x+C} dz \int H_{u+F}(z, dy) G_B(y, A).$$

Thus

$$\int_{x+C} G_B(z, A)\, dz = \frac{1}{|D|} \int_{x+C} dz \int_D du \int H_{u+F}(z, g) G_B(y, A).$$

By Proposition 18.3

$$\int_{x+C} G_B(z, A)\, dz \leqslant \frac{|C - D|}{|D|} \int_{\bar{D}+F} G_B(y, A)\, dy < \infty$$

as desired.

PROPOSITION 18.5. — *Let* $B \in \mathcal{B}_2$. *Let* f *be a continuous non-negative function on* $\mathfrak{G}$ *such that for some compact neighborhood* P *of the origin of* $\mathfrak{G}$ *the function* f_P *defined by*

$$f_P(x) = \max_{y \in x+P} f(y), \qquad x \in \mathfrak{G},$$

is integrable. Then $G_B f$ *is integrable on compacts.*

Proof. — Observe that

$$\begin{aligned} |P| G_P f(x) &= |P| \int G_B(x, dy) f(y) \\ &= \int dz \int_{z+P} G_B(x, dy) f(y) \leqslant \int dz\, f_P(z) G_B(x, z + P). \end{aligned}$$

Consequently

$$\begin{aligned} |P| \int_C G_B f(x)\, dx \leqslant \int dz\, f_P(z) \int_C G_B(x, z + P)\, dx \\ = \int dz\, f_P(z) \int_{z+P} \tilde{G}_B(x, C)\, dx \end{aligned}$$

and the desired result follows from Proposition 18.4 applied to the dual process.

PROPOSITION 18.6. — *There exist functions* $f \in \mathcal{F}^+$ *and* $g \in \mathcal{F}^+$ *such that* $J(f) > J(g) > 0$ *and* $g - f$ *is non-negative outside of some sufficiently large compact set* A.

Proof. — This result follows easily from the example on $\mathfrak{G} = R$ given on page 48 of [8] and the arguments used in proving Theorem 3.4 of [7].

Let $B \in \mathcal{B}_2$. For $f \in \mathcal{F}$ and $\lambda > 0$ we have the usual identity

$$(18.6) \quad A^\lambda f(x) - H_B^\lambda A^\lambda f(x) = - G_B^\lambda f(x) + L_B^\lambda(x) J(f).$$

THEOREM 18.3. — *Let* $B \in \mathcal{B}_2$. *Then*

$$(18.7) \qquad \lim_{\lambda \downarrow 0} L_B^\lambda(x) = L_B(x), \qquad x \in \mathfrak{G},$$

exists and L_B *is non-negative, finite a.e. and integrable on compacts. For any* $C \in \mathcal{B}$

$$\lim_{\lambda \downarrow 0} \int_C L_B^\lambda(x)\, dx = \int_C L_B(x)\, dx. \tag{18.8}$$

If $B \in \mathcal{B}_3$, *then* L_B *is bounded on compacts and the convergence in* (18.7) *is uniform on compacts.*

Proof. — Let $B \in \mathcal{B}_2$ and let $f \in \mathcal{F}$ satisfy the assumptions of Proposition 18.5. Now by monotone convergence

$$\lim_{\lambda \downarrow 0} G_B^\lambda f(x) = G_B f(x), \qquad x \in \mathfrak{G}.$$

By Theorem 17.2

$$\lim_{\lambda \downarrow 0} A^\lambda f(x) = Af(x), \qquad x \in \mathfrak{G},$$

uniformly for x in compacts and hence

$$\lim_{\lambda \downarrow 0} H_B^\lambda A^\lambda f(x) = H_B Af(x), \qquad x \in \mathfrak{G}.$$

Thus by (18.6)

$$\lim_{\lambda \downarrow 0} L_B^\lambda(x) = L_B(x)$$

exists for $x \in \mathfrak{G}$. Since L_B^λ is non-negative so is L_B. Also $L_B(x) < \infty$ if and only if $G_B f$ is finite. Since $G_B f$ is integrable on compacts it and L_B are both finite a.e. $x \in \mathfrak{G}$. If both $L_B(x)$ and $G_B f(x)$ are finite, then

$$Af(x) - H_B Af(x) = - G_B f(x) + L_B(x) J(f).$$

Since $G_B f$ is integrable on compacts, so is L_B. By monotone convergence

$$\lim_{\lambda \downarrow 0} \int_C G_B^\lambda f(x)\, dx = \int_C G_B f(x)\, dx$$

and (18.8) now follows from (18.6).

Suppose now that $B \in \mathcal{B}_3$. Then for $f \in \mathcal{F}$

$$\lim_{\lambda \downarrow 0} (G_B^\lambda f(x) - L_B^\lambda(x) J(f)) = Af(x) - H_B Af(x), \quad x \in \mathfrak{G}.$$

Let f and g be as in Proposition 18.6 and let C be compact. Then since $f \leqslant g$ outside of some sufficiently large compact set

$$\lim_{\lambda \downarrow 0} \sup \sup_{x \in C} (G_B^\lambda f(x) - G_B^\lambda g(x)) < \infty.$$

This implies that

$$\lim_{\lambda \downarrow 0} \sup \sup_{x \in C} L_B^\lambda(x)(J(f) - J(g)) < \infty, \qquad x \in \mathfrak{G}.$$

Since $J(f) > J(g)$ we see that

$$\lim_{\lambda \downarrow 0} \sup \sup_{x \in C} L_B^\lambda(x) < \infty.$$

This implies that L_B is finite everywhere and in fact bounded on compacts.

For $\lambda > 0$, $G_B^\lambda f(x) \leqslant G_B f(x)$. Thus

$$\lim_{\lambda \downarrow 0} \sup \sup_{x \in C} (L^\lambda{}_B(x) - L_B(x)) \leqslant 0. \tag{18.9}$$

Since $f \leqslant g$ outside of some compact

$$\lim_{\lambda \downarrow 0} \sup \sup_{x \in C} (G_B f(x) - G_B^\lambda f(x)) \leqslant \lim_{\lambda \downarrow 0} \sup \sup_{x \in C} (G_B g(x) - G_B^\lambda g(x)).$$

This implies that

$$\lim_{\lambda \downarrow 0} \sup \sup_{x \in C} J(f)(L_B(x) - L_B^\lambda(x)) \leqslant \lim_{\lambda \downarrow 0} \sup \sup_{x \in C} J(g)(L_B(x) - L_B^\lambda(x)).$$

Since $J(f) > J(g) \geqslant 0$ and L_B is bounded on compacts

$$\lim_{\lambda \downarrow 0} \sup \sup_{x \in C} (L_B(x) - L_B^\lambda(x)) \leqslant 0.$$

Together with (18.9) this implies that $L_B^\lambda(x) \to L_B(x)$ as $\lambda \downarrow 0$ uniformly on compacts, as desired.

THEOREM 18.4. — *Let* $B \in \mathcal{B}_2$ *and* $f \in \mathcal{F}^*$. *Then*

$$Af(x) - H_B Af(x) = - G_B f(x) + L_B(x) J(f), \quad x \in \mathfrak{G}, \tag{18.9}$$

with the understanding that if $J(f) = 0$ *the term* $L_B(x)J(f)$ *is defined to be zero for all* $x \in \mathfrak{G}$ *(even if* $L_B(x) = \infty$ *!). In particular* $G_B f$ *is integrable on compacts and if* $B \in \mathcal{B}_3$, *then* $G_B f$ *is bounded on compacts.*

Proof. — This result follows easily from equation (18.6) and Theorem 18.4.

In the type II case set

$$L_B^{\pm}(x) = L_B(x) \pm \sigma^{-2}(\psi(x) - H_B\psi(x)).$$

Then $L_B^{\pm}$ is integrable on compacts and finite at exactly those values of $x_{\pm}$ where L_B is finite. In particular $L_B^{\pm}$ is finite a.e. If $B \in \mathcal{B}_3$ then $L_B^{\pm}$ is bounded on compacts. Also $L_B^{\pm}$ is non-negative, as is evident from the following result.

THEOREM 18.5. — *Let* $B \in \mathcal{B}_2$, $f \in \Phi^*$, *and* $\varphi \in \Phi$. *If the process is type* I *recurrent, then*

$$(18.10) \qquad \lim_{y \to \infty} G_B f_y(x) = L_B(x) J(f), \quad L_B(x) < \infty,$$

and

$$(18.11) \qquad \lim_{y \to \infty} (\varphi, G_B f_y) = J(f)(\varphi, L_B).$$

If the process is type II *recurrent, then*

$$(18.12) \qquad \lim_{y \to +\infty} G_B f_y(x) = L_B^{\pm}(x) J(f), \quad L_B(x) < \infty,$$

and

$$(18.13) \qquad \lim_{y \to +\infty} (\varphi, G_B f_y) = J(f)(\varphi, L_B^{\pm}).$$

If $B \in \mathcal{B}_3$, *then* (18.10) *and* (18.12) *hold for all* $x \in \mathfrak{G}$ *and the convergence is uniform on compacts.*

Proof. — For $f \in \mathcal{F}^*$, this result follows immediately from Theorems 17.2 and 18.3 and the identity valid for $L_B(x) < \infty$:

$$A f_y(x) - A f_y(0) - H_B(A f_y - A f_y(0))(x) \\ = - G_B f_y(x) + L_B(x) J(f).$$

In the non-singular case we are done. In general however, we must replace the collection $\mathcal{F}$ by C_c. This is easily done by a standard « unsmoothness » argument based on Theorem 3.4 of [7].

19. Asymptotic Behavior of $G_B f(x)$ and $H_B f(x)$.

Continuing our study of recurrent processes, we obtain from Theorem 18.5 and duality

THEOREM 19.1. — *Let* $B \in \mathcal{B}_2$, $f \in \Phi$ *and* $\varphi \in \Phi^*$. *If the process is type* I *recurrent, then*

$$\lim_{x \to \infty} (\varphi_x, G_B f) = J(\varphi)(f, \tilde{L}_B).$$

If the process is type II *recurrent, then*

$$\lim_{x \to \pm\infty} (\varphi_x, G_B f) = J(\varphi)(f, \tilde{L}_B^{\pm}).$$

The existence of limiting hitting distributions is most readily established by reduction to the discrete time case.

THEOREM 19.2. — *Let* $B \in \mathcal{B}_1$. *If the process is type* I *recurrent there is a probability measure* μ_B *supported by* $\bar{B}$ *and such that for* $f \in \Phi$ *and* $\varphi \in \Phi^*$

$$\lim_{x \to \infty} (\varphi_x, H_B f) = J(\varphi)(f, \mu_B).$$

If the process is type II *recurrent there are probability measures* μ_B^+ *and* μ_B^- *supported by* $\bar{B}$ *such that for* $f \in \Phi$ *and* $\varphi \in C_c$

$$\lim_{x \to \pm\infty} (\varphi_x, H_B f) = J(\varphi)(f, \mu_B^{\pm}).$$

Remark. — In the type II case we set $\mu_B = (\mu_B^+ + \mu_B^-)/2$.

Proof. — By Proposition 5.3 we can assume that the random walk obtained by looking at the process at integer times is a recurrent random walk on $\mathfrak{G}$. Its type is I or II according as the recurrent process is type I or II. Let

$$T_B^1 = \min[n \geqslant 0 | X_n \in B]$$

and let

$$H_B^1 f(x) = E_x[f(X_{T_B^1}); T_B^1 < \infty].$$

LEMMA 19.1. — *Let* C *be a compact subset of* $\mathfrak{G}$ *and* $\varepsilon > 0$. *Then there is a compact subset* K *of* $\mathfrak{G}$ *such that* $C \subseteq K$ *and*

$$\limsup_{x \to \infty} P_x(X_{T_K^1} \in C) \leqslant \varepsilon.$$

Proof. — Let C_1 be a compact set containing C and such that $|\partial C_1| = 0$. Let n be a positive integer such that $\varepsilon n \geqslant 1$. Let $C_2, \ldots, C_n$ be translates of C_1 such that $C_1, \ldots, C_n$ are disjoint. Let L be a compact set containing $C_1 \cup \cdots \cup C_n$ and such that $|\partial L| = 0$.

Suppose the random walk is type I. Then by Theorem 5.7 of [7] for $1 \leqslant k \leqslant n$

$$c_k = \lim_{x \to \infty} P_x(X_{T^1_L} \in C_k)$$

exists. Clearly $c_1 + \cdots + c_n \leqslant 1$. Thus there is a j such that $1 \leqslant j \leqslant n$ and $c_j \leqslant 1/n \leqslant \varepsilon$. Let x_j be such that $C_j = x_j + C_1$ and set $K = L - x_j$. Then

$$\lim_{x \to \infty} P_x(X_{T^1_K} < C_1) = c_j \leqslant \varepsilon.$$

Consequently

$$\limsup_{x \to \infty} P_x(X_{T^1_K} \in C) \leqslant \varepsilon,$$

as desired. The proof in the type II case requires only obvious modifications of the proof in the type I case.

LEMMA 19.2. — *Let* $B > \mathcal{B}_1$ *and let* $\varepsilon > 0$. *Then there is a compact set* K *such that for* $f \in \Phi$

$$\limsup_{x \to \infty} |H_B f(x) - H^1_K H_B f(x)| \leqslant \varepsilon \|f\|.$$

Proof. — There is a compact set $C \subseteq B$ such that

$$P_y(X_t \in C \quad \text{for} \quad 0 \leqslant t \leqslant 1) \geqslant 1/2, \qquad y \in \overline{B}.$$

By Lemma 19.1 there is a compact subset K of $\mathfrak{G}$ such that $C \subseteq K$ and

$$\limsup_{x \to \infty} P_x(X_{T^1_K} \in C) \leqslant \varepsilon/4.$$

It follows that

$$\limsup_{x \to \infty} P_x(T_B \leqslant T^1_K) \leqslant \varepsilon/2,$$

from which the conclusion of the lemma follows immediately.

Proof of Theorem 19.2. Suppose the process is type I. Then by Theorem 5.6 of [7]

$$\lim_{x \to \infty} (\varphi_x, H^1_K H_B f) = J(\varphi)(H_B f, \mu^1_K),$$

where μ_K^1 is the limiting hitting distribution of K for the random walk. Let $\varepsilon_n \to 0$ as $n \to \infty$ and let K_n be the corresponding compact sets in Lemma 19.2. There are probability measures μ_n supported by $\overline{B}$ such that $(H_B f, \mu_{K_n}^1) = (f, \mu_n)$. We now have that

$$\lim_{x \to \infty} (\varphi_x, H_{K_n}^1 H_B f) = J(\varphi)(f, \mu_n)$$

and

$$\limsup_{x \to \infty} |\varphi_x, H_{K_n}^1 H_B f - H_B f| \leqslant \varepsilon_n \|f\| J(\varphi).$$

Consequently

$$\limsup_{x \to \infty} |(\varphi_x, H_B f) - J(\varphi)(f, \mu_n)| \leqslant \varepsilon_n \|f\| J(\varphi).$$

This implies that (f, μ_n) is a Cauchy sequence in n for each $f \in \Phi$. Thus there is a number c_f such that

$$\lim_{n \to \infty} (f, \mu_n) = c_f.$$

It Follows by Corollary 4 Dunford-Schwartz [3, p. 160] that for some probability measure μ_B supported by $\overline{B}$

$$\lim_{n \to \infty} (f, \mu_n) = (f, \mu_B).$$

Therefore

$$\lim_{x \to \infty} (\varphi_x, H_B f) = (f, \mu_B)$$

as desired. Only obvious modifications are required to complete the proof in the type II case.

Theorem 19.3. — *Let* $B \in \mathcal{B}_1^*$ *and* $f \in \Phi^*$. *If the process is type* I *recurrent, then*

$$\lim_{x \to \infty} H_B f(x) = (f, \mu_B).$$

Proof. — Once we know that the appropriate limits of $H_B f$ exist we can identify these limits by means of Theorem 19.2. The proof that these limits exist reduces to the corresponding discrete time results in [7] by using the same argument used to prove Theorem 19.2. In the general case, we use Theorem 9.1. which states that if $B \in \mathcal{B}_4$, then $H_B f$ is continuous a.e. $x \in \mathfrak{G}$.

THEOREM 19.4. — *Let* $B \in \mathscr{B}_1^*$ *and* $f \in \Phi^*$. *If the process is type* I *recurrent, then*

$$\lim_{x \to \infty} G_B f(x) = (f, \tilde{L}_B).$$

If the process is type II *recurrent, then*

$$\lim_{x \to \pm \infty} G_B f(x) = (f, \tilde{L}_B^{\pm}).$$

Proof. — Once we know that the indicated limits exist we can immediately identify them by means of Theorem 19.1.

To show that the limits do exist choose a compact set $C \in \mathscr{B}_1^*$ such that $\overline{B} \subseteq C$ and f is supported by C. This can be done by Theorem 9.4. Then $G_B f = H_C G_B f$. In the non-singular case the existance of the desired limits now follows immediately from Theorem 19.3. In the singular case, one way of proceding is to choose a big enough compact set K such that $|\partial K| = 0$ and $H_C G_B f(x)$ is close to $H_K^1 H_C G_B f(x) = H_K^1 G_B f(x)$ and using the fact from Theorem 9.1 that $G_B f$ is continuous a.e. $x \in \mathfrak{G}$.

20. Robin's Constant.

This section is devoted to associating a number $k(B)$ to sets in $\mathscr{B}$ such that $-\infty \leqslant k(B) < \infty$. It will turn out that $k(B) > -\infty$ if and only if $B \in \mathscr{B}_2$. The constants $k(B)$ enter in a natural way when we study the time dependent behavior of recurrent processes in the following several sections.

PROPOSITION 20.1. — *Let* $f \in \mathscr{F}$. *Then for* $B \in \mathscr{B}_3$, $G_B f$ *is a bounded function and for* $B \in \mathscr{B}_2$ *and* C *compact*

$$\int_{x+C} G_B f(y)\, dy$$

is bounded in x.

Proof. — Let $f \in \mathscr{F}$, $B \in \mathscr{B}_2$, and C be compact. Let $f_1 \in \mathscr{F}$ with $J(f_1) = 1$. Set $g = f - J(f) f_1$. Then $g \in \mathscr{F}$, $J(g) = 0$, and on $\{x | L_B(x) < \infty\}$

$$Gg - H_B G g = G_B f - J(f) G_B f_1.$$

Thus by Theorem 17.3

$$\int_{x+C} G_B f(y)\, dy - J(f) \int_{x+C} G_B f_1(y)\, dy$$

is bounded in x. Consequently in order to prove the conclusion for $f \in \mathcal{F}$ it suffices to prove it for one such f_1.

By Proposition 18.6 we can find $f_1 \in \mathcal{F}^+$ and $f_2 \in \mathcal{F}^+$ such that $J(f_1) = 1$, $J(f_2) > 1$ and $f_2 \leqslant f_1$ outside some compact set A. Let M be the maximum of f_2 on A. Then for some $N < \infty$ and all $x \in \mathfrak{G}$

$$J(f_2) \int_{x+C} G_B f_1(y)\, dy - N \leqslant \int_{x+C} G_B f_2(y)\, dy$$
$$\leqslant M \int_{x+C} G_B(y, A)\, dy + \int_{x+C} G_B f_1(y)\, dy.$$

Thus for $x \in \mathfrak{G}$

$$0 \leqslant \int_{x+C} G_B f_1(y)\, dy \leqslant \left(N + M \int_{x+C} G_B(y, A)\, dy\right) \Big/ (J(f_2) - 1)$$

Since by Proposition 18.4

$$\int_{x+C} G_B(y, A)\, dy$$

is bounded in x, it follows that

$$\int_{x+C} G_B f_1(y)\, dy$$

is bounded in x, as desired.

The proof that $G_B f$ is bounded for $B \in \mathcal{B}_3$ is the same, except that it is no longer necessary to integrate over C.

Proposition 20.2. — *Let $\mathfrak{G}_1$ be a compactly generated open subgroup of $\mathfrak{G}$ and let ψ be a continuous homomorphism from $\mathfrak{G}_1$ onto a closed d-dimensional subgroup of some Euclidean space R^d such that $\psi(x) = 0$ if and only of x is a compact element of $\mathfrak{G}_1$. In the type* II *case we let $\mathfrak{G}_1 = \mathfrak{G}$ and ψ as usual. Let $f \in \mathcal{F}^*$. If $B \in \mathcal{B}_3$, then in the type* I *case*

$$\lim_{x \to \infty,\, x \in \mathfrak{G}_1} (\psi(x))^{-1} G_B f_x(x) = 0$$

and in the type II *case*

$$\lim_{x \to \infty} |\psi(x)|^{-1} G_B f_x(x) = 2\sigma^{-2} J(f).$$

If $B \in \mathcal{B}_2$ *and* C *is compact, then in the type* I *case*

$$\lim_{x \to \infty,\, x \in \mathfrak{G}_1} |\psi(x)|^{-1} \int_{x+C} G_B f_x(z)\, dz = 0$$

and in the type II *case*

$$\lim_{x \to \infty} |\psi(x)|^{-1} \int_{x+C} G_B f_x(z)\, dz = 2\sigma^{-2}|C|\, J(f).$$

Proof. — From Theorem 18.4 we see that if $L_B(z) < \infty$, then

$$Af(z - x) - H_B A f_x(z) = - G_B f_x(z) + L_B(z) J(f).$$

Consequently

$$\begin{aligned} G_B f_x(z) = (Af(x) + Af(-x)) &+ (Af(z) - Af(x) - Af(z - x)) \\ &+ (H_B A f_x(z) - A f_x(0)) - H_B A f(z) + G_B f(z). \end{aligned}$$

Now $Af(z) - Af(x)$ and $Af(z - x)$ are bounded as $z - x$ range over a compact. Also $H_B A f_x(z) - A f_x(0)$ stays bounded for $x,\ z \in \mathfrak{G}$ (with $L_B(z) < \infty$ and hence $H_B 1(z) = 1$). By Proposition 20.1, $G_B f$ is bounded if $B \in \mathcal{B}_3$ and otherwise

$$\int_{x+C} G_B f(z)\, dz$$

stays bounded. In the type II case it follows from Theorem 17.2 that

$$\lim_{x \to \infty} |\psi(x)|^{-1}(Af(x) + Af(-x)) = 2\sigma^{-2} J(f).$$

In the type I case it follows from Theorem 17.2 that

$$\lim_{x \to \infty,\, x \in \mathfrak{G}_1} (\psi(x))^{-1}(Af(x) + Af(-x)) = 0$$

From these results the proposition follows immediately.

Proposition 20 3 — *Let* $\mathfrak{G}_1$ *and* ψ *be as in Proposition* 20.2 *and let* A *be compact. If* $B \in \mathcal{B}_3$, *then in the type* I *case*

$$\lim_{x \to \infty,\, x \in \mathfrak{G}_1} (\psi(x))^{-1} G_B(x, x + A) = 0$$

and in the type II *case*

$$\limsup_{x \to \infty} |\psi(x)|^{-1} G_B(x, x + A) < \infty.$$

If $B \in \mathcal{B}_2$ *and* C *is compact, then in the type* I *case*

$$\lim_{x \to \infty,\, x \in \mathfrak{G}_1} (\psi(x))^{-1} \int_{x+C} G_B(z, x + A)\, dz = 0$$

and in the type II *case*

$$\limsup_{x \to \infty} |\psi(x)|^{-1} \int_{x+C} G_B(z, x + A)\, dz < \infty.$$

Proof. — This result follows immediately from Proposition 20.2. With more work one can show that in the type II case the actual limits exist if $|\partial A| = 0$.

Proposition 20.4. — *Let* $\mathfrak{G}_1$ *and* ψ *be as in Proposition* 20.2 *and let* A *be compact. If* $B \in \mathcal{B}_3$, *then*

$$G_B(x, y + A)/(1 + |\psi(y)|)$$

is bounded in $x \in \mathfrak{G}$ *and* $y \in \mathfrak{G}_1$. *If* $B \in \mathcal{B}_2$ *and* C *is compact, then*

$$\int_{x+C} G_B(z, y + A)/(1 + |\psi(y)|$$

is bounded for $x \in \mathfrak{G}$ *and* $y \in \mathfrak{G}_1$.

Proof. — If $B \in \mathcal{B}_3$ the result follows immediately from Proposition 20.4. Suppose $B \in \mathcal{B}_2$ and C is compact. Let E be a compact set such that $A \subseteq \mathring{E}$. Let D be a compact set having positive measure and such that $A - D \subseteq E$. Then for $\mathfrak{U} \in D$

$$\int_{x+C} dz\, G_B(z, y + A) \leqslant \int_{x+C} dz\, H_{y+u+E}(z, dv) G_B(v, y + A).$$

Consequently by Proposition 18.3

$$\int_{x+C} dz\, G_D(z, y + A) \leqslant \frac{|C - D|}{|D|} \int_{y+D+E} dv\, G_B(v, y + A)$$

and the desired result now by Proposition 20.3.

Proposition 20.5. — *Let* $\mathfrak{G}_1$ *and* ψ *be as in Proposition* 20.2. *Let* $f \in \mathcal{F}^+$ *be supported by* $\mathfrak{G}_1$ *and such that* $(\psi(x))^4 f(x)$ *is bounded on* $\mathfrak{G}_1$. *If* $B \in \mathcal{B}_4$, *then in the type* I *case*

$$\lim_{x \to \infty} G_B f(x) = (f, \tilde{L}_B)$$

and in the type II *case*

$$\lim_{x \to \pm\infty} G_B f(x) = (f, \tilde{L}_B^{\pm}),$$

both limits being finite. If $B \in \mathcal{B}_2$ *and* $\varphi \in C_c$, *then in the type* I *case*

$$\lim_{x \to \infty} (\varphi_x, G_B f) = J(\varphi)(f, \tilde{L}_B)$$

and in the type II *case*

$$\lim_{x \to \pm\infty} (\varphi_x, G_B f) = J(\varphi)(f, \tilde{L}_B^{\pm}),$$

both limits being finite.

Proof. — Suppose $B \in \mathcal{B}_2$ and $\varphi \in C_c$. Now

$$(\varphi_x, G_B f) = \int \varphi_x(z) \int_{\mathfrak{G}_1} G_B(x, dy) f(y).$$

Let E be a compact subset of $\mathfrak{G}_1$ with $|E| > 0$. By Proposition 20.4 there is an $M < \infty$ such that for y in $\mathfrak{G}_1$ and sufficiently large

$$\int_{\mathfrak{G}} \varphi_x(z) G_B(z, y + E)\, dz \leqslant M|\psi(y)|, \qquad x \in \mathfrak{G}.$$

We can also assume that M is such that

$$f(u) \leqslant M(\psi(y))^{-4}, \qquad u > y + E, \qquad y \in \mathfrak{G}_1.$$

Then there is a compact set D such that for $x \in \mathfrak{G}$

$$\int_{\mathfrak{G}} \varphi_x(z)\, dz \int_{D^c \cap \mathfrak{G}_1} dy \int_{y+E} G_B(z, du) f(u) \leqslant M^2 J(\varphi) \int_{D^c \cap \mathfrak{G}_1} |\psi(y)|^{-3}\, dy$$

which can be made arbitrarily small by making D sufficiently large (since necessarily $d \leqslant 2$ in the recurrent case). Let $D_1 \supset D$ be a compact set such that $(D_1^c \cap \mathfrak{G}_1) - E \subseteq D^c$. Then

$$\begin{aligned}\int_{\mathfrak{G}} \varphi_x(z)\, dz \int_{D^c \cap \mathfrak{G}_1} dy \int_{y+E} G_B(z, du) f(u) &= \int_{\mathfrak{G}} \varphi_x(z)\, dz \int_{\mathfrak{G}_1} G_B(z, du) f(u) |D^c \cap (u - E)| \\ &\geqslant |E| \int_{\mathfrak{G}} \varphi_x(z)\, dz \int_{D_1^c} G_B(z, du) f(u).\end{aligned}$$

In otherwords, given $\varepsilon > 0$ we can find a compact set D_1 such that

$$\int_{\mathfrak{G}} \varphi_x(z)\, dz \int_{D_1^c} G_B(z, du) f(u) \leqslant \varepsilon, \qquad x \in \mathfrak{G}.$$

By Theorem 19.1 in the type I case

$$\lim_{x \to \infty} \int_{\mathfrak{G}} \varphi_x(z)\, dz \int_{D_1} G_B(z, du) f(u) = J(\varphi) \int_{D_1} f(x) \tilde{L}_B(x)\, dx$$

and in the type II case

$$\lim_{x \to \pm\infty} \int_{\mathfrak{G}} \varphi_x(z)\, dz \int_{D_1} G_B(z, du) f(u) = J(\varphi) \int_{D_1} f(x) \tilde{L}_B^{\pm}(x)\, dx.$$

Since ε can be made arbitrarily small, the conclusion of the proposition follows.

If $B \in \mathcal{B}_4$ the same proof works except that we need not integrate with φ, we use Theorem 19.4 instead of Theorem 19.1, and we choose D_1 such that $|\partial D_1| = 0$.

THEOREM 20.1. — *Let* $B \in \mathcal{B}_1^*$. *In the type* I *case there is a finite constant* $k(B)$ *such that for* $f \in \mathcal{F}^*$

$$\lim_{x \to \infty} (Af(x) - L_B(x) J(f)) = k(B) J(f).$$

In the type II *case there exist finite constants* $k^{\pm}(B)$ *such that for* $f \in \mathcal{F}^*$

$$\lim_{x \to \pm\infty} (Af(x) - L_B(x) J(f)) = k^{\pm}(B) J(f) \mp \sigma^{-2} K(f).$$

Let $B \in \mathcal{B}_2$. *In the type* I *case there is a finite constant* $k(B)$ *such that for* $f \in \mathcal{F}$ *and* $\varphi \in C_c$

$$\lim_{x \to \infty} (\varphi_x, Af - J(f) L_B) = J(\varphi) k(B) J(f).$$

In the type II *case there are finite constants* $k^{\pm}(B)$ *such that for* $f \in \mathcal{F}$ *and* $\varphi \in C_c$

$$\lim_{x \to \pm\infty} (\varphi_x, Af - J(f) L_B) = J(\varphi)(k^{\pm}(B) J(f) \mp \sigma^{-2} K(f)).$$

Remark. — The Robin's Constant is defined as $k(B)$ in the type I case and

$$k(B) = (k^+(B) + k^-(B))/2$$

in the type II case.

Proof. — One can find an $f_1 \in \mathcal{F}$ of the form of Proposition 20.5 with $J(f_1) = 1$. It can also be assumed that in the type II case $K(f_1) = 0$. Recall that

$$Af_1(x) - H_B Af_1(x) = -G_B f_1(X) + L_B(x).$$

Suppose $B \in \mathcal{B}_2$ and $\varphi \in C_c$. Then by Theorem 19.2 and Proposition 20.5, in the type I case there is a constant $k(B)$ such that

$$\lim_{x \to \infty} (\varphi_x, Af_1 - L_B) = J(\varphi)k(B),$$

and in the type II case there exist finite constants $k^\pm(B)$ such that

$$\lim_{x \to \pm\infty} (\varphi_x, Af_1 - L_B) = J(\varphi)k^\pm(B).$$

The desired result now follows by Corollary 17.1.

The proof for $B \in \mathcal{B}_1^*$ is similar.

THEOREM 20.2. — *Let* $B \in \mathcal{B}_1^*$ *and* $f \in \mathcal{F}^*$. *In the type* I *case*

$$\lim_{x \to \infty} G_B f(x) = -k(B)J(f) + (Af, \mu_B)$$

and in the type II *case*

$$\lim_{x \to \pm\infty} G_B f(x) = -k^\pm(B)J(f) \pm \sigma^{-2}K(f) + (Af, \mu_B^\pm).$$

Let $B \in \mathcal{B}_2$, $f \in \mathcal{F}$, *and* $\varphi \in C_c$. *In the type* I *case*

$$\lim_{x \to \infty} (\varphi_x, G_B f) = J(\varphi)(-k(B)J(f) + (Af, \mu_B)$$

and in the type II *case*

$$\lim_{x \to \infty} (\varphi_x, G_B f) = J(\varphi)(-k^\pm(B)J(f) \pm \sigma^{-2}K(f) + (Af, \mu^\pm)).$$

Proof. — This result follows immediately from Theorems 18.4, 19.2, 19.3 and 20.1.

THEOREM 20.3. — *Let* $B \in \mathcal{B}_1^*$ *and* $f \in \mathcal{F}^*$. *Then*

$$(f, \tilde{L}_B) = -k(B)J(f) + (Af, \mu_B)$$

and in the type II *case*

$$(f, \tilde{L}_B^\pm) = -k^\pm(B)J(f) + \sigma^{-2}K(f) + (Af, \mu_B^\pm).$$

Proof. — If f satisfies the assumptions of Proposition 20.5 the result follows from Proposition 20.5 and Theorem 20.2. The general case can easily be reduced to this special case by arguing as in the proof of Theorem 5.12 of [7], since Proposition 5.2 of [7] extends in an obvious manner to the continuous time recurrent potential operator A.

COROLLARY 20.1. — *Let* $B \in \mathcal{B}_1^*$ *and* $f \in \mathcal{F}^*$. *In the type* I *case*

$$\lim_{x \to \infty} G_B f(x) = (f, \tilde{L}_B)$$

and in the type II *case*

$$\lim_{x \to +\infty} G_B f(x) = (f, L_B^{\pm}).$$

Let $B \in \mathcal{B}_2$, $f \in \mathcal{F}$, *and* $\varphi \in C_c$. *In the type* I *case*

$$\lim_{x \to \infty} (\varphi_x, G_B f) = J(\varphi)(f, \tilde{L}_B)$$

and in the type II *case*

$$\lim_{x \to +\infty} (\varphi_x, G_B f) = J(\varphi)(f, \tilde{L}_B^{\pm}).$$

Proof. — This result follows immediately from Theorems 20.2 and 20.3.

PROPOSITION 20.6. — *Let* $B \in \mathcal{B}$ *and let* $B_n \supseteq B$, $B_n \downarrow$ *and* $P_x(\lim_{n \to \infty} T_{B_n} = T_B) = 1$ *a.e.* $x \in \mathfrak{G}$. *If* $B \in \mathcal{B}_1$ *and* $\varphi \in C_c$, *then in the type* I *case*

$$\lim_{n \to \infty} (\varphi, \mu_{B_n}) = (\varphi, \psi_B)$$

and in the type II *case*

$$\lim_{n \to \infty} (\varphi, \mu_{B_n}^{\pm}) = (\varphi, \psi_B^{\pm}).$$

Proof. — Let C be a compact set such that $\mathring{C}$ contains $\bar{B}$ and $\bar{B}_n$, $n \geqslant 1$. Consider first the type I case. Now $(\varphi, \mu_B) = (H_B\varphi, \mu_C)$. Let D be a non-empty relatively compact open set such that $B - D \subseteq C$. Then

$$(\varphi, \mu_B) = \frac{1}{|D|}\int_D dy\,(H_B\varphi, \mu_{y+C})$$
$$= \frac{1}{|D|}\int \mu_C\,(dz)\int_D dy\,H_B\varphi(y - z).$$

If also $B_n - D \subseteq C$, $n \geqslant 1$, then

$$(\varphi, \mu_{B_n}) = \frac{1}{|D|}\int \mu_C\,(dz)\int_D dy\,H_B\varphi(y + z).$$

By quasi-left-continuity it follows that

$$\lim_{n\to\infty} H_{B_n}\varphi(x) = H_B\varphi(x) \qquad \text{a.e.} \qquad x \in \mathfrak{G}$$

and hence that $(\varphi, \mu_{B_n}) \to (\varphi, \mu_B)$ as $n \to \infty$.
The proof in the type II case is similar.

Proposition 20.7. — *Let* B *and* B_n, $n \geqslant 1$, *be as in Proposition* 20.6 *with* $B \in \mathcal{B}_2$. *Let* $\varphi \in \Phi$. *Then*

$$\lim_{n\to\infty} (\varphi, L_{B_n}) = (\varphi, L_B)$$

and in the type II *case*

$$\lim_{n\to\infty} (\varphi, L^{\pm}_{B_n}) = (\varphi, L^{\pm}_{B}).$$

Proof. — Choose $f \in \mathcal{F}^+$ with $J(f) = 1$. Then

$$Af(x) - H_BAf(x) = -G_Bf(x) + L_B(x)$$

and

$$Af(x) - H_{B_n}Af(x) = -G_{B_n}f(x) + L_{B_n}(x).$$

Now

$$\lim_{n\to\infty} H_{B_n}Af(x) = H_BAf(x) \qquad \text{and} \qquad \lim_{n\to\infty} G_{B_n}f(x) = G_Bf(x)$$

for almost all $x \in \mathfrak{G}$. which implies that

$$\lim_{n\to\infty} (\varphi, L_{B_n}) = (\varphi, L_B).$$

This implies the desired result in the type I case. The result in the type II case now follows from the formula

$$L^{\pm}_B(x) = L_B(x) \pm \sigma^{-2}(\psi(x) - H_B\psi(x)).$$

PROPOSITION 20.8. — *Let* B *and* B_n, $n \geqslant 1$, *be as in Proposition* 20.6 *with* $B \notin \mathcal{B}_2$. *Let* $\varphi \in C_c^+$ *with* $J(\varphi) > 0$. *Then*

$$\lim_{n\to\infty} (\varphi, L_{B_n}) = \infty$$

and in the type II *case*

$$\lim_{n\to\infty} (\varphi, L^{\pm}_{B_n}) = \infty.$$

Proof. — Choose $f \in \mathcal{J}^+$ such that $J(f) = 1$. Then

$$Af(x) - H_{B_n}Af(x) = - G_{B_n}f(x) + L_{B_n}(x).$$

Now $(\varphi, H_{B_n}Af)$ is bounded in n and

$$\lim_{n\to\infty} (\varphi, G_{B_n}) = \infty,$$

from which the desired results follow immediately.

We can extend the definition of $k(B)$ to all sets in $\mathcal{B}$ by setting $k(B) = -\infty$ whenever $B \in \mathcal{B}$ but $B \notin \mathcal{B}_2$. Similarly in the type II case we extend $k^{\pm}(B)$ to all sets in $\mathcal{B}$ by setting $k^{\pm}(B) = -\infty$ if $B \in \mathcal{B}$ but $B \notin \mathcal{B}_2$.

THEOREM 20.4. — *Let* B *and* B_n, $n \geqslant 1$, *be as in Proposition* 20.6. *Then*

$$\lim_{n\to\infty} k(B_n) = k(B)$$

and in the type II *case*

$$\lim_{n\to\infty} k^{\pm}(B_n) = k^{\pm}(B).$$

Proof. — Let $f \in \mathcal{J}^+$ with $J(f) = 1$. We can assume that $B_n \in \mathcal{B}_2$ for $n \geqslant 1$. Then by Theorem 20.3

$$k(B_n) = -(f, \tilde{L}_{B_n}) + (\mu_{B_n}, Af). \tag{20.1}$$

Suppose first that $B \in \mathcal{B}_2$. It follows from Proposition 20.6 that

$$\lim_{n\to\infty} (\mu_{B_n}, Af) = (\mu_B, Af).$$

By Theorem 18.5, $\tilde{L}_{B_n} \uparrow$ as $n \uparrow$. By Theorem 20.3 $(f, \tilde{L}_B)$

is finite. Since f is non-negative it now follows by Proposition 20.7 that

$$\lim_{n\to\infty} (f, \tilde{L}_{B_n}) = (f, L_B).$$

Thus

$$\lim_{n\to\infty} k(B_n) = -(f, L_B) + (\mu_B, Af) = k(B)$$

as desired. The proof in the type II case is similar.

Suppose next that $B \in \mathcal{B}$ but $B \notin \mathcal{B}_2$. We will prove that in the type I case

$$\lim_{n\to\infty} k(B_n) = -\infty.$$

By Proposition 20.8

$$\lim_{n\to\infty} (f, \tilde{L}_{B_n}) = \infty.$$

Since (μ_{B_n}, Af) is bounded in n it follows from equation (20.1) that $k(B_n) \to -\infty$ as $n \to \infty$. The proof in the type II case is similar.

Proposition 20.9. — *Let* $B \in \mathcal{B}_2$ *and let* $C \in \mathcal{B}_3$ *contain* B. *Suppose first that* $B \in \mathcal{B}_3$. *Then*

$$k(C) - k(B) = \tilde{k}(C) - \tilde{k}(B) = (L_B, \mu_C).$$

Suppose next that $\overline{B} \subseteq \mathring{C}$ *and let* D *be a relatively compact non-empty open set such that* $B - D \subseteq C$. *Then*

$$k(C) - k(B) = \tilde{k}(C) - \tilde{k}(B) = \frac{1}{|D|} \int_D du (L_B, \mu_{u+C}).$$

Proof. — In the type II case by limits at ∞ we mean the average of the limits at $+\infty$ and $-\infty$.

Suppose $\overline{B} \subseteq \mathring{C}$ and let D be as in the statement of the proposition. Let $f, \varphi \in C_c$ sith $J(f) = J(\varphi) = 1$. Then for $u \in D$

$$(\varphi_x, G_{u+C} f_y) = (\varphi_x, G_B f_y) - (\varphi_x, H_{u+C} G_B f_y).$$

Consequently

$$\int_D du (\varphi_x, G_{u+C} f_y) = |D| (\varphi_x, G_B f_y) - \int_D du (\varphi_x, H_{u+C} G_B f_y).$$

Letting $y \to \infty$, we get from Theorem 18.5,

$$\int_D du(\varphi_x, L_{u+C}) = |D|(\varphi_x, L_B) - \int_D du(\varphi_x, H_{u+C}L_B).$$

Letting $x \to \infty$ we get from Theorems 19.2 and 20.1

$$|D|(k(C) - k(B)) = \int_D du(L_B, \mu_{u+C}).$$

Reversing the process and first letting $x \to \infty$, we get from Theorems 19.1 and 19.2

$$\int_D du(f_y, \tilde{L}_{u+C}) = |D|(f_y, \tilde{L}_B) - \int_D du(G_B f_y, \mu_{u+C}).$$

Letting $y \to \infty$ we get from Theorems 18.5 and 20.1 that

$$|D|(\tilde{k}(C) - \tilde{k}(B)) = \int_C du(L_B, \mu_{u+C}),$$

which completes the proof of the theorem.

THEOREM 20.5. — *Let* B, B_1, *and* B_2 *all be in* $\mathscr{B}_2$. *Then* $\tilde{k}(B) = k(B) = k(-B)$. *If* $B_1 \subseteq B_2$, *then* $k(B_1) \leqslant k(B_2)$. *Finally* $k(B_1 \cup B_2) \leqslant k(B_1) + k(B_2) - k(B_1 \cap B_2)$.

Proof. — It is obvious from the definition of the dual process that $k(-B) = \tilde{k}(B)$. Let $C \in \mathscr{B}_3$ be such that $-C = C$ and $\overline{B} \subseteq \mathring{C}$.

Then $\tilde{k}(C) = k(-C) = k(C)$ so by Proposition 20.9

$$\tilde{k}(B) = k(B) + \tilde{k}(C) - k(C) = k(B).$$

Suppose $C \subset \mathscr{B}_3$ and $(\overline{B}_1 \cup \overline{B}_2) \subseteq \mathring{C}$. Then by Proposition 20.9

$$k(B_2) - k(B_1) = \frac{1}{|D|} \int_D du(L_{B_1} - L_{B_2}, \mu_{u+C}) \geqslant 0.$$

Finally observe that

$$1_{[T_{B_1} \geqslant t]} + 1_{[T_{B_2} \geqslant t]} = 1_{[T_{B_1} \geqslant t] \cup [T_{B_2} \geqslant t]} + 1_{[T_{B_1} \geqslant t] \cap [T_{B_2} \geqslant t]}$$
$$\leqslant 1_{[T_{B_1 \cap B_2} \geqslant t]} + 1_{[T_{B_1 \cup B_2} \geqslant t]}].$$

Choose $f \in C_c^+$ with $J(f) = 1$. Then

$$G_{B_1} + G_{B_2} f \leqslant G_{B_1 \cup B_2} f + G_{B_1 \cup B_2} f.$$

From Theorem 18.5 it follows that if $B_1 \cap B_2 \in \mathcal{B}_2$, then

$$L_{B_1 \cup B_2} \geqslant L_{B_1} + L_{B_2} - L_{B_1 \cup B_2}$$

and by Proposition 20.9

$$k(B_1 \cup B_2) \leqslant k(B_1) + k(B_2) - k(B_1 \cap B_2).$$

If $B_1 \cap B_2 \notin \mathcal{B}_2$, then $k(B_1 \cap B_2) = -\infty$ and this inequality still holds. This completes the proof of the theorem.

21. Time Dependent Behavior (Recurrent Case).

Throughout this section X_t will denote a recurrent process. For $B \in \mathcal{B}$ and $A \in \mathcal{B}$ recall that

$$E_B(t, A) = \int_{\mathfrak{G}} dx\, P_x(T_B \leqslant t, X_{T_B} \in A)\, dx.$$

We also set for $t \geqslant 0$

$$E_B(t) = E_B(t, \overline{B}) = E_B(t, \mathfrak{G}) = \int_{\mathfrak{G}} dx\, P_x(T_B \leqslant t).$$

Then $E_B(t, A)$ and $E_B(t)$ are zero unless $B \in \mathcal{B}_1$. $E_B(t, .)$ defines a measure on $\overline{B}$ having total mass $E_B(t)$.

Proposition 21.1. — *For* $t \geqslant 0$ *and* $h \geqslant 0$

$$E_B(t + h, A) - E_B(t, A) = \int_{\mathfrak{G}} P_x(\tilde{T}_B > t) P_x(T_B \leqslant h, X_{T_B} \in A)\, dx.$$

Proof. — This result follows from the computations

$$\begin{aligned} E_B(t + h, A) &- E_B(t, A) \\ &= \int_{\mathfrak{G}} dy\, P_y(t < T_B \leqslant t + h, X_{T_B} \in A)\, dy \\ &= \int_{\mathfrak{G}} dy \int_{\mathfrak{G}} P_y(T_B > t, X_t \in dx)\, P_x(T_B \leqslant h, X_{T_B} \in A) \\ &= \int_{\mathfrak{G}} P_x(\tilde{T}_B > t) P_x(T_B \leqslant h, X_{T_B} \in A)\, dx. \end{aligned}$$

Proposition 21.2. — *If* $B \in \mathcal{B}_1$, *then*

$$\lim_{t \to \infty} E_B(t) = \infty,$$

$$\lim_{t \to \infty} (E_B(t + h) - E_B(t)) = 0, \qquad h \geqslant 0,$$

and

$$\lim_{t\to\infty} \frac{E_B(t+s)}{E_B(t)} = 1$$

uniformly for s *in compacts.*

Proof. — These results follow immediately from Proposition 21.1 ans the definition of $E_B(t)$, since $P_x(T_B < \infty) = 1$ a.e. $x \in \mathfrak{G}$ for $B \in \mathcal{B}_1$.

PROPOSITION 21.3. — *If* $B \in \mathcal{B}_1$ *and* $C \in \mathcal{B}_1$, *then*

$$\lim_{t\to\infty} \frac{E_B(t)}{E_C(t)} = 1.$$

Proof. — We can assume that $B \subseteq \mathring{C}$. Let D be a relatively compact non-empty open set such that $B - D \subseteq C$. Then for $s \geqslant 0$ and $u \in D$

$$E_B(t+s) \geqslant \int_{\mathfrak{G}} E_{u+C}(t, dy) P_y(T_B \leqslant s)$$
$$= \int_{\mathfrak{G}} E_C(t, dy) P_{u+y}(T_B \leqslant s).$$

Consequently

$$E_B(t+s) \geqslant \frac{1}{|D|} \int_{\mathfrak{G}} E_C(t, dy) \int_{y+D} P_u(T_B \leqslant s)\, du.$$

Choose $0 < \varepsilon < 1$. There is an $s > 0$ such that

$$\frac{1}{|D|} \int_{y+D} P_u(T_B \leqslant s)\, du \geqslant 1 - \varepsilon, \quad y \in \overline{C}.$$

Then $E_B(t+s) \geqslant (1-\varepsilon) E_C(t)$. By Proposition 21.2

$$\liminf_{t\to\infty} E_B(t)/E_C(t) \geqslant 1.$$

Since $E_B(t) \leqslant E_C(t)$, it follows that

$$\lim_{t\to\infty} E_B(t)/E_C(t) = 1,$$

as desired.

PROPOSITION 21.4. — *Let* $A \in \mathcal{B}$ *and* $0 \leqslant t < \infty$. *Then for* $B \in \mathcal{B}_3$

$$\int_0^t P_x(T_B > s, X_{T_B} \in A)\, ds$$
$$= \int_{B'} G_B(x, dy) P_y(T_B \leqslant t, X_{T_B} \in A), \quad x \in \mathfrak{G},$$

and for $B \in \mathcal{B}_2$ *and* $C \in \mathcal{B}$

$$\int_C dx \int_0^t P_x(T_B > s, X_{T_B} \in A)\, ds$$
$$= \int_C dx \int_{B'} G_B(x, dy) P_y(T_B \leqslant t, X_{T_B} \in A), \quad x \in \mathfrak{G}.$$

Proof. — Let $B \in \mathcal{B}_3$. Then

$$\int_{B'} G_B(x, dy) P_y(T_B \leqslant t, X_{T_B} \in A)$$
$$= \int_0^\infty du \int_{B'} Q_B^u(x, dy) P_y(T_B \leqslant t, X_{T_B} \in A)$$
$$= \int_0^\infty du\, P_x(u \leqslant T_B \leqslant u + t, X_{T_B} \in A)$$
$$= \int_0^t ds\, P_x(T_B > s, X_{T_B} \in A).$$

The proof of the corresponding result for $B \in \mathcal{B}_2$ is similar.

Most of the remaining results in this section will be obtained in the type I case, the corresponding results in the type II case being deferred to Section 22.

THEOREM 21.1. — *Suppose the process is type* I. *If* $B \in \mathcal{B}_3$, *then*

$$\lim_{t \to \infty} \int_0^t ds\, P_x(T_B > s)/E_B(t) = L_B(x)$$

uniformly for x *in compacts. If* $B \in \mathcal{B}_2$ *and* $C \in \mathcal{B}$, *then*

$$\lim_{t \to \infty} \int_C dx \int_0^t ds\, P_x(T_B > s)/E_B(t) = \int_C L_B(x)\, dx.$$

Proof. — Let $B \in \mathcal{B}_3$. By Proposition 31.4

$$\int_0^t ds\, P_x(T_B > s) = \int G_B(x, dy) P_y(T_B \leqslant t), \quad x \in \mathfrak{G}.$$

Choose $D \in \mathcal{C}$ with $|D| > 0$. Then

$$|D| \int G_B(x, dy) P_y(T_B \leqslant t) = \int_{\mathfrak{G}} dy \int_{y+D} G_B(x, dz) P_z(T_B \leqslant t).$$

Choose $0 < \varepsilon < 1$. We can find a $t_0 > 0$ such that for $y \in \mathfrak{G}$, $z \in y + D$ and $t \geqslant 0$

$$(1 - \varepsilon) P_y(T_B \leqslant t - t_0) \leqslant P_z(T_B \leqslant t) \leqslant (1 + \varepsilon) P_y(T_B \leqslant t + t_0).$$

It follows from Theorem 18.5 that, as x ranges over a compact set, for y sufficiently large

$$(L_B(x) - \varepsilon)|D| \leqslant G_B(x, y + D) \leqslant (L_B(x) + \varepsilon)|D|.$$

Thus for some $M < \infty$

$$
\begin{aligned}
|D| \int G_B(x, dy) P_y(T_B \leqslant t) & \\
&\leqslant (1 + \varepsilon) \int P_y(T_B \leqslant t + t_0) G_B(x, y + D)\, dy \\
&\leqslant M + (1 + \varepsilon)(L_B(x) + \varepsilon)|D| E_B(t + t_0)
\end{aligned}
$$

and similarly for $t \geqslant t_0$

$$
\begin{aligned}
|D| \int G_B(x, dy) P_y(T_B \leqslant t) & \\
&\geqslant -M + (1 - \varepsilon)(L_B(x) - \varepsilon)|D| E_B(t - t_0).
\end{aligned}
$$

The conclusion of the theorem now follows from Proposition 21.2.

Suppose now that $B \in \mathcal{B}_2$ and $C \in \mathcal{B}$. Choose $B_1 \in \mathcal{B}_3$ such that $B_1 \supseteq B$. Since $P_x(T_B \geqslant s) \geqslant P_x(T_{B_1} \geqslant s)$ it follows that

$$\liminf_{t \to \infty} \int_C dx \int_0^t ds\ P_x(T_B > s)/E_{B_1}(t) = \int_C L_B(x)\, dx.$$

It now follows from Propositions 20.7 and 21.3 that

$$\liminf_{t \to \infty} \int_C dx \int_0^t ds\ P_x(T_B > s)/E_B(t) \leqslant \int_C L_B(x)\, dx.$$

On the other hand

$$P_z(T_B \leqslant t) \leqslant P_z(T_{B_1} \leqslant t)$$

and by imitating the proof of Theorem 21.1 for the special case $B \in \mathcal{B}_3$, we can show that

$$\limsup_{t \to \infty} \int_A dx \int_0^t ds\ P_x(T_B > s)/E_{B_1}(t) \leqslant \int_C L_B(x)\, dx$$

and since, by Proposition 21.3, $E_B(t)/E_{B_1}(t) \to 1$ as $t \to \infty$ the conclusion of the theorem now follows.

THEOREM 21.2. — *Suppose the process is type* I. *If* $B \in \mathcal{B}_1^*$, *then for* $f \in \Phi^*$

$$\lim_{t \to \infty} \int_0^t ds\ E_x(f(X_{T_B}); T_B > s)/E_B(t) = L_B(x)(f, \mu_B)$$

uniformly for x *in compacts.*

Proof. — This result follows easily from Theorems 19.3 and 21.1 and the facts that $E_B(t) \to \infty$ as $t \to \infty$ and for any compact set K

$$\int_0^\infty ds\, P_x(X_s \in K \quad \text{and} \quad T_B > s) = G_B(x, K)$$

is bounded for x in compacts.

PROPOSITION 21.5. — *If* $B \in \mathcal{B}_3$, *then for any* $t > 0$ *and* $\varepsilon > 0$ *there is a compact set* K *such that*

$$\int_{K'} G_B(x, dy) P_y(T_B \leqslant t) \leqslant \varepsilon, \qquad x \in \mathfrak{G}.$$

If $B \in \mathcal{B}_2$ and $C \in \mathcal{B}$, then for any $\varepsilon > 0$ there is a compact set K such that

$$\int_C dx \int_{K'} G_B(x, dy) P_y(T_B \leqslant t) \leqslant \varepsilon, \qquad x < \mathfrak{G}.$$

We start the proof of this proposition with

LEMMA 21.1. — *Let* $B \in \mathcal{B}_3$ *and* $K \in \mathcal{B}$. *Then*

$$\int_{K'} G_B(x, dy) P_y(T_B \leqslant t)$$
$$\leqslant t P_x(X_s \in K' \textit{ for some } s \textit{ such that } T_B - t \leqslant s \leqslant T_B).$$

Proof. — Let $t > 0$. Then for $u \geqslant 0$

$$\sum_{n=0}^\infty \int_{K'} Q_B^{u+nt}(x, dy) P_y(T_B \leqslant t)$$
$$\leqslant P_x(X_s \in K' \text{ for some } s \text{ such that } T_B - t \leqslant s \leqslant T_B).$$

If we integrate u in the left side of this inequality from 0 to t we get

$$\int_0^\infty du \int_{K'} Q_B^u(x, dy) P_y(T_B \leqslant t) = \int_{K'} G_B(x, dy) P_y(T_B \leqslant t),$$

from which the lemma follows.

Proof of Proposition 21.5. — Let $B \in \mathcal{B}_3$ and $t > 0$ and choose $\varepsilon > 0$. There is a compact set $K_1 \supseteq B$ such that

$$P_y(T_B \leqslant t) \leqslant \varepsilon/3t, \qquad y \in K_1'.$$

By Theorem 19.2 and the argument used in proving Lemma

19.1 it follows that we can choose a compact set K_2 such that

$$\limsup_{x\to\infty} H_{K_2}(x, K_1) < \varepsilon/3t.$$

There is a compact set $K_3 \supseteq K_2$ such that

$$P_x(X_s \in K_3' \text{ for some } s \text{ such that } T_B - t \leqslant s \leqslant T_B) \leqslant \varepsilon/3t$$

for all $x \in K_2$. These inequalities imply that

$$\limsup_{x\to\infty} P_x(X_s \in K_3' \text{ for some } s \text{ such that } T_B - t \leqslant s \leqslant T_B) < \varepsilon/t.$$

By Lemma 21.1 it follows that there is a compact set L such that

$$\int_{K'} G_B(x, dy) P_y(T_B \leqslant t) \leqslant \varepsilon, \qquad x \in L^C.$$

It is easy to see that there is a compact set $K \supseteq K_3$ such that

$$\int_{K'} G_B(x, dy) P_y(T_B \leqslant t) \leqslant \varepsilon, \qquad x \in L.$$

The set K is the desired set.

The proof of the proposition for $B \in \mathcal{B}_2$ rurs along similar lines, but uses additionally Proposition 18.3.

THEOREM 21.2. — *Let* $B \in \mathcal{B}_3$. *Then for* $t > 0$ *and* $A \in \mathcal{B}$

$$\int \tilde{L}_B(y) P_y(T_B \leqslant t, X_{T_B} \in A)\, dy = t\mu_B(A),$$

and in the type II *case*

$$\int L_B^{\pm}(y) P_y(T_B \leqslant t, X_{T_B} \in A)\, dy = t\mu_B^{\pm}(A).$$

Proof. — By Proposition 21.4

$$\int_0^t ds\, P_x(T_B > s, X_{T_B} \in A) = \int_{B'} G_B(x, dy) P_y(T_B \leqslant t, X_{T_B} \in A).$$

Choose $\varepsilon > 0$. By Proposition 21.5 there is a compact set K such that

$$\int_{K'} G_B(x, dy) P_y(T_B \leqslant t, X_{T_B} \in A) \leqslant \varepsilon, \qquad x \in \mathfrak{G}.$$

Choose $C \in \mathfrak{A}$ such that $|C| > 0$. Then

$$\int_{x+C} dz \int_K G_B(z, dy) P_y(T_B \leqslant t, X_{T_B} \in A) = \int_K \tilde{G}_B(y, x + C) P_y(T_B \leqslant t, X_{T_B} \in A).$$

In the type I case as $x \to \infty$ the last term approaches, by Theorem 18.5,

$$|C| \int_K \tilde{L}_B(y) P_y(T_B \leqslant t, X_{T_B} \in A).$$

On the other hand by Theorem 19.2

$$\lim_{x \to \infty} \int_{x+C} dz \int_0^t ds\, P_z(T_B > s, X_{T_B} \in A) = |C| t \mu_B(A).$$

Since ε can be made arbitrarily small

$$\int_{\mathfrak{G}} \tilde{L}_B(y) P_y(T_B \leqslant t, X_{T_B} \in A) = t\mu_B(A).$$

In the type II case we need only let $x \to \pm\infty$ and use the same argument.

THEOREM 21.3. — *Suppose the process is type* I. *Then for* $B \in \mathcal{B}_1$ *and* $A \in \mathcal{B}$.

$$\lim_{t \to \infty} E_B(t, A)/E_B(t) = \mu_B(A).$$

Proof. — Suppose first that $B \in \mathcal{B}_2$. By Proposition 21.1 for $A \in \mathcal{B}$

$$E_B(s+1, A) - E_B(s, A) = \int dx\, P_x(\tilde{T}_B > s) P_x(T_B \leqslant 1, X_{T_B} \in A)$$

and hence

$$\int_0^t (E_B(s+1, A) - E_B(s, A))\, ds = \int_{\mathfrak{G}} dx\, P_x(T_B \leqslant 1, X_{T_B} \in A) \int_0^t ds\, P_x(\tilde{T}_B > s).$$

Let K be compact. Then by Theorem 21.1

$$\lim_{t \to \infty} (E_B(t))^{-1} \int_K dx\, P_x(T_B \leqslant 1, X_{T_B} \in A) \int_0^t ds\, P_x(\tilde{T}_B > s) = \int_K dx\, \tilde{L}_B(x) P_x(T_B \leqslant 1, X_{T_B} \in A).$$

Moreover

$$\int_{K'} dx\, P_x(T_B \leqslant 1, X_{T_B} \in A) \int_0^t ds\, P_x(\tilde{T}_B > s) \leqslant \int_{K'} dx\, P_x(T_B \leqslant 1) \int_0^t dx\, P_x(\tilde{T}_B > s).$$

Since

$$\int_0^t (E_B(s+1) - E_B(s))\, ds \sim E_B(t)$$

and by Theorem 21.2

$$\int \tilde{L}_B(x)P_x(T_B \leqslant 1)\,dx = 1,$$

it follows that

$$\lim_{K \uparrow \mathfrak{G}} \limsup_{t\to\infty} \frac{1}{E_B(t)} \int_{K'} P_x(T_B \leqslant 1) \int_0^t P_x(T_B > s)\,ds = 0$$

and thus

$$\lim_{t\to\infty} \int_0^t (E_B(s+1, A) - E_B(s, A))\,ds/E_B(t)$$
$$= \int dx\, \tilde{L}_B(x)P_x(T_B \leqslant 1, X_{T_B} \in A) = \mu_B(A)$$

and therefore that

$$\lim_{t\to\infty} \frac{E_B(t, A)}{E_B(t)} = \mu_B(A).$$

Suppose now that $B \in \mathscr{B}_1$. Let B_1 be a compact set in $\mathscr{B}_2$ containing B. Then

$$E_B(t, A) = \int_0^t \int_{B_1} E_{B_1}(ds, dy)P_y(T_B \leqslant t - s, X_{T_B} \in A)$$
$$= \int_{B_1} E_{B_1}(t, dy)H_B(y, A)$$
$$- \int_0^t \int_{B_1} E_{B_1}(ds, dy)P_y(T_B > t - s, X_{T_B} \in A).$$

Now by Proposition 21.3

$$\lim_{t\to\infty} \int_{B_1} E_{B_1}(t, dy)H_B(y, A)/E_B(t)$$
$$= \lim_{t\to\infty} \int_{B_1} E_{B_1}(t, dy)H_B(y, A)/E_{B_1}(t)$$
$$= \int_{B_1} \mu_{B_1}(dy)H_B(y, A) = \mu_B(A).$$

Moreover

$$\limsup_{t\to\infty} \int_0^t \int_{B_1} E_{B_1}(ds, dy)P_y(T_B > t - s, X_{T_B} \in A)/E_B(t)$$
$$\leqslant \limsup_{t\to\infty} \int_0^t \int_{B_1} E_{B_1}(ds, dy)P_y(T_B > t - s)/E_B(t)$$
$$= \limsup_{t\to\infty} (E_{B_1}(t) - E_B(t))/E_B(t) = 0,$$

from which the conclusion of the theorem follows.

Proposition 21.6. — *Let* $B \in \mathcal{B}_1$ *and* $C \in \mathcal{B}_1$. *Then*

$$\lim_{t \to \infty} \frac{\int_{0-}^{t} E_B(t-s)\,E_B\,(ds)}{\int_{0-}^{t} E_C(t-s)\,E_C\,(ds)} = 1.$$

Proof. — This result is a direct consequence of Proposition 21.3. According to that proposition

$$\int_{0-}^{t} E_B(t-s)E_B\,(ds) \sim \int_{0-}^{t} E_C(t-s)E_B\,(ds)$$
$$= \int_{0-}^{t} E_B(t-s)E_C\,(ds) \sim \int_{0-}^{t} E_C(t-s)E_C\,(ds)$$

According to Proposition 21.6 we can find a non-decreasing function $g(t)$, $t \geqslant 0$, such that for $B \in \mathcal{B}_1$

$$\lim_{t \to \infty} \int_{0-}^{t} E_B(t-s)E_B\,(ds)/g(t) = 1.$$

Theorem 21.4. — *Suppose the process is type* I. *Then for* $C \in \mathcal{B}_2$ *and* $B \in \mathcal{B}$.

$$\lim_{t \to \infty} \int_{0}^{t} (E_C(s) - E_B(s))\,ds/g(t) = k(C) - k(B).$$

Proof. — We use the notation $f_1(t) \sim f_2(t)$ in this proof to mean that

$$\lim_{t \to \infty} (f_1(t) - f_2(t))/g(t) = 0.$$

Suppose first that $B \in \mathcal{B}_2$ and $\overline{B} \subseteq \mathring{C}$. Let D be a relatively compact non-empty open set such that $B - D \subseteq C$. Then

$$|D|(E_C(s) - E_B(s)) = \int_D du \int_{0-}^{s} \int_{\overline{C}} E_{u+C}\,(dr, dy) P_y(T_B > s - r)$$
$$= \int_{0-}^{s} \int_{\overline{C}} E_C\,(dr, dy) \int_{y+D} du\, P_u(T_B > s - r).$$

Consequently

$$|D| \int_{0}^{t} (E_C(s) - E_B(s))\,ds$$
$$= \int_{0-}^{t} \int_{\overline{C}} E_C\,(ds, dy) \int_{y+D} du \int_{0}^{t-s} P_u(T_B > r)\,dr,$$

which, by Theorem 21.1 and Proposition 21.3, is asymptotic to

$$\begin{aligned}
\int_{0-}^{t} \mathrm{E_C}(t-s) \int_{\bar{\mathrm{C}}} \mathrm{E_C}\,(ds, dy) \int_{y+\mathrm{D}} \mathrm{L_B}(u)\, du
&= \int_{0-}^{t} \mathrm{E_C}(t-s)\, ds \int_{\bar{\mathrm{C}}} \mathrm{E_C}(s, dy) \int_{y+\mathrm{D}} \mathrm{L_B}(u)\, du \\
&= \int_{0-}^{t} \left(\int_{\bar{\mathrm{C}}} \mathrm{E_C}(t-s, dy) \int_{y+\mathrm{D}} \mathrm{L_B}(u)\, du\right) \mathrm{E_C}\,(ds) \\
&\sim \int_{0}^{t} \mathrm{E_C}(t-s) \mathrm{E_C}\,(ds) \int_{\bar{\mathrm{C}}} \mu_{\mathrm{C}}\,(dy) \int_{y+\mathrm{D}} \mathrm{L_B}(u)\, du \\
&\sim g(t) \int_{\mathrm{D}} du(\mathrm{L_B}, \mu_{u+\mathrm{C}}) \\
&= g(t)|\mathrm{D}|(k(\mathrm{C}) - k(\mathrm{B}))
\end{aligned}$$

by Proposition 20.9.

From this it follows that the conclusion of the theorem holds whenever $\mathrm{B} \in \mathcal{B}_2$. To complete the proof of the theorem we need only show that if $\mathrm{B} \in \mathcal{B}$ but $\mathrm{B} \notin \mathcal{B}_2$, then

$$(21.1) \qquad \lim_{t \to \infty} \int_{0}^{t} (\mathrm{E_C}(s) - \mathrm{E_B}(s))\, ds/g(t) = \infty.$$

But by Theorem 20.4 there exist sets $\mathrm{B}_n \in \mathcal{B}_2$ with $\mathrm{B}_n \supseteq \mathrm{B}$ and $\lim_{n \to \infty} k(\mathrm{B}_n) = -\infty$. Thus

$$\begin{aligned}
&\liminf_{t \to \infty} \int_{0}^{t} (\mathrm{E_C}(s) - \mathrm{E_B}(s))\, ds/g(t) \\
&\qquad \geqslant \lim_{t \to \infty} \int_{0}^{t} \mathrm{E_C}(s) - \mathrm{E}_{\mathrm{B}_n}(s))\, ds/g(t) = k(\mathrm{C}) - k(\mathrm{B}_n)
\end{aligned}$$

and hence (21.1) holds, as desired.

22. Stronger Results on the Time Dependent Behavior (Recurrent Case).

Throughout this section X_t will be a recurrent i.d. process that satisfies the additional

Condition 2. For some $g \in (\mathcal{F}^*)^+$ with $\mathrm{J}(g) = 1$

$$\lim_{\lambda \downarrow 0} \lambda^{1-1/\alpha} \mathrm{H}\left(\frac{1}{\lambda}\right)^{-1} \mathrm{G}^{\lambda} g(x) = 1$$

uniformly in x on compacts for some constant α, $1 \leqslant \alpha \leqslant 2$ and some slowly varying function H.

By essentially the same argument as used in [7] (see § 13) we can show that this condition is satisfied for every type II process with $\alpha = 2$ and H the constant function $(2\sigma^2)^{-1/2}$. An easy Abelian argument shows that this condition holds when $\mathfrak{G} = R^1 \oplus H$ or $Z^1 \oplus H$ and the process $\psi(X_t)$ is in the domain of attraction of a stable law of exponent α and thus, in particular, for the stable processes themselves.

For $B \in \mathcal{B}$, $A \in \mathcal{B}$ set

$$e_B^h(t, A) = E_B(t+h, A) - E_B(t, A)$$

and set

$$e_B(t) = e_B^1(t, B).$$

PROPOSITION 22.1. — *For any* $B \in \mathcal{B}$,

$$E_B(t) \sim \frac{t^{1/\alpha}}{H(t)\Gamma\left(1+\dfrac{1}{\alpha}\right)}$$

and

$$e_B^h(t) \sim \frac{ht^{-1+1/\alpha}}{H(t)\Gamma(1/\alpha)}.$$

In particular, for any type II *process*

$$\lim_{t\to\infty} \frac{E_B(t)}{\sqrt{t}} = 2(2/\pi)^{1/2}\sigma$$

and

$$\lim_{t\to\infty} e_B(t)\sqrt{t} = (2/\pi)^{1/2}\sigma.$$

Proof. — It follows from (3.7) that

$$E_B^\lambda G^\lambda g = (1, H_B^\lambda G^\lambda g) = (g, \tilde{H}_B^\lambda \tilde{G}^\lambda 1)$$
$$\frac{1}{\lambda}\int_{\mathfrak{G}} g(x)E_x(e^{-\lambda\tilde{T}_B})\, dx.$$

Thus

$$E_B^\lambda(\overline{B}) \sim \frac{\lambda^{-1/\alpha}}{H\left(\dfrac{1}{\lambda}\right)}$$

and thus by Karamata's theorem.

$$E_B(t) \sim \frac{t^{1/\alpha}}{H(t)\Gamma\left(1+\dfrac{1}{\alpha}\right)}.$$

Since

$$e_B^h(t) = \int_{\mathfrak{G}} \tilde{P}_x(T_B > t) P_x(T_B \leqslant h)\, dx$$

is monotonic in t it follows that

$$e_B^h(t) \sim \frac{ht^{-1+1/\alpha}}{H(t)\Gamma(1/\alpha)}$$

as desired.

THEOREM 22.1. — *For any* $B \in \mathcal{B}_3$, *uniformly in* x *in compacts*,

$$P_x(T_B > t) \sim \frac{L_B(x) t^{-1+1/\alpha}}{H(t)\Gamma(1/\alpha)}$$

and for any $B \in \mathcal{B}_2$ and any $C \in \mathcal{B}$

$$\int_C P_x(T_B > t)\, dx \sim \left[\int_C L_B(x)\, dx\right] \frac{t^{-1+1/\alpha}}{H(t)\Gamma(1/\alpha)}.$$

Proof. — By definition of $L_B^\lambda(x)$

$$\int_0^\infty P_x(T_B > t) e^{-\lambda t}\, dt = \frac{L_B^\lambda(x)}{\lambda G^\lambda g(0)}$$

and thus, uniformly in x on compacts,

$$\int_0^\infty P_x(T_B > t) e^{-\lambda t}\, dt \sim L_B(x) \left(\frac{1}{\lambda}\right)^{1/\alpha} H\left(\frac{1}{\lambda}\right)^{-1}.$$

The result for $B \in \mathcal{B}_3$ now follows by the usual Tauberian arguments. The proof for $B \in \mathcal{B}_2$ is similar.

COROLLARY 22.1. — *For any type* II *process*

$$\lim_{t \to \infty} \sqrt{t}\, P_x(T_B > t) = \sqrt{\frac{2}{\pi}}\, \sigma L_B(x)$$

uniformly in x *on compacts whenever* $B \in \mathcal{B}_3$. *For* $B \in \mathcal{B}_2$,

$$\lim_{t \to \infty} \sqrt{t} \int_C P_x(T_B > t)\, dx \sim \sqrt{\frac{2}{\pi}}\, \sigma \int_C L_B(x)\, dx.$$

Proof. — Immediate.

Corollary 22.2 — *If the process is type* I *and Condition* 2 *holds then for any* $B \in \mathcal{B}_1$ *and any* $A \in \mathcal{B}$

$$e_B^h(t, A) \sim \mu_B(A) h \frac{t^{-1+1/\alpha}}{H(t)\Gamma(1/\alpha)}.$$

Proof. — The result follows from Theorem 21.3 and Proposition 22.1 and the fact that $e_B^h(t, A)$ is monotonic in t by a standard Tauberian argument.

Theorem 22.2. — *Suppose the process is type* II. *Then for any* $B \in \mathcal{B}_1$

$$\lim_{t \to \infty} \sqrt{t}\, e_B(t, A) = (2/\pi)^{1/2} \sigma \mu_B(A).$$

Proof. — First suppose $B \in \mathcal{B}_2$. We can write

$$\frac{e_B(t, A)}{e_B(t)} = \frac{1}{e_B(t)} \int_{\mathfrak{G}} P_x(\tilde{T}_B > t) P_x(T_B \leqslant 1, X_{T_B} \in A)\, dx.$$

From Corollary 22.1

$$(22.1) \quad \lim_{K \uparrow \mathfrak{G}} \lim_{t \to \infty} \frac{1}{e_B(t)} \int_K P_x(\tilde{T}_B > t) P_x(T_B \leqslant 1, X_{T_B} \in A)\, dx$$
$$= \int_{\mathfrak{G}} \tilde{L}_B(x) P_x(T_B \leqslant 1, X_{T_B} \in A)\, dx = \mu_B(A).$$

In particular this is true for $A = \overline{B}$, and thus is must be that

$$(22.2) \quad \lim_{K \uparrow \mathfrak{G}} \lim_{t \to \infty} \frac{1}{e_B(t)} \int_{K'} P_x(\tilde{T}_B > t) P_x(T_B \leqslant 1)\, dx = 0.$$

Thus the theorem is true for any $B \in \mathcal{B}_2$. Now let $B \in \mathcal{B}_1$ and choose $B_1 \in \mathcal{B}_2$ such that $B_1 \supset B$. Since the theorem is true for B_1 we see from (22.1) and an Abelian argument that

$$E_{B_1}(t, A) \sim E_{B_1}(t) \mu_{B_1}(A).$$

Arguing as in the conclusion of the proof of Theorem 21.3 we see that

$$E_B(t, A) \sim E_B(t) \mu_B(A) \sim \mu_B(A) t^{1/2} 2 (2/\pi)^{1/2} \sigma.$$

Since $e_B(t, A)$ is monotonic in t it follows that

$$e_B(t, A) \sim \mu_B(A) t^{-1/2} (2/\pi)^{1/2} \sigma$$

as desired.

THEOREM 22.3. — *Suppose* U *Condition* 2 *is satisfied. Then for and* $C \in \mathcal{B}_2$ *and* $B \in \mathcal{B}$

$$\lim_{t\to\infty} \frac{[E_C(t) - E_B(t)]}{q(t)} = k(B) - k(A) \tag{22.3}$$

where

$$q(t) = \int_0^t e_C(s)e_C(t-s)\, ds \sim \frac{t^{-1+2/\alpha}}{H(t)^2\Gamma\left(\frac{2}{\alpha}\right)}. \tag{22.4}$$

In particular, for every type II *process*

$$\lim_{t\to\infty} [E_C(t) - E_B(t)] = (2\sigma^2)(k(C) - k(B)).$$

Proof. — First assume that $B \in \mathcal{B}_2$ and that C is such that $\overline{B} \subset \mathring{C}$. Let D be an open neighborhood of 0 such that $\overline{B} - D \subset C$. Observe that

$$|D|\,[E_C(t) - E_B(t)] = \int_{\overline{C}}\int_0^t E_C(ds, dy) \int_{D+y} du\, P_u(T_B > t-s).$$

By Theorem 22.1 and Corollary 22.1

$$\begin{aligned}
\int_{\overline{C}}\int_0^t & E_C(ds, dy) \int_{D+y} du\, P_u(T_B > t-s) \\
&\sim \int_{\overline{C}}\int_0^t E_C(ds, dy) e_B(t-s) \int_{D+y} L_B(u)\, du \\
&\sim \sum_{k=1}^{[t]} \int_{\overline{C}}\int_{k-1}^k E_C(ds, dy) e_B(t-s) \int_{D+y} L_B(u)\, du \\
&\sim \sum_{k=1}^{[t]} \int_{\overline{C}} e_C(k-1, dy) \int_{D+y} L_B(u)\, du\, e_B(t-k+1) \\
&\sim \sum_{k=1}^{[t]} e_C(k-1) e_C([t]-k+1) \int_{\overline{C}} \mu_C(dy) \int_{D+y} L_B(u)\, du.
\end{aligned}$$

The desired result follows from this by Proposition 20.7 and the fact that

$$\sum_{k=1}^{[t]} e_C(k-1)e_C([t]-k+1) \sim \int_0^t e_C(s)e_C(t-s)\, ds = q(t).$$

Thus (22.3) holds for any $B \in \mathcal{B}_2$ and C as above and consequently for any $C \in \mathcal{B}_2$ and $B \in \mathcal{B}_2$. Now suppose $B \in \mathcal{B}$

and $B \notin \mathcal{B}_2$. We must then show that

$$\lim_{t\to\infty} \frac{E_C(t) - E_B(t)}{q(t)} = \infty.$$

The proof of this fact can be carried out by essentially the same argument as used in the proof of Theorem 21.4 so we omit the details.

THEOREM 22.4. — *Let* $B \in B_1^*$ *and* $f \in \Phi^*$. *Then if the process is type* I

$$E_x[f(X_{T_B}); T_B > t] \sim \frac{L_B(x)(\mu_B, f)t^{-1+1/\alpha}}{H(t)\Gamma(1/\alpha)}$$

uniformly for x *in compacts.*

Proof. — The result follows from Theorem 21.2 and Proposition 22.1 by a familiar Tauberian argument.

We now turn our attention to establishing the corresponding result in the type II case. To this end we will need to extend a result of Belkin [1] from integer valued random walks to type II processes. The proof is essentially that of Belkin.

PROPOSITION 22.2. — *Let* X_t *be a type* II *process and let* $B \in \mathcal{B}_3$. *Then*

$$\lim_{t\to\infty} \sqrt{t}P_0(\psi(X_t)/\sqrt{t} \leqslant b; T_B > t) = (2/\pi)^{1/2}\sigma \int_{-\infty}^{b} f_B(a)\, da$$

where f_B *is*

$$f_B(a) = \frac{1}{2\sigma^2} e^{-a^2/2\sigma^2} \left[|a| L_B(0) - \frac{aE_0\psi(X_{T_B})}{\sigma^2}\right].$$

Proof. — Let $\varphi_t(\theta) = E_0(e^{i\theta\psi(X_t)})$ and let $\varphi(\theta) = e^{-\theta^2\sigma^2/2}$. We can write

$$\begin{aligned}(22.6)\quad E_0(\exp(i\theta\psi(X_t)/\sqrt{t}); T_B > t) &= \varphi_t(\theta/\sqrt{t}) \\ &- \int_0^t P_0(T_B \in ds)\varphi_{t-s}(\theta/\sqrt{t}) \\ &+ \int_{\overline{\psi(B)}} \int_0^t P_0(T_B \in ds, \psi(X_s) \in dy) \\ &[\varphi_{t-s}(\theta/\sqrt{t}) - e^{i\theta(y/\sqrt{t})}\varphi_{t-s}(\theta/\sqrt{t})].\end{aligned}$$

Let $M(\theta) = \log \varphi_1(\theta)$. Then $\varphi_t(\theta) = e^{tM(\theta)}$. Integration by parts and some easy computations show that

$$(22.7)\quad \varphi_t(\theta/\sqrt{t}) - \int_0^t P_0(T_B \in ds)\varphi_{t-s}(\theta/\sqrt{t}) = P_0(T_B > t) + \int_0^t P_0(T_B > s)M(\theta/\sqrt{t})\varphi_{t-s}(\theta/\sqrt{t})\, ds.$$

Let $\varepsilon > 0$ be given. Then uniformly in s, $\varepsilon t \leqslant s \leqslant (1-\varepsilon)t$

$$P_0(T_B > s)\sqrt{t} \sim L_B(0)(2/\pi)^{1/2}\sigma(t/s)^{1/2}.$$

Also $tM(\theta/\sqrt{t}) \to -\theta^2\sigma^2/2$ and for $0 \leqslant s \leqslant (1-\varepsilon)t$

$$\varphi_{t-s}(\theta/\sqrt{t}) - \varphi(\theta(1-s/t)^{1/2}) \to 0.$$

Thus

$$(22.8)\quad \sqrt{t}\int_{\varepsilon t}^{(1-\varepsilon)t} P_0(T_B > s)\varphi_{t-s}(\theta/\sqrt{t})M(\theta/\sqrt{t})\, ds + L_B(0)(2/\pi)^{1/2}\sigma\int_{\varepsilon t}^{(1-\varepsilon)t} (t/s)^{1/2}\frac{\theta^2\sigma^2}{2t}\varphi(\theta\sqrt{1-s/t})\, ds \to 0.$$

But

$$(22.9)\quad \int_{\varepsilon t}^{(1-\varepsilon)t} (t/s)^{1/2}\frac{\theta^2\sigma^2}{2t}\varphi(\theta(1-s/t)^{1/2})\, ds \to \int_{\varepsilon}^{1-\varepsilon} x^{-1/2}\frac{\theta^2\sigma^2}{2}\varphi(\theta(1-x)^{1/2})\, dx.$$

In addition there is a t_0 such that for all $s \geqslant t_0$,

$$P_0(T_B > s)\sqrt{s} \leqslant K < \infty.$$

Thus for some $K' < \infty$

$$(22.10)\quad \left|\int_{t_0}^{\varepsilon t} P_0(T_B > s)\sqrt{t}\, M(\theta/\sqrt{t})\varphi_{t-s}(\theta/\sqrt{t})\, ds\right| \leqslant K'\frac{1}{\sqrt{t}}\int_{t_0}^{\varepsilon t} s^{-1/2}\, ds = 2K'\frac{1}{\sqrt{t}}[\sqrt{\varepsilon t} - \sqrt{t_0}].$$

Also

$$(22.11)\quad \left|\int_0^{t_0} ds\, P_0(T_B > s)\sqrt{t}\, M(\theta/\sqrt{t})\varphi_{t-s}\right| = 0(1/\sqrt{t}).$$

Similarly for some constant C

$$(22.12)\quad \varlimsup_{t\to\infty}\left|\int_{(1-\varepsilon)t}^{t} P_0(T_B > s)\sqrt{t}\, M(\theta/\sqrt{t})t\varphi_{t-s}(\theta/\sqrt{t})\, ds\right| \leqslant C\varepsilon(1-\varepsilon)^{-1/2}.$$

It follows from (22.8)-(22.12) that

$$(22.13)\quad \lim_{t\to\infty} \sqrt{t} \int_0^t P_0(T_B > s)\varphi_{t-s}(\theta/\sqrt{t})M(\theta/\sqrt{t})\, ds$$
$$= -(2/\pi)\sigma L_B(0) \int_0^1 \frac{\theta^2\sigma^2}{2\sqrt{x}} \exp\left(-\frac{\theta^2\sigma^2(1-x)}{2}\right) dx.$$

Simple computations and integration by parts show that

$$(22.14)\quad \int_{\overline{\psi(B)}} \int_0^t P_0(T_B \in ds,\ \psi(X_s) \in dy)\varphi_{t-s}(\theta/\sqrt{t}) \times (1 - e^{i\theta y/\sqrt{t}})$$
$$= -\int_{\overline{\psi(B)}} P_0(T_B > t,\ \psi(X_{T_B}) \in dy)(1 - e^{i\theta y/\sqrt{t}})$$
$$+ \varphi_t(\theta/\sqrt{t}) \int_{\overline{\psi(B)}} P_0(\psi(X_{T_B}) \in dy)(1 - e^{i\theta y/\sqrt{t}})$$
$$- \int_{\overline{\psi(B)}} \int_0^t P_0(T_B > s,\ \psi(X_{T_B}) \in dy)M(\theta/\sqrt{t}) \times (1 - e^{i\theta y/\sqrt{t}})\varphi_{t-s}(\theta/\sqrt{t})\, ds$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Now in view of Corollary 22.1

$$(22.15)\quad |\mathrm{I}|\sqrt{t} = 0(t^{-1/2})$$

and also

$$(22.16)\quad |\mathrm{III}|\sqrt{t} = 0(t^{-1/2}).$$

On the other hand

$$(22.17)\quad \lim_{t\to\infty} \sqrt{t}\, \mathrm{II} = -i\theta\varphi(\theta)E_0\psi(X_{T_B}).$$

Thus from (22.14)-(22.17) we obtain

$$(22.18)\quad \lim_{t\to\infty} \sqrt{t} \int_{\overline{\psi(B)}} \int_0^t P_0(T_B \in ds,\ \psi(X_s) \in dy)\varphi_{t-s}(\theta/\sqrt{t})$$
$$(1 - e^{i\theta y/\sqrt{t}}) = -i\theta E_0\psi(X_{T_B}) \exp\left(-\frac{\theta^2\sigma^2}{2}\right).$$

Thus by (22.6), (22.7), (22.13) and (22.18) we see that

$$\lim_{t\to\infty} \left(\frac{\pi}{2\sigma^2}\right)^{1/2} \sqrt{t}E_0(e^{i\theta\psi(X_t)/\sqrt{t}};\ T_B > t)$$
$$= L_B(0)\left[1 - \int_0^1 \frac{\sigma^2\theta^2}{2\sqrt{x}} \exp\left(-\frac{\theta^2\sigma^2(1-x)}{2}\right) dx\right]$$
$$- i\theta\left(\frac{\pi}{2\sigma^2}\right)^{1/2} E_0\psi(X_{T_B}) \exp\left(-\frac{\theta^2\sigma^2}{2}\right).$$

The Fourier transform on the right (see Belkin [1]) can be shown to be that of the function

$$f_{\mathrm{B}}(a) = \frac{1}{2\sigma^2} \exp\left(-\frac{a^2}{2\sigma^2}\right)\left[|a|\mathrm{L}_{\mathrm{B}}(0) - \frac{a}{\sigma^2}\mathrm{E}_0\psi(\mathrm{X}_{\mathrm{T}_{\mathrm{B}}})\right].$$

This establishes the proposition.

Corollary 22.3. — *Let* $\mathrm{B} \in \mathcal{B}_3$ *and let* X_t *be a type* II *process. Then for any* $x \in \mathfrak{G}$,

$$\lim_{t\to\infty} \sqrt{t}\,\mathrm{P}_x(\mathrm{X}_t \in \mathfrak{G}^{\pm};\ \mathrm{T}_{\mathrm{B}} > t) = \left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^{1/2} \sigma \mathrm{L}_{\mathrm{B}}^{\pm}(x).$$

Proof. — Since

$$\mathrm{P}_x(\psi(\mathrm{X}_t) > 0;\ \mathrm{T}_{\mathrm{B}} > t) = \mathrm{P}_0\left(\frac{\psi(\mathrm{X}_t)}{\sqrt{t}} > \frac{-\psi(x)}{\sqrt{t}};\ \mathrm{T}_{\mathrm{B}-x} > t\right)$$

it follows from Proposition 22.2 that

$$\begin{aligned}
&\lim_{t\to\infty}\sqrt{t}\,\mathrm{P}_x(\psi(\mathrm{X}_t) > 0;\ \mathrm{T}_{\mathrm{B}} > t) \\
&\qquad = \left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^{1/2} \sigma\left[\mathrm{L}_{\mathrm{B}-x}(0) - \frac{1}{\sigma^2}\mathrm{E}_0\psi(\mathrm{X}_{\mathrm{T}_{\mathrm{B}-x}})\right] \\
&\qquad = \left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^{1/2} \sigma\left[\mathrm{L}_{\mathrm{B}}(x) - \frac{1}{\sigma^2}\mathrm{E}_x[\psi(\mathrm{X}_{\mathrm{T}_{\mathrm{B}}}) - \psi(x)]\right] \\
&\qquad = \left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^{1/2} \sigma \mathrm{L}_{\mathrm{B}}^{+}(x).
\end{aligned}$$

The proof for $\mathrm{X}_t \in \mathfrak{G}^-$ is similar.

Corollary 22.4. — *Let* $\mathrm{B} \in \mathcal{B}_1$. *Then for any* $\mathrm{A} \in \mathcal{B}$ *for a type* II *process*

$$\lim_{t\to\infty}\sqrt{t}\int_{\mathfrak{G}^{\pm}} \mathrm{P}_x(t < \mathrm{T}_{\mathrm{B}} \leqslant t + h;\ \mathrm{X}_{\mathrm{T}_{\mathrm{B}}} \in \mathrm{A})_x = h\mu_{\mathrm{B}}^{\pm}(\mathrm{A})\left(\frac{2}{\pi}\right)^{1/2}\sigma/2.$$

Proof. — First suppose that $\mathrm{B} \in \mathcal{B}_3$. We can write

$$\begin{aligned}
&\int_{\mathfrak{G}^+} \mathrm{P}_x(t < \mathrm{T}_{\mathrm{B}} \leqslant t + h;\ \mathrm{X}_{\mathrm{T}_{\mathrm{B}}} \in \mathrm{A})\, dx \\
&\qquad = \int_{\mathfrak{G}} \tilde{\mathrm{P}}_x(\mathrm{T}_{\mathrm{B}} > t;\ \mathrm{X}_t \in \mathfrak{G}^+)\mathrm{P}_x(\mathrm{T}_{\mathrm{B}} \leqslant h,\ \mathrm{X}_{\mathrm{T}_{\mathrm{B}}} \in \mathrm{A})\, dx.
\end{aligned}$$

From Theorem 22.1 we see that for any compact set K

$$\varlimsup_{t\to\infty} \sup_{x\in\mathrm{K}} \tilde{\mathrm{P}}_x(\mathrm{T_B} > t,\ \mathrm{X}_t \in \mathfrak{G}^+)\sqrt{t} \leqslant \varlimsup_{t\to\infty} \sup_{x\in\mathrm{K}} \mathrm{P}_x(\mathrm{T_B} > t)\sqrt{t} < \infty$$

and thus

$$\lim_{\mathrm{K}\uparrow\mathfrak{G}} \lim_{t\to\infty} \sqrt{t}\int_{\mathrm{K}} \tilde{\mathrm{P}}_x(\mathrm{T_B} > t,\ \mathrm{X}_t \in \mathfrak{G}^+)\mathrm{P}_x(\mathrm{T_B} \leqslant h,\ \mathrm{X}_{\mathrm{T_B}} \in \mathrm{A})\,dx$$
$$= \lim_{\mathrm{K}\uparrow\mathfrak{G}} \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\sigma}{2}\right)\int_{\mathrm{K}} \tilde{\mathrm{L}}_{\mathrm{B}}^{+}(x)\mathrm{P}_x(\mathrm{T_B} \leqslant h,\ \mathrm{X}_{\mathrm{T_B}} \in \mathrm{A}) = \mu_{\mathrm{B}}^{+}(\mathrm{A}).$$

From (22.2) we see that

$$\lim_{\mathrm{K}\uparrow\mathfrak{G}} \lim_{t\to\infty} \sqrt{t}\int_{\mathrm{K}'} \tilde{\mathrm{P}}_x(\mathrm{T_B} > t;\ \mathrm{X}_t \in \mathfrak{G}^+)\mathrm{P}_x(\mathrm{T_B} \leqslant h,\ \mathrm{X}_{\mathrm{T_B}} \in \mathrm{A})\,dx = 0.$$

Now let $\mathrm{B} \in \mathcal{B}_1$ and choose $\mathrm{B}_1 \in \mathcal{B}_3$ such that $\mathrm{B}_1 \supset \mathrm{B}$. From what has already been proved and an Abelian argument

$$\int_{\mathfrak{G}^+} \mathrm{P}_x(\mathrm{T}_{\mathrm{B}_1} \leqslant t,\ \mathrm{X}_{\mathrm{T}_{\mathrm{B}_1}} \in \mathrm{A})\,dx \sim \left(\frac{2}{\pi}\right)^{1/2} \sigma\mu_{\mathrm{B}_1}^{+}(\mathrm{A})\sqrt{t}.$$

Using this and arguing as in the proof of Theorem 21.3 we find that

$$\int_{\mathfrak{G}^+} \mathrm{P}_x(\mathrm{T_B} \leqslant t,\ \mathrm{X}_{\mathrm{T_B}} \in \mathrm{A})\,dx \sim \left(\frac{2}{\pi}\right)^{1/2} \sigma\mu_{\mathrm{B}}^{+}(\mathrm{A})\sqrt{t}$$

and then by a Tuberian argument that

$$\int_{\mathfrak{G}^+} \mathrm{P}_x(t < \mathrm{T_B} \leqslant t + h,\ \mathrm{X}_{\mathrm{T_B}} \in \mathrm{A})\,dx \sim h\left(\frac{\sigma}{2}\right)\left(\frac{2}{\pi}\right)^{1/2} \sigma\mu_{\mathrm{B}}^{+}(\mathrm{A})t^{-1/2}$$

as desired.

We may now establish the analogue of Theorem 22.4 for a type II process.

THEOREM 22.5. — *Let* X_t *be a type* II *process and let* $\mathrm{B} \in \mathcal{B}_1^*$. *Then for* $f \in \Phi^*$.

$$\lim_{t\to\infty} \mathrm{E}_x[f(\mathrm{X}_{\mathrm{T_B}});\ \mathrm{T_B} > t]\sqrt{t}$$
$$= \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\sigma}{2}\right)[\mathrm{H_B}f(+\infty)\mathrm{L}_{\mathrm{B}}^{+}(x) + \mathrm{H_B}f(-\infty)\mathrm{L}_{\mathrm{B}}^{-}(x)].$$

Proof. — We can write

$$\begin{aligned}
E_x[f(X_{T_B}); T_B > t] &= \int_{\mathfrak{G}} P_x(T_B > t, X_t \in dy) H_B f(y) \\
&= \int_{\mathfrak{G}^+} P_x(T_B > t, X_t \in dy)[H_B f(y) - H_B f(+\infty)] \\
&+ \int_{\mathfrak{G}^-} P_x(T_B > t, X_t \in dy)[H_B f(y) - H_B f(-\infty)] \\
&+ H_B f(+\infty) P_x(T_B > t, X_t \in \mathfrak{G}^+) \\
&+ H_B f(-\infty) P_x(T_B > t, X_t \in \mathfrak{G}^-).
\end{aligned}$$

It follows from Corollary 22.3 that as $t \to \infty$ the last two terms are asymptotic to

$$[H_B f(+\infty) L_B^+(x) + H_B f(-\infty) L_B^-(x)](\sigma/2)(2/\pi)^{1/2} t^{-1/2}.$$

Let $\varepsilon > 0$ be given. Then there is a compact subset K of $\mathfrak{G}$ such that $K = K_1 \oplus H$, $K_1 \subset R^1$, $|\partial(K_1 \cap \mathfrak{G}^+)|_{R^1} = 0$ and $|H_B f(x) - H_B f(+\infty)| < \varepsilon$ for $x \in \mathfrak{G}^+ \cap K'$. Thus

$$\begin{aligned}
&\int_{\mathfrak{G}^+} P_x(T_B > t, X_t \in dy)|H_B f(y) - H_B f(-\infty)| \\
&\qquad \leqslant \varepsilon P_x(T_B > t) + 2\|f\| P_x\left(T > t_B, \frac{\psi(X_t)}{\sqrt{t}} \in t^{-1/2} K_1 \cap \mathfrak{G}^+\right).
\end{aligned}$$

By Proposition 22.2 and Corollary 22.1 we then see that

$$\begin{aligned}
&\overline{\lim_{t\to\infty}} \sqrt{t} \int_{\mathfrak{G}^+} P_x(T_B > t, X_t \in dy)|H_B f(y) - H_B f(+\infty)| \\
&\qquad\qquad \leqslant \left(\frac{2}{\pi}\right)^{1/2} L_B(x)\sigma\varepsilon.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\overline{\lim_{t\to\infty}} \sqrt{t} \int_{\mathfrak{G}^-} P_x(T_B > t, X_t \in dy)|H_B f(y) - H_B f(-\infty)| \\
&\qquad\qquad \leqslant \left(\frac{2}{\pi}\right)^{1/2} L_B(x)\sigma\varepsilon.
\end{aligned}$$

This establishes the theorem.

23. Invariant Functions for Killed Processes.

Let B be a Borel set and let Q_B^t be the transition operator for the process killed on B, i.e. $Q_B^t f(x) = E_x[f(X_t); T_B > t]$. Using the fact that

$$\{f(X_{t+s}); T_B > t + s\} = \{f(X_t \circ \theta_s); T_B \circ \theta_s > t, T_B > s\}$$

it follows that for f a bounded measurable function

$$Q_B^{t+s}f(x) = E_x[E_{X(s)}[f(X_t); T_B > t]; T_B > s] = Q_B^s(Q_B^t)f(x)$$

so Q_B^t has the semi-group property.

DEFINITION. — *A measurable function f is said to be Q_B^t invariant if $Q_B^t f = f$ for all $t > 0$. A measurable function is called essentially Q_B^t invariant if for each $t > 0$, $Q_B^t f(x) = f(x)$* a.e.

Our first task in this section will be to find all the bounded invariant and essentially invariant functions.

THEOREM 23.1 — *For any Borel set B the function $P_x(T_B = \infty)$ is a bounded Q_B^t invariant function. For a non-singular process the only bounded Q_B^t invariant functions are $\alpha P_x(T_B = \infty)$ for α a constant. If B' is relatively compact, then 0 is the only Q_B^t invariant function. This is also the case if the process is recurrent and $P_x(T_B = \infty) \equiv 0$. In general, the only bounded essentially Q_B^t invariant functions f are $f(x) = \alpha P_x(T_B = \infty)$ a.e.*

Remark. — In the general case even if we assume f is a bounded Q_B^t invariant function the most we can conclude is that $f(x) = \alpha P_x(T_B = \infty)$ a.e.

We will prove this theorem by a sequence of lemmas.

LEMMA 23.1. — *Let B be a Borel set. Then $P_x(T_B = \infty)$ is a bounded Q_B^t invariant function.*

Proof. — Since $Q_B^t \Phi_B(x) = P_x(t < T_B < \infty)$, we see that

$$\int_{\mathfrak{G}} Q_B^t(x, dy) P_y(T_B = \infty) = P_x(T_B > t) - P_x(t < T_B < \infty) = P_x(T_B = \infty)$$

as desired.

LEMMA 23.2. — *Suppose h is bounded, measurable, and for each $t > 0$, $P^t h(x) = h(x)$ a.e. Then for some constant α, $h(x) = \alpha$ a.e.*

Proof. — Let $\varphi \in C_c$, and set $\psi(x) = \int_{\mathfrak{G}} \varphi(t) h(t + x)\, dt$. An easy computation shows that $P^t\psi(x) = \psi(x)$ for all $x \in \mathfrak{G}$ and all $t > 0$. But then $\lambda G^\lambda \psi = \psi$, and as $\lambda G^\lambda(0, dx)$ is a probability measure on $\mathfrak{G}$ whose support is Σ it follows from the Choquet-Deny theorem and the basic assumption that for some constant K, $\psi(x) = \mathrm{K}$ a.e. As h is bounded and $\varphi \in \mathrm{C}_c$ the function ψ is continuous and thus $\psi(x) = \mathrm{K}$ for all $x \in \mathfrak{G}$. Thus $\psi(x) = \psi(0)$ for all $x \in \mathfrak{G}$. Hence for any $\varphi \in \mathrm{C}_c$

$$\int_{\mathfrak{G}} \varphi(t)[f(t+x) - f(t)]\, dt = 0$$

and thus for some constant α, $f(x) = \alpha$ a.e.

Lemma 23.3. — *Suppose f is bounded and essentially Q_B^t invariant. Then $f(x) + \|f\|_\infty \mathrm{P}_x(\mathrm{T}_\mathrm{B} = \infty) \geqslant 0$ a.e. If f is bounded and Q_B^t invariant, then this inequality holds everywhere.*

Proof. — Note that for a.e. x (all x if f is invariant)

$$|f(x)| = |\mathrm{Q}_\mathrm{B}^t f(x)| \leqslant \|f\|_\infty \mathrm{P}_x(\mathrm{T}_\mathrm{B} > t)$$

and thus for a.e. x (all x if f is invariant)

$$|f(x)| \leqslant \|f\|_\infty \mathrm{P}_x(\mathrm{T}_\mathrm{B} = \infty).$$

Lemma 23.4. — *Assume $f \geqslant 0$ a.e. and essentially Q_B^t invariant. Then $\mathrm{P}^{n+1}f \geqslant \mathrm{P}^n f$ a.e., $n = 1, 2, \ldots$ and so $\lim\limits_{n\to\infty} \mathrm{P}^n f = h$ exists a.e. The function h is essentially P^t invariant.*

Proof. — For any real $t > 0$, $\mathrm{P}^t f \geqslant \mathrm{Q}_\mathrm{B}^t f = f$ a.e. and so $\mathrm{P}^{t+s} f \geqslant \mathrm{P}^s f$ a.e. Thus for some measurable $h \geqslant 0$, $\mathrm{P}^n f \uparrow h$ a.e. Let $\varphi \in \Phi^+$. Then

$$(\varphi, \mathrm{P}^{t+s} f) = (\varphi \mathrm{P}^t, \mathrm{P}^s f) \geqslant (\varphi \mathrm{P}^t, f) = (\varphi, \mathrm{P}^t f)$$

and so $\lim\limits_{t\to\infty} (\varphi, \mathrm{P}^t f) = a \leqslant +\infty$ exists. By monotone convergence $a = (\varphi, h)$. Thus by monotone convergence again $(\varphi, h) = \lim\limits_{n\to\infty} (\varphi, \mathrm{P}^{t+n} f) = \lim\limits_{n} (\varphi \mathrm{P}^t, \mathrm{P}^n f) = (\varphi \mathrm{P}^t, h) = (\varphi, \mathrm{P}^t h)$. Consequently $h = \mathrm{P}^t h$ a.e.

We may now establish Theorem 23.1.

Proof of Theorem 23.1. — By Lemma 23.1 and 23.3 it suffices to prove the theorem for $f \geqslant 0$ a.e. Suppose this is the case. Now for $n = 1, 2, \ldots$

$$\begin{aligned} (23.1) \qquad \mathrm{P}^n f(x) &= \mathrm{Q}_\mathrm{B}^n f(x) + \mathrm{E}_x[f(\mathrm{X}_n);\, \mathrm{T}_\mathrm{B} \leqslant n] \\ &= f(x) + \mathrm{E}_x[f(\mathrm{X}_n);\, \mathrm{T}_\mathrm{B} \leqslant n] \quad \text{a.e.} \end{aligned}$$

By Lemma 23.4 $P^n f \uparrow h$ a.e. and thus the limit

$$\lim_{n\to\infty} E_x[f(X_n);\ T_B \leqslant n] = h_1(x)$$

exists a.e. Observing that $h_1(x) \leqslant \|f\| P_x(T_B \leqslant \infty)$ we see that $Q_B^t h_1(x) \leqslant \|f\| P_x(t < T_B < \infty)$ so $Q_B^t h_1(x) \downarrow 0$. Thus a.e. x,

$$f(x) = \lim_n Q_B^n h(x). \tag{23.2}$$

Now since f is bounded so is h (because $P^t f \leqslant \|f\|$) and thus by Lemmas 23.4 and 23.2 $h(x) = \alpha$ a.e. x for some constant α. Let $\varphi \in \Phi^+$. Then from (23.2) and (3.10) we see that

$$\begin{aligned}(\varphi, f) &= \lim_{n\to\infty} (\varphi, Q_B^n h) = \lim_{n\to\infty} (h, \tilde{Q}_B^n \varphi) = \alpha \lim_{n\to\infty} (1, \tilde{Q}_B^n \varphi) \\ &= \alpha \lim_{n\to\infty} (\varphi, Q_B^n 1) = \alpha \int_{\mathfrak{G}} P_x(T_B = \infty)\varphi(x)\, dx,\end{aligned}$$

and thus $f(x) = \alpha P_x(T_B = \infty)$ a.e. Suppose that $f \geqslant 0$ is Q_B^t invariant. Then $P^t f \uparrow h$ everywhere and h is a bounded invariant function. If the process is non-singular $h(x) = \alpha$ for all x. Using (23.1), which holds for all x if f is Q_B^t invariant, we see that

$$h = f + H_B h$$

and as $h = \alpha$ it follows that $f(x) = \alpha P_x(T_B = \infty)$ for all x. If B' is relatively compact and f is a bounded Q_B^t invariant function then for all $t > 0$ and all $x \in \mathfrak{G}$,

$$|f(x)| \leqslant \|f\| P_x(T_B > t).$$

Now $P_x(T_B > t) \downarrow 0$ so $f \equiv 0$. Also if the process is recurrent and $P_x(T_B = \infty) = 0$ for all x then $f \equiv 0$.

This establishes the theorem.

PROPOSITION 23.1. — *Let* B *be relatively compact set. For a type* I *transient process the functions* $\alpha P_x(T_B = \infty)$ *are the only bounded* Q_B^t *invariant functions having a limit at* ∞. *For a type* II *transient process* $\alpha P_x(T_B = \infty)$ *are the only bounded* Q_B^t *invariant function having a limit as either* $x \to +\infty$ *if* $m > 0$ *or as* $x \to -\infty$ *if* $m < 0$.

Proof. — By (3.19) there is a compact set K such that $P_x(T_B < \infty) \leqslant 2G(x, K)$. By the renewal theorem, in the type I case, $\lim_{x \to \infty} P_x(T_B \leqslant \infty) = 0$, while in the type II case with $m > 0$, $\lim_{x \to +\infty} P_x(T_B < \infty) = 0$. Thus $P_x(T_B = \infty)$ has the desired limit properties. Let f be a bounded Q_B^t invariant function having the stated limit properties. Suppose the process is type I. Then as

$$f(x) = \int_{\mathfrak{G}} Q_B^t(x, dy)[f(y) - f(\infty)] + f(\infty)P_x(T_B > t)$$

it follows that $f(x) = f(\infty)P_x(T_B = \infty)$. Now suppose that the process is type II with $m > 0$. Then

$$f(x) = \int_{\mathfrak{G}-} Q_B^t(x, dy)f(y) + \int_{\mathfrak{G}+} Q_B^t(s, dy)[f(y) - f(+\infty)]\, dy + f(+\infty)Q_B^t(x, \mathfrak{G}^+).$$

Since $P_x(X_t \to +\infty) = 1$ and f is bounded it follows that the first two terms on the right converge to 0 as $t \to \infty$. Moreover

$$P_x(T_B > t,\ X_t \in \mathfrak{G}^+) = P_x(T_B > t) - P_x(T_B > t,\ X_t \in \mathfrak{G}^-)$$

and thus

$$\lim_{t \to \infty} P_x(T_B > t,\ X_t \in \mathfrak{G}^+) = P_x(T_B = \infty).$$

The proof in the type II case with $m < 0$ is similar. This establishes the proposition.

For recurrent processes $P_x(T_B = \infty) = 0$ a.e. for all sets $B \in \mathcal{B}_1$ and so to get non-trivial Q_B^t invariant functions we must drop the requirement that the desired function is bounded. We therefore now turn our attention to finding all Q_B^t invariant and essentially Q_B^t invariant functions that are locally bounded and bounded from below. Our first task will be to show that there are such functions. Now for an arbitrary Borel set there may not be any functions bounded from below and Q_B^t invariant that are not also bounded. This happens for example whenever B' is relatively compact. Hence to obtain non-trivial results we will restrict our attention to relatively compact sets.

An essential tool used in our investigation will be to show that Green's function of B for the i.d. process are dominated by a corresponding quantity for a random walk.

PROPOSITION 23.2. — *Let* B *be relatively compact set such that* $\dot{\mathrm{B}} \neq \varphi$. *Let* $\mathrm{U}_\mathrm{B}^t(x, \mathrm{A})$ *be defined by*

$$\mathrm{U}_\mathrm{B}^t(x, \mathrm{A}) = \sum_{n=1}^{\infty} \mathrm{P}_x(\mathrm{X}_{tn} \in \mathrm{A}\,;\ \mathrm{T}_\mathrm{B}^t > nt)$$

where $\mathrm{T}_\mathrm{B}^t = \inf\{nt > 0 : \mathrm{X}_{nt} \in \mathrm{B}\}$. *Then given* A_1 *relatively compact there exists* $\mathrm{A} \supset \mathrm{A}_1$, A *compact and* $\beta < \infty$ *such that*

$$\mathrm{G}_\mathrm{B}(x, \mathrm{A}_1) \leqslant \frac{\beta}{t}\, \mathrm{U}_\mathrm{B}^t(x, \mathrm{A}). \tag{23.3}$$

Proof. — Set

$$\xi_n = \int_{nt}^{(n+1)t} 1_{\mathrm{A}_1}(\mathrm{X}_s)\, ds$$

and let $\mathrm{I}_n = 1$ if $\mathrm{T}_\mathrm{B}^t = nt$ and let $\mathrm{I}_n = 0$ otherwise. Then

$$\begin{aligned}
\mathrm{E}_x \int_0^{\mathrm{T}_\mathrm{B}^t} 1_{\mathrm{A}_1}(\mathrm{X}_s)\, ds &= \sum_{n=0}^{\infty} \mathrm{E}_x \left[\int_0^{nt} 1_{\mathrm{A}_1}(\mathrm{X}_s)\, ds\,;\ \mathrm{T}_\mathrm{B}^t = nt\right] \\
&= \sum_{n=0}^{\infty} \mathrm{E}_x \left[\sum_{k=0}^{\infty} \xi_k \mathrm{I}_n\right] \\
&= \mathrm{E}_x \left[\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \xi_k \mathrm{I}_n\right] = \mathrm{E}_x \sum_{k=0}^{\infty} \xi_k \sum_{n=k+1}^{\infty} \mathrm{I}_n \\
&= \sum_{k=0}^{\infty} \mathrm{E}_x \left[\xi_k \sum_{n=k+1}^{\infty} \mathrm{I}_n\right] = \sum_{k=0}^{\infty} \mathrm{E}_x[\xi_k\,;\ \mathrm{T}_\mathrm{B}^t > kt].
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{E}_x \int_0^{\mathrm{T}_\mathrm{B}^t} 1_{\mathrm{A}_1}(\mathrm{X}_s)\, ds &= \sum_{n=0}^{\infty} \mathrm{E}_x \left[\int_{nt}^{(n+1)t} 1_{\mathrm{A}_1}(\mathrm{X}_s)\, ds\,;\ \mathrm{T}_\mathrm{B}^t > nt\right] \\
&= \sum_{n=0}^{\infty} \int_{nt}^{(n+1)t} \mathrm{E}_x[1_{\mathrm{A}_1}(\mathrm{X}_s)\,;\ \mathrm{T}_\mathrm{B}^t > nt]\, ds.
\end{aligned}$$

Let $nt \leqslant s \leqslant (n+1)t$. Then for any set A,

$$\begin{aligned}
&(23.5)\quad \mathrm{P}_x(\mathrm{X}_{(n+1)t} \in \mathrm{A}\,;\ \mathrm{T}_\mathrm{B}^t > nt) \\
&= \int_{\mathfrak{G}} \mathrm{P}_x(\mathrm{X}_{nt} \in dy,\ \mathrm{T}_\mathrm{B}^t > nt)\mathrm{P}_y(\mathrm{X}_t \in \mathrm{A}) \\
&= \int_{\mathfrak{G}} \mathrm{P}_x(\mathrm{X}_{nt} \in dy\,;\ \mathrm{T}_\mathrm{B}^t > nt) \int_{\mathfrak{G}} \mathrm{P}_y(\mathrm{X}_{s-nt} \in dz)\mathrm{P}_s(\mathrm{X}_{(n+1)t-s} \in \mathrm{A}) \\
&\geqslant \int_{\mathfrak{G}} \mathrm{P}_x(\mathrm{X}_{nt} \in dy\,;\mathrm{T}_\mathrm{B} > nt) \int_{\mathrm{A}_1} \mathrm{P}_y(\mathrm{X}_{s-nt} \in dz)\mathrm{P}_z(\mathrm{X}_{(n+1)t-s} \in \mathrm{A}).
\end{aligned}$$

Now since $\bar{A}_1$ is compact we can choose A compact, $A \supset \bar{A}_1$, such that $P_z(X_s \in A,$ all $s \leqslant t) \geqslant \beta > 0$ for all $z \in \bar{A}_1$. So choosen we see from (23.5) that for $nt \leqslant s \leqslant (n+1)t$,

$$P_x(X_{(n+1)t} \in A, T_B^t > nt) \geqslant P_x(X_s \in A_1, T_B^t > nt)\beta$$

and thus

$$\sum_{n=0}^{\infty} \int_{nt}^{(n+1)t} P_x(X_s \in A_1 ; T_B^t > nt)\, ds$$
$$\leqslant \frac{t}{\beta} \sum_{n=0}^{\infty} P_x(X_{(n+1)t} \in A ; T_B^t > nt)$$
$$= \frac{t}{\beta} \sum_{n=1}^{\infty} P_x(X_{(n+1)t} \in A ; T_B^t \geqslant (n+1)t) \leqslant \frac{t}{\beta} U_B^t(x, A).$$

From (23.4) we then see that

$$E_x \int_0^{T_B^t} 1_{A_1}(X_s)\, ds \leqslant \frac{t}{\beta} U_B^t(x, A).$$

Since $T_B^t \geqslant T_B$ it follows that

$$E_x \int_0^{T_B} 1_{A_1}(X_s)\, ds \leqslant E_x \int_0^{T_B^t} 1_{A_1}(X_s)\, ds.$$

This establishes the proposition.

Theorem 23.2. — (Recurrent process). *Let* $B \in \mathcal{B}_3$. *Then* L_B *is* Q_B^t *invariant and in the type* II *case* L_B^+ *and* L_B^- *are* Q_B^t *invariant.*

Let $B \in \mathcal{B}_2$. *Then* L_B *is essentially* Q_B^t *invariant and in the type* II *case* L_B^+ *and* L_B^- *are essentially* Q_B^t *invariant.*

We begin the proof of this theorem with some results for discrete time recurrent random walks on G. The notation for such random walks is that of [7].

Lemma 23.5. — (Recurrent random walks on $\mathfrak{G}$). *Let* $B \in \mathcal{B}$ *have a non-empty interior. Then* $P\mathcal{L}_B = L_B$.

Proof. — Let A be defined by $A = D - I$. Observe that $PH_B = \Pi_B$ and $U_B = PG_B + \Pi_B$. and recall from Theorem 10.1 of [7] that $PA = A + I$. Choose $f \in \mathcal{F}$ with

$J(f) = 1$. By letting P act on the basic identity 10.1 of [7] we get that

$$Af + f - \Pi_B Af = P\mathfrak{L}_B - PG_B f.$$

The identity (5.15) of [7] can be rewritten as

$$Af + f - \Pi_B Af = L_B - PG_B f.$$

By subtraction it follows that $P\mathfrak{L}_B = L_B$, as desired.

LEMMA 23.6. — (Recurrent random walk on $\mathfrak{G}$) *Let* $B \in \mathcal{B}$ *have a non-empty interior. Then for any* $x \in \mathfrak{G}$, $A \in \mathcal{B}$ *and* $\varepsilon > 0$ *there is a compact set* K *such that*

$$\int_{K'} P(x, dz) G_B(z, y + A) \leqslant \varepsilon, \qquad y \in \mathfrak{G}.$$

Proof. — We can assume that $|\partial A| = 0$. Now

$$U_B = PG_B + \Pi_B$$

and hence

$$U_B(x, y + A) = \int P(x, dz) G_B(z, y + A) + \Pi_B(x, y + A).$$

If y is sufficiently large $\Pi_B(x, y + A) = 0$, so that for any compact set K_1

$$\begin{aligned} U_B(x, y + A) &= \int P(x, dz) G_B(z, y + A) \\ &= \int_{K_1} P(x, dz) G_B(z, y + A) \\ &\quad + \int_{K_1'} P(x, dz) G_B(z, y + A). \end{aligned}$$

Consider first the type I case. Choose $\varepsilon > 0$ and let K_1 be a compact set such that

$$|A| (L_B(x) - \int_{K_1} P(x, dz) \mathfrak{L}_B(z)) \leqslant \varepsilon/2.$$

This can be done by Lemma 23.5 since $\mathfrak{L}_B$ is non-negative. Now

$$\lim_{y \to \infty} U_B(x, y + A) = L_B(x) |A|$$

and

$$\lim_{y \to \infty} \int_{K_1} P(x, dz) G_B(z, y + A) = \int_{K_1} P(x, dz) \mathfrak{L}_B(z) |A|.$$

Thus there is a compact set D such that

$$\int_{K_1'} P(x, dz) G_B(z, y + A) \leqslant \varepsilon, \qquad y \in D'.$$

There is a compact set $K \supseteq K_1$ such that

$$\int_{K'} P(x, dz) G_B(z, y + A) \leqslant \varepsilon, \qquad y \in D.$$

The set K is the desired set. The proof in the type II case is similar.

LEMMA 23.7. — *Let* $B \in \mathcal{B}$ *have non-empty interior and let* t *be such that* S_t *generates* $\mathfrak{G}$. *Then for any* $x \in \mathfrak{G}$, $A \in \mathcal{B}$ *and* $\varepsilon > 0$ *there is a compact set* K *such that*

$$\int_{K'} P^t(x, dz) G_B(z, y + A) \leqslant \varepsilon, \qquad y \in \mathfrak{G}.$$

Proof. — This result follows by applying Lemma 23.6 to the random walk obtained by looking at the process at integer multiples of t and using Proposition 23.2.

LEMMA 23.8. — *The conclusion of Lemma* 23.7 *holds if* B *is merely assumed to be in* $\mathcal{B}_3$.

Proof. — Let B_1 be a compact set containing B and having a non-empty interior. Then

$$\begin{aligned} G_B(x, y + A) - G_{B_1}(x, y + A) &= \int_{B_1} H_{B_1}(x, dz) G_B(z, y + A) \\ &\leqslant \sup_{z \in B_1} \sup_{y \in G} G_B(z, y + A) < \infty \end{aligned}$$

and the desired result now follows from Lemma 23.7.

LEMMA 23.9. — *Let* $B \in \mathcal{B}_3$ *and let* t *be such that* S_t *generates* $\mathfrak{G}$. *Then for any* $A \in \mathcal{B}$, $C \in \mathcal{B}$ *and* $\varepsilon > 0$ *there is a compact set* K *such that*

$$\int_C dx \int_{K'} P^t(x, dz) G_B(z, y + A) \leqslant \varepsilon, \qquad y \in \mathfrak{G}.$$

Proof. — This follows by essentially the same arguments as led to Lemma 23.8 except that in the proof of the proper analog of Lemma 23.6 (where we integrate on $x \in C$) we use the fact that

$$\int_{C} L_B(x)\,dx = \int_{C} dx \int_{\mathfrak{G}} P(x, dz)\mathcal{L}_B(z).$$

LEMMA 23.10. — *Let* $B \in \mathcal{B}_2$ *and let* t *be such that* S_t *generates* $\mathfrak{G}$. *Then for any* $A \in \mathcal{B}$, $C \in \mathcal{B}$, *and* $\varepsilon > 0$ *there is a compact set* K *such that*

$$\int_{C} dx \int_{K'} P^t(x, dz) G_B(z, y + A) \leqslant \varepsilon, \qquad y \in \mathfrak{G}.$$

Proof. — Let B_1 be a compact set such that $\overline{B} \subseteq \mathring{B}_1$. There is a compact set D having positive measure and such that $B - D \subseteq B_1$. It follows easily from Lemma 23.9 that there is a compact set K_1 such that

$$\frac{1}{|D|}\int_{D} du \int_{C} dx \int_{K_1'} P^t(x, dz) G_{u+B_1}(z, y + A) \leqslant \frac{\varepsilon}{2}, \qquad u \in \mathfrak{G}.$$

Let K_2 be a compact set such that $K_1' - C \supseteq K_2'$. Then

$$\frac{1}{|D|}\int_{K_2'} P^t(0, dz) \int_{D} du \int_{z+C} dx\, G_{u+B_1}(x, y + A) \leqslant \frac{\varepsilon}{2}, \qquad y \in \mathfrak{G}.$$

By Proposition 19.3

$$\begin{aligned} 0 \leqslant \int_{z+C} dx\, G_B(x, y + A) &- \frac{1}{|D|}\int_{D} du \int_{z+C} dx\, G_{u+B_1}(x, y + A) \\ &= \frac{1}{|D|}\int du \int_{z+C} dx \int H_{u+B_1}(x, dv) G_B(v, y + A) \\ &\leqslant \frac{|C - D|}{|D|}\int_{B_1+D} dv\, G_B(v, y + A), \end{aligned}$$

which by Theorem 18.5, is bounded uniformly in y and z. Consequently there is a compact set K_3 such that

$$\int_{K_3'} P^t(0, dz) \int_{z+C} dx\, G_B(x, y + A) \leqslant \varepsilon, \qquad y \in \mathfrak{G}.$$

Let K be a compact set such that $K' - C \subseteq K_3'$. Then

$$\begin{aligned} &\int_{C} dx \int_{K'} P^t(x, dz) G_B(z, y + A) \\ &\qquad \leqslant \int_{K_3'} P^t(0, dz) \int_{z+C} G_B(x, y + A) \leqslant \varepsilon, \quad y \in \mathfrak{G}, \end{aligned}$$

as desired.

Proof of Theorem 23.2. — Let $B \in \mathscr{B}_3$ and suppose the process is type I. Then

$$\int Q_B^t(x, dz) G_B(z, y + A) = G_B(x, y + A) - \int_0^t Q_B^s(x, y + A)\, ds.$$

Let $A \in \mathscr{B}$ with $|A| > 0$ and $|\partial A| = 0$. It follows that

$$\lim_{y \to \infty} \int Q_B^t(x, dz) G_B(z, y + A) = \lim_{y \to \infty} G_B(x, y + A) = L_B(x)|A|.$$

If K is compact then

$$\lim_{y \to \infty} \int_{K'} Q_B^t(x, dz) G_B(z, y + A) = |A| \int_K Q_B^t(x, dz) L_B(z).$$

Since $Q_B^t(x, dz) \leqslant P^t(x, dz)$ it follows from Lemma 23.8 that if S_t generates $\mathfrak{G}$, then

$$L_B(x) = \int Q_B^t(x, dz) L_B(z).$$

Now S_t generates $\mathfrak{G}$ except for countably many values of t. Since $Q_B^{s+t} = Q_B^s Q_B^t$ it follows that L_B is Q_B^t invariant for all $t > 0$.

The proof in the type II case is similar. The proof for $B \in \mathscr{B}_2$ is also similar except that Lemma 23.10 is used instead of Lemma 23.8. We get directly that if S_t generates $\mathfrak{G}$, then

$$\int_C dx\, (L_B(x) - Q_B^t L_B(x))\, dx = 0, \quad C \in \mathscr{B}.$$

It follows that $L_B(x) = Q_B^t L_B(x)$ a.e. $x \in \mathfrak{G}$. Again the extension to all $t > 0$ follows from the semi-group property of Q_B^t.

Now that we know that there are locally bounded Q_B^t invariant functions that are bounded from below we shall investigate the uniqueness of such functions. In general the best we can hope to do is show that in say the type I case multiples of L_B are the essentially unique such functions. Since

$$f + M \geqslant 0 \quad \text{for some} \quad 0 \leqslant M < \infty, Q_B^t(f + M) = f + Q_B^t M \downarrow f$$

a.e. and thus every Q_B^t invariant function that is bounded from below must in fact be $\geqslant 0$ a.e. Thus with no loss in generality we can assume that f is such a function.

THEOREM 23.3. — *Let* $B \in \mathfrak{B}_4$ *and let* $h \geqslant 0$ *a.e. be locally integrable and such that* $Q_B^t h = h$ *a.e. for* $t > 0$. *Then in the type* I *case* $h = CL_B$ *a.e. and in the type* II *case*

$$h = C_1 L_B^+ + C_2 L_B^- \quad a.e.$$

Proof. — Consider a type I recurrent process. Let $K_1 \subset \mathring{K}_2 \subset K_2 \subset \cdots$ be compacts, $\bigcup_n K_n = \mathfrak{G}$. Define h_n by $h_n = \min(h, nG(., K_n))$ and set

$$\delta_n = (h_n - Q_B^1 h_n). \tag{23.6}$$

Then

$$Q_B \delta_n = Q_B^s h_n - Q_B^{s+1} h_n$$

so

$$\int_0^t Q_B^s \delta_n \, ds = \int_0^1 Q_B^s h_n \, ds - \int_t^{t+1} Q_B^s h_n \, ds. \tag{23.7}$$

But

$$Q_B^s h_n \leqslant n Q_B^s G_B 1_{K_n} = n \int_s^\infty Q_B^t 1_{K_n} \, dt.$$

Hence if $t \leqslant s \leqslant t+1$,

$$Q_B^s h_n \leqslant n \int_t^\infty Q_B^s 1_{K_n} \, ds \downarrow 0, \qquad t \to \infty.$$

Thus

$$\lim_{t\to\infty} \int_t^{t+1} Q_B^s h_n \, ds = 0. \tag{23.8}$$

Hence from (23.7) we see that

$$G_B \delta_n = \int_0^1 Q_B^s h_n \, ds. \tag{23.9}$$

Since $h_n \uparrow h$ and $Q_B^s h = h$ a.e. we see that

$$G_B \delta_n \uparrow \int_0^1 Q_B^s h \, ds = h, \quad \text{a.e.} \tag{23.10}$$

Let K be compact. Then from (24.12)

$$\int_K \delta_n(x) \, dx = \int_K (h_n(x) - Q_B^1 h_n(x) \, dx.$$

Thus

$$\lim_{n\to\infty} \int_K \delta_n(x) \, dx = \int_K h - \int_K Q_B^1 h = 0. \tag{23.11}$$

Now let $\varphi \in C_c^+$ be such that $(L_B, \varphi) > 0$. Since $\tilde{G}_B\varphi \to (L_B, \varphi)$ there is a compact A such that $\tilde{G}_B\varphi(x) \geqslant \delta > 0$, $x \notin A$. Define measures $\gamma_n(dy)$ by

$$\gamma_n(dy) = \tilde{G}_B\varphi(y)\delta_n(y)\,dy.$$

Then

$$\gamma_n(\mathfrak{G}) = (\tilde{G}_B\varphi, \delta_n) = (\varphi, G_B\delta_n) \leqslant (\varphi, h) < \infty.$$

Thus for any $f \in C_c^+$,

$$(f, G_B\delta_n) = (\delta_n, \tilde{G}_B f) = \int_A \tilde{G}_B f(x)\delta_n(x)\,dx + \int_{A'} \frac{\tilde{G}_B f(x)}{\tilde{G}_B\varphi(x)}\gamma_n(dx).$$

By (23.10), (23.11) and the fact that $\tilde{G}_B f$ is bounded on A we see that

$$(f, h) = \lim_{n\to\infty} \int_{A'} \frac{\tilde{G}_B f(x)}{\tilde{G}_B\varphi(x)}\gamma_n(dx). \tag{23.13}$$

By (23.11) for any compact set K, $\gamma_n(K) \to 0$. Also $\gamma_n(\mathfrak{G}) = (\varphi, G_B\delta_n) \to (h, \psi)$ so we that for any compact K, $\gamma_n(K') \to (\varphi, h)$. Let $\varepsilon > 0$ be given. Then there is a compact set $K \supset A$ such that

$$\left|\frac{\tilde{G}_B\varphi(x)}{\tilde{G}_B f(x)} - \frac{\tilde{G}_B\varphi(\infty)}{\tilde{G}_B f(\infty)}\right| < \varepsilon, \qquad x \notin K.$$

Thus from (23.13) we see that

$$\left|(f, h) - \frac{\tilde{G}_B f(\infty)}{\tilde{G}_B\varphi(\infty)}(\varphi, h)\right| \leqslant \varepsilon(\varphi, h).$$

Thus as ε is arbitrary

$$(f, h) = (L_B, f)\frac{(\varphi, h)}{(L_B, \varphi)}. \tag{23.14}$$

Since (23.14) is true for all $f \in C_c^+$ it follows that

$$h = \frac{(\varphi, h)}{(L_B, \varphi)} L_B \quad \text{a.e.}$$

Consider now a type II recurrent process. We know that $L_B^\pm \geqslant 0$. Suppose $L_B^+ = 0$ a.e. Then

$$L_B(x) = \sigma^{-2}\int_B H_B(x, dz)\psi(z - x) \quad \text{a.e.}$$

and thus

$$L_B(x) + L_B(-x) = \sigma^{-2} \int_B [H_B(x, dz) + H_B(-x, dz)]\psi(z) \quad \text{a.e.}$$

However, the left hand side tends to ∞ as $x \to \infty$ while the right hand side is bounded. Thus $L_B^+ = 0$ a.e. is impossible so $L_B^+ > 0$ on a set of positive measure. A similar argument shows that $L_B^- > 0$ on a set of positive measure. Using Urysohn's lemma we may find $\varphi \in C_c^+$ such that $(L_B^\pm, \varphi) > 0$. Since $\lim_{x \to \pm\infty} \tilde{G}_B\varphi(x) = (L_B^\pm, \varphi)$ we see that there is a compact set A such that $\tilde{G}_B\varphi(x) \geqslant \delta > 0$ for $x \notin A$. Let γ_n be as before. From (23.18) we see that there is a subsequence $n_j \to \infty$ such that $\gamma_{n_j}(\mathfrak{G}^+) \to \alpha^+$, $\gamma_{n_j}(\mathfrak{G}^-) \to \alpha^-$ where $\alpha^+ + \alpha^- = (h, \varphi)$. Then as $\gamma_n(K \cap \mathfrak{G}^\pm) \leqslant \gamma_n(K) \to 0$ for any compact K we see that

$$\lim_{n_j \to \infty} \gamma_{n_j}(\mathfrak{G}^\pm \cap K') = \alpha^\pm. \tag{23.15}$$

We can choose K compact such that for $x \notin K \cap \mathfrak{G}^+$

$$\left|\frac{\tilde{G}_B f(x)}{\tilde{G}_B\varphi(x)} - \frac{\tilde{G}_B f(+\infty)}{\tilde{G}_B\varphi(+\infty)}\right| < \varepsilon \tag{23.16}$$

and for $x \notin K \cap \mathfrak{G}^-$

$$\left|\frac{\tilde{G}_B f(x)}{\tilde{G}_B\varphi(x)} - \frac{\tilde{G}_B f(-\infty)}{\tilde{G}_B\varphi(-\infty)}\right| < \varepsilon. \tag{23.17}$$

It now follows from (23.13) and (23.15)-(23.17) that

$$\begin{aligned}(h, f) &= \alpha_1 \frac{\tilde{G}_B f(+\infty)}{\tilde{G}_B\varphi(+\infty)} + \alpha_2 \frac{\tilde{G}_B f(-\infty)}{\tilde{G}_B\varphi(-\infty)} \\ &= \alpha_1 \frac{(L_B^+, f)}{(L_B, \varphi)} + \alpha_2 \frac{(L_B^-, f)}{(L_B^-, \varphi)}.\end{aligned}$$

Since this is true for all $f \in C_c^+$ and α_1 and α_2 are independent of f we see that

$$h = \frac{\alpha_1}{(L_B^+, \varphi)} L_B^+, + \frac{\alpha_2}{(L_B^-, \varphi)} L_B^- \quad \text{a.e.}$$

We can extend these uniqueness results to sets in $\mathcal{B}_3$ as follows.

THEOREM 23.4. — *Let* $B \in \mathcal{B}_3$. *Then in the type* I *recurrent case the only* Q_B^t *invariant functions that are locally bounded and bounded from below coincide with* CL_B *a.e. In the type* II *recurrent case the only such functions coincide a.e. with* $C_1L_B^+ + C_2L_B^-$.

Proof. — Since $B \in \mathcal{B}_3$, $P_x(T_B = \infty) \equiv 0$. Now for some M $Mf + M > 0$ and so $Q_B^t(f + M) = f + Q_B^t M \downarrow f$, $t \to \infty$. Thus $f = Q_B^\infty(f + M) \geqslant 0$. Choose $A \in \mathcal{B}_4$, $A \supset B$. Then

$$f(x) = Q_B^t f(x) = Q_A^t f(x) + \int_0^t \int_{\bar{A}} P_x(T_B \in ds, X_{T_A} \in dy) Q_B^{t-s} f(y)$$
$$= Q_A^t f(x) + E_x[f(X_{T_A}); T_A \leqslant t].$$

It follows that $\lim_{t \to \infty} Q_A^t f = Q_A^\infty f$ exists and is a Q_A^t invariant function (by dominated convergence since $Q_A^t f \downarrow$) and that

$$f = Q_A^\infty f + H_A f.$$

Hence by Theorem 23.3 in the type I case $Q_A^\infty f = Cl_A$ a.e. and in the type II case $Q_A^\infty f = C_1 L_A^+ + C_2 L_A^-$ a.e. It is easily checked that

$$L_B = L_A + H_A L_B$$

and in the type II case that

$$L_B^\pm = L_A^\pm + H_A L_B^\pm.$$

Consequently in the type I case

$$f - CL_B = H_A(f - CL_B) \quad \text{a.e.}$$

Thus $f - CL_B$ is essentially bounded and Q_B^t invariant. Since $B \in \mathcal{B}_3$ the only such function is essentially 0. Thus $f = CL_B$ a.e. A similar argument shows that in the type II case $f = C_1 L_B^+ + CL_B^-$ a.e. This establishes the theorem.

24. Poisson Type Equations.

Let B be a closed set. The process $X_t(B) = X_{t \wedge T_B}$ is process X_t stopped when it hits B. Its transition operator ${}_BP^t$ is given by

$$ {}_BP^t f(x) = Q_B^t f(x) + E_x[f(X_{T_B}); T_B \leqslant t].$$

An easy computation shows that for any measurable function f bounded from below ${}_{\mathrm{B}}\mathrm{P}^{t+s}f = {}_{\mathrm{B}}\mathrm{P}^t({}_{\mathrm{B}}\mathrm{P}^s f)$ so ${}_{\mathrm{B}}\mathrm{P}^t$ has the semi-group property.

Proposition 24.1. — *The function* $f = h + \mathrm{H_B}\varphi$ *where* φ *is bounded from below and* h *is bounded from below and* $\mathrm{Q}^t_{\mathrm{B}}$ *invariant is bounded from below and* ${}_{\mathrm{B}}\mathrm{P}^t$ *invariant. Conversely every such* ${}_{\mathrm{B}}\mathrm{P}^t$ *invariant function is of this form.*

Proof. — Suppose h is $\mathrm{Q}^t_{\mathrm{B}}$ invariant. Since

$$\mathrm{P}_x(\mathrm{X}_{\mathrm{T_B}} \in \mathrm{B} \mid \mathrm{T_B} < \infty) = 1 \quad \text{and} \quad h = 0$$

on B we see that h is ${}_{\mathrm{B}}\mathrm{P}^t$ invariant. Also

$$\begin{aligned} {}_{\mathrm{B}}\mathrm{P}^t\mathrm{H_B}\varphi(x) &= \mathrm{Q}^t_{\mathrm{B}}\mathrm{H_B}\varphi(x) + \mathrm{E}_x[\mathrm{H_B}\varphi(\mathrm{X}_{\mathrm{T_B}});\ \mathrm{T_B} \leqslant t] \\ &= \mathrm{E}_x[\varphi(\mathrm{X}_{\mathrm{T_B}});\ t < \mathrm{T_B} < \infty] + \mathrm{E}_x[\varphi(\mathrm{X}_{\mathrm{T_B}});\ \mathrm{T_B} \leqslant t] \\ &= \mathrm{H_B}\varphi(x). \end{aligned}$$

Suppose now that $f \geqslant \mathrm{M} > -\infty$ and ${}_{\mathrm{B}}\mathrm{P}^t$ invariant. Then as $f - \mathrm{M} \geqslant 0$ and

$${}_{\mathrm{B}}\mathrm{P}^t(f - \mathrm{M}) = {}_{\mathrm{B}}\mathrm{P}^t f - \mathrm{M}\mathrm{P}_x(\mathrm{T_B} > t) - \mathrm{M}\mathrm{P}_x(\mathrm{T_B} \leqslant t) = f - \mathrm{M}$$

it suffices to consider $f \geqslant 0$. But then

$$f(x) = \mathrm{Q}^t_{\mathrm{B}}f(x) + \mathrm{E}_x[f(\mathrm{X}_{\mathrm{T_B}});\ \mathrm{T_B} \leqslant t]$$

and $\mathrm{E}_x[f(\mathrm{X}_{\mathrm{T_B}});\ \mathrm{T_B} \leqslant t] \uparrow \mathrm{H_B}f(x)$. Consequently $\mathrm{Q}^t_{\mathrm{B}}f \downarrow \mathrm{Q}^\infty_{\mathrm{B}}f$ and dominated convergence shows that $\mathrm{Q}^\infty_{\mathrm{B}}f$ is $\mathrm{Q}^t_{\mathrm{B}}$ invariant. This establishes the proposition.

Functions invariant for ${}_{\mathrm{B}}\mathrm{P}^t$ play the role of functions harmonic on B′ as we shall now explain.

Define the operator Δ_{B} on the measurable functions as follows. The domain $\mathrm{D}(\Delta_{\mathrm{B}})$ of Δ_{B} consists in all measurable functions f such that

$$\sup_{x \in \mathfrak{G}} \sup_{0 < t \leqslant 1} \left| \frac{{}_{\mathrm{B}}\mathrm{P}^t f(x) - f(x)}{t} \right| = \mathrm{M} < \infty \tag{24.1}$$

and

$$\lim_{t \downarrow 0} \frac{{}_{\mathrm{B}}\mathrm{P}^t f(x) - f(x)}{t}$$

exists. For $f \in \mathrm{D}(\Delta_{\mathrm{B}})$

$$\Delta_{\mathrm{B}} f(x) = \lim_{t \downarrow 0} \frac{{}_{\mathrm{B}}\mathrm{P}^t f(x) - f(x)}{t}. \tag{24.2}$$

When restricted to the bounded Borel functions Δ_B is just the weak infinitesimal generator of the semi-group ${}_BP^t$ on the space of bounded Borel Functions. For our purposes however we will need to apply Δ_B to measurable functions that are just bounded from below. Henceforth in this section all measurable functions f will be assumed to be bounded from below. It follows from (24.1) and the semi-group property of ${}_BP^tf$ that

$$|{}_BP^tf(x) - f(x)| \leqslant Mt \tag{24.3}$$

so

$$|{}_BP^tf| \leqslant |f| + Mt.$$

Thus the Laplace transform ${}_BR^\lambda$ of ${}_BP^tf$ is well defined.

Proposition 24.2. — *If f is bounded from below and $f \in D(\Delta_B)$ then ${}_BP^tf$ is continuous in t for $t > 0$ and right continuous at* 0. *Also*

$$\lim_{h \downarrow 0} \frac{{}_BP^{t+h}f - {}_BP^t}{h} = P^t\Delta_B f, \tag{24.4}$$

$${}_BP^tf - f = \int_0^t P^s\Delta_B f\, ds, \tag{24.5}$$

and

$${}_BR^\lambda(\lambda - \Delta_B)f = f. \tag{24.6}$$

Proof. — It follows from (24.1) that for $t \geqslant 0$ and $0 < h \leqslant 1$

$$|{}_BP^{t+h}f - {}_BP^tf| \leqslant Mh$$

so ${}_BP^tf$ is right continuous for $t \geqslant 0$. Moreover for $t > 0$ and $0 < h < 1$ and $t - h > 0$,

$$|{}_BP^{t-h}f - {}_BP^tf| \leqslant \int_{\mathfrak{G}} {}_BP^{t-h}(x, dy)|f(y) - P^hf(y)| \leqslant Mh$$

so ${}_BP^tf$ is left continuous for $t > 0$. Equation (24.4) follows at once by dominated convergence from (24.1) and (24.2). Now

$$\frac{1}{h}\int_0^t {}_BP^s({}_BP^hf - f)\, ds = \frac{1}{h}\int_t^{t+h} {}_BP^sf\, ds - \frac{1}{h}\int_0^h {}_BP^sf\, ds. \tag{24.7}$$

By dominated convergence the left hand side converges to $\int_0^t P^s\Delta_B f\ ds$ as $h \downarrow 0$. Using the continuity properties of ${}_BP^s f$ we see that the right hand side of (24.7) converges to ${}_BP^t f - f$ as $h \downarrow 0$. This establishes (24.5). Equation (24.6) follows by taking Laplace transforms.

COROLLARY 24.1. — *A function f bounded from below is ${}_BP^t$ invariant if and only if $\Delta_B f(x) = 0$ for all $x \in \mathfrak{G}$.*

Proof. — Immediate from (24.5).

If, in particular, we apply Proposition 24.2 to the closed set $B = \emptyset$ we obtain the following.

COROLLARY 24.2. — *Let $\Delta_\emptyset = \Delta$. If f is bounded from below and $f \in D(\Delta)$ then $P^t f$ is continuous for $t > 0$ and right continuous at $t = 0$. Also*

$$\lim_{h \downarrow 0} \frac{P^{t+h}f - P^t f}{h} = P^t\,\Delta f, \tag{24.8}$$

$$P^t f - f = \int_0^t P^s\,\Delta f\,ds \tag{24.9}$$

and

$$G^\lambda(\lambda - \Delta)f = f. \tag{24.10}$$

In general a function $f \in D(\Delta_B)$ need not be in $D(\Delta)$. However if $f \in D(\Delta)$ then we have the following.

PROPOSITION 24.3. — *Let $f \in D(\Delta)$ and let B be a closed set. If $\Delta f(x) = 0$ for $x \in B'$ then $f \in D(\Delta_B)$ and $\Delta_B f = 0$.*

Proof. — Set $\varphi = \Delta f$. Then as $\varphi = 0$ on B' we see that

$$G^\lambda\varphi = H_B^\lambda G^\lambda\varphi$$

and as $f \in D(\Delta)$ we see by (24.10) that

$$f = G^\lambda(\lambda - \Delta)f = \lambda G^\lambda f - G^\lambda\varphi. \tag{24.11}$$

Also

$$G^\lambda f = G_B^\lambda f + H_B^\lambda G^\lambda f$$

Hence

$$\begin{aligned} f + G^\lambda\varphi &= \lambda[G_B^\lambda f + H_B^\lambda G^\lambda f] \\ &= \lambda G_B^\lambda f + H_B^\lambda(f + G^\lambda\varphi) = \lambda G_B^\lambda f + H_B^\lambda f + G^\lambda\varphi. \end{aligned}$$

Thus

$$f = \lambda G^{\lambda}_{B}f + H^{\lambda}_{B}f = {}_{B}R^{\lambda}f.$$

Hence a.e. t,

$$f = {}_{B}P^{t}f. \tag{24.12}$$

Since ${}_{B}P^{t+s}f = {}_{B}P^{t}\,{}_{B}P^{s}f$ (24.12) must hold for all $t > 0$. Indeed given any $t > 0$ we can find t_1, t_2 such that $t = t_1 + t_2$ and ${}_{B}P^{t_1}f = f1$, ${}_{B}P^{t_2}f = f$, so the conclusion follows by the semi-group property.

Let φ be a measurable function. A function $f \in D(\Delta)$ solves Poisson's equation for φ if $\Delta f = \varphi$. We shall investigate the solutions of Poisson's equation for $\varphi \in C_c$ that are bounded from below and continuous.

Proposition 24.4. — *Suppose the process X_t is transient and let $\varphi \in C_c$. The function $G\varphi$ is a bounded continuous solution of $\Delta f = -\varphi$. The only bounded continuous solutions of $\Delta f = -\varphi$ are $f = G\varphi + \alpha$ for α a constant. Moreover every continuous solution bounded from below is given by $f = h + G\varphi$ where h is continuous, bounded from below and $\Delta h = 0$.*

Proof. — Since

$$P^{t}G\varphi = G\varphi - \int_0^t P^{s}\varphi\, ds$$

and $P^{s}\varphi \to \varphi$ uniformly as $s \to 0$ it follows that $G\varphi \in D(\Delta)$ and $\Delta G\varphi = -\varphi$. Suppose f is continuous, bounded from below, and $\Delta f = -\varphi$. Then $f - G\varphi$ is bounded from below, continuous and satisfies the equation $\Delta(f - G\varphi) = 0$. Hence by Corollary 24.1 (with $B = \varnothing$) $f - G\varphi$ is P^{t} invariant. If f is assumed to be bounded then $f - G\varphi$ is a continuous, bounded, P^{t} invariant function so $f - G\varphi \equiv \alpha$ for some constant α.

We now turn to consider Poissons equation for recurrent processes. The main difference with the transient case is that now there need be no solution in the general case and even in the non-singular case there is no solution bounded from below if $J(\varphi) < 0$.

Let us first show that potentials $A\varphi$ provide solutions in the non-singular case.

PROPOSITION 24.5. — *Let* X_t *be a non-singular recurrent processes and let* $\varphi \in C_c$. *Then* $\Delta A\varphi = \varphi$.

Proof. — By replacing φ with $-\varphi$ if necessary we can assume that $J(\varphi) \geqslant 0$. Then by Theorem 17.7 there is a constant M such that $A\varphi \geqslant -M$. Now an easy computation shows that

$$P^t A^\lambda \varphi = A^\lambda \varphi + \int_0^\infty e^{-\lambda s}[P^s\varphi - P^{t+s}\varphi]\, ds.$$

Since $P^t\varphi \to 0$ as $t \to \infty$ we see that

$$\lim_{\lambda \downarrow 0} P^t A^\lambda \varphi = A\varphi + \int_0^t P^s\varphi\, ds. \tag{24.13}$$

By Fatou's lemma

$$g(x) = \lim_{\lambda \downarrow 0} P^t A^\lambda \varphi(x) - P^t A\varphi(x) \geqslant 0$$

But then

$$\begin{aligned} g_y(x) = g(x-y) &= \lim_{\lambda \downarrow 0} P^t A^\lambda \varphi(x-y) - P^t A\varphi(x-y) \\ &= \lim_{\lambda \downarrow 0} P^t A^\lambda \varphi_y(x) - P^t A\varphi_y(x), \end{aligned}$$

so

$$g_y(x) - g(x) = \lim_{\lambda \downarrow 0} P^t A^\lambda(\varphi_y - \varphi)(x) - P^t A(\varphi_y - \varphi)(x).$$

By Theorem 17.5, for fixed y, $\sup\limits_{0<\lambda\leqslant 1} \sup\limits_{x \in \mathfrak{G}} |A^\lambda(\varphi_y - \varphi)(x)| < \infty$ and thus by bounded convergence $g_y(x) - g(x) = 0$. Hence $g(x) \equiv g_0 \geqslant 0$ for some constant g_0. From (24.13) we then see that

$$P^t A\varphi = A\varphi + \int_0^t P^s\varphi\, ds - g_0. \tag{24.14}$$

Thus

$$P^{(n+1)t} A\varphi = P^{nt} A\varphi + P^{nt}\int_0^t P^s\varphi\, ds - g_0. \tag{24.15}$$

Using (24.14) and (24.15) we easily get that

$$P^{(n+1)t} A\varphi = A\varphi - (n+1)g_0 + \sum_{j=0}^n P^{jt}\int_0^t P^s\varphi\, ds.$$

Since $A\varphi + M \geqslant 0$

$$(24.16)\quad 0 \leqslant \frac{P^{(n+1)t}[A\varphi + M]}{n+1} = \frac{A\varphi + M}{n+1} - g_0 + \frac{1}{n+1}\sum_{j=0}^{n} P^{jt}\int_0^t P^s\varphi\, ds.$$

Since $P^s\varphi \to 0$ as $s \to \infty$

$$\lim_{t\to\infty} \frac{1}{n+1}\sum_{j=0}^{n} P^{jt}\int_0^t P^s\varphi\, ds = 0$$

and so the right hand side of (24.16) converges to $-g_0$ as $n \to \infty$. Since the left hand side $\geqslant 0$ for all n we see that $-g_0 \geqslant 0$ so $g_0 \leqslant 0$. Thus $g_0 = 0$. Thus by (24.14)

$$P^tA\varphi - A\varphi = \int_0^t P^s\varphi\, ds$$

and as $\varphi \in C_c$ it follows that $A\varphi \in D(\Delta)$ and $\Delta A\varphi = \varphi$.

THEOREM 24.1. — *Suppose X_t is a recurrent non-singular process and let $\varphi \in C_c$. In order that the equation $\Delta f = \varphi$ have a continuous solution that is bounded from below it is necessary that $J(\varphi) \geqslant 0$. In that case for a type I process the only such solutions are $f = A\varphi + \beta$ for β a constant. For a type II process the only such solutions are*

$$f = A\varphi + (\alpha J(\varphi)/\sigma^2)\psi + \beta$$

for α a constant such that $|\alpha| \leqslant 1$.

Proof. — Suppose f is a continuous solution bounded from below. Let $B \in B_4$ be a compact set containing the support of φ. Then $\Delta f(x) = 0$ for $x \in B'$ so by Proposition 24.3 f is a ${}_BP^t$ invariant function. Consequently by Proposition 24.1 and Theorem 24.4 for a type I process $f = aL_B + H_Bf$ a.e. while for a type II process

$$f = aL_B^+ + bL_B^- + H_Bf$$

a.e. Using essentially the same argument used to establish Lemmas 10.8 and 10.9 in [7] it follows that $J(\varphi)$ must be $\geqslant 0$ and $f = A\varphi - \beta$ a.e. in the type I case and $f = A\varphi + (\alpha J(\varphi)/\sigma^2)\psi - \beta$ a.e. for $|\alpha| \leqslant 1$ in the type I case. Since f, $A\varphi$ and ψ are continuous the theorem is established.

We will next investigate the Poisson equation for the stopped process $X_B(t)$. Let $C_c(B')$ denote the continuous functions having compact support contained in B'.

PROPOSITION 24.6. — *Suppose X_t is a transient process. Let B be a closed set and let φ be a measurable function that is bounded on B and let $h \in C_c(B')$. Then for any constant α, the function $\psi(x) = \alpha P_x(T_B = \infty) + H_B\varphi + G_B h$ is a bounded solution of the equation system $\Delta_B f = -h$, $f(x) = \varphi(x)$, $x \in B$. In the non-singular case these are the only bounded solutions and in the general case every bounded solution coincides with one of these functions a.e. If B' is relatively compact the unique solution is $\psi = H_B\varphi + G_B h$. If B is compact then ψ is the unique solution having a limit at ∞ in the type* I *case and a limit at $+\infty$ in the type* II *case with $m > 0$.*

Proof. — Observe that $G_B h = \int_0^\infty Q_B^t h\, dt$. An easy computation shows that

$$Q_B^t G_B h = G_B h - \int_0^t Q_B^s h\, ds$$

and since for $x \notin B$,

$$\lim_{s \downarrow 0} Q_B^s h(x) = \lim_{s \downarrow 0} E_x[h(X_s);\, T_B > s] = h(x) P_x(T_B > 0) = h(x)$$

we see that

$$\lim_{t \downarrow 0} \frac{Q_B^t G_B h - G_B h}{t} = -h.$$

But then as $E_x[G_B h(X_{T_B});\, T_B \leqslant t] = 0$, $G_B h \in D(\Delta_B)$ and $\Delta_B G_B h = -h$. The function $\psi = \alpha P_x(T_B = \infty) + H_B\varphi$ is ${}_B P^t$ invariant and so $\Delta_B \psi = -h$. It is clear that $\psi = \varphi$ on B. Let f be any bounded solution. Then $f - G_B h$ is a bounded solution of $\Delta_B g = 0$, $g = \varphi$ on B. By Proposition 24.1 only bounded solutions of

$$\Delta_B g = 0 \qquad \text{are} \qquad g = g' + H_B g = g' + H_B \varphi;$$

where g' is a bounded Q_B^t invariant function. The remaining assertions now follow from Theorem 23.1 and Proposition 23.1.

Turning our attention to the recurrent case we have the following,

PROPOSITION 24.7. — *Suppose* X_t *is a recurrent process and let* $B \in \mathcal{B}_3$ *be compact. Then if* φ *is a bounded measurable function on* B *and* $h \in C_c$ *the function* $\psi = H_B\varphi + G_B h$ *is the unique bounded solution of* $\Delta_B f = -h$, $f = \varphi$ *on* B. *The function* ψ *is also the unique solution of this system if* B *is a closed set with such that* B' *is relatively compact.*

Proof. — The proof is similar to the previous proposition. We omit the details.

In general the functions $H_B\varphi$, $G_B h$ and $P_x(T_B = \infty)$ do not possess any continuity properties so our solutions to the Poisson equation $\Delta_B f = -h$, $h = \varphi$ on B do not have continuity properties in the ordinary sense. [In § 25 we will show that for strong Feller processes the above functions, for φ a continuous function, do have desirable continuity properties.] However we will now show that our solutions possess a certain stochastic continuity.

PROPOSITION 24.8. — *Let* B *be a closed set and let* φ *be a Borel function that is bounded on* B. *Then for any sequence* τ_n *of stopping times such that* $\tau_n \uparrow T_B$ *a.s.* P_x

$$\lim_{n\to\infty} H_B\varphi(X_{\tau_n}) = \varphi(X_{T_B}) \quad a.s. \quad P_x \quad on \quad [T_B < \infty]$$

and

$$P_{X_{\tau_n}}(T_B = \infty) = 1 \quad a.s. \quad P_x \quad on \quad [T_B = \infty].$$

Proof. — By quasi-left continuity $X_{\tau_n} \to X_{T_B}$ a.s. P_x on $[T_B < \infty]$. Let $\mathcal{F}_{\tau_n}$ be the σ field associated with the time τ_n [see [2], Chapter I for details], and let $\mathcal{F} = \sigma\left(\bigcup_n \mathcal{F}_{\tau_n}\right)$. Since

$$H_B\varphi(X_{\tau_n}) = E_{X_{\tau_n}}[\varphi(X_{T_B});\ T_B < \infty] = E_x[\varphi(X_{T_B});\ T_B < \infty|\mathcal{F}_{\tau_n}]$$

it follows by a well known result on conditional expectation that a.s. P_x,

$$\lim_{n\to\infty} H_B\varphi(X_{\tau_n}) = E_x[\varphi(X_{T_B});\ T_B < \infty|\mathcal{F}].$$

Since $X_{\tau_n} \to X_{T_B}$ a.s. P_x on $[T_B < \infty]$ and X_{τ_n} is $\mathcal{F}_{\tau_n}$ measurable we see that $\varphi(X_{T_B})$ is $\mathcal{F}$ measurable and $[T_B < \infty \in \mathcal{F}$. Hence a.s. P_x

$$E_x[\varphi(X_{T_B}); T_B < \infty | \mathcal{F}] = \varphi(X_{T_B}) \quad \text{on} \quad [T_B < \infty].$$

The proof in the second assertion in the proposition is similar.

PROPOSITION 24.9. — *Let* B *be a closed set such that* $\sup_x G_B(x, K) = M(K) < \infty$ *for each compact set* K. *Then for any sequence* τ_n *of stopping times such that* $\tau_n \uparrow T_B$ *a.s.* P *and for any* $h \in \Phi$

$$\lim_{n\to\infty} G_B h(X_{\tau_n}) = 0 \qquad a.s. \qquad P_x.$$

Proof. — Let τ_n and $\mathcal{F}_{\tau_n}$ be as in Proposition 24.8. Let $\varepsilon > 0$ be given. Then a.s. P_x

$$\varliminf_{n\to\infty} P_x(T_B \leqslant \tau_n + \varepsilon | \mathcal{F}_{\tau_n}) \geqslant \lim_{n\to\infty} P_x(T_B \leqslant \tau_k + \varepsilon | \mathcal{F}_{\tau_n}) = 1_{[T_B \leqslant \tau_k + \varepsilon]}$$

and thus a.s. P_x

$$\lim_{n\to\infty} P_x(T_B \leqslant \tau_n + \varepsilon | \mathcal{F}_{\tau_n}) = 1.$$

Let K be compact and contain the support of h. Then

$$\begin{aligned} E_{X_{\tau_n}}\left[\int_0^{T_B} |h(X_t)|\,dt\right] &\leqslant \|h\|\varepsilon + M(K)\|h\| P_{X_{\tau_n}}(T_B > \varepsilon) \\ &= \|h\| [\varepsilon + M(K) P_x(T_B > \tau_n + \varepsilon | \mathcal{F}_{\tau_n}). \end{aligned}$$

Hence a.s. P_x,

$$\varlimsup_{n\to\infty} |G_B h(X_{\tau_n})| \leqslant \|h\|\varepsilon$$

as desired.

So far we have extended to notion of a harmonic function on B′ from the point of view that such a function is one whose Laplacian is zero on B′. Classically such functions also can be characterized by means of an averaging property. The extension of this idea is as follows.

DEFINITION 24.1. — *Let* B *be a closed set. A universally measurable function* f *on* $\mathfrak{G}$ *is said to be harmonic on* B′ *if for any open set* U *having compact closure contained in* B′, $f(x) = E_x[f(X_{T_{U'}})]$, $x \in U$. *A harmonic function on* B′

is said to be stochastically regular if for every sequence τ_n *of stopping times such that* $\tau_n \uparrow T_B$ a.s. P_x, $f(X_{\tau_n}) \to f(X_{T_B})$ a.s. P_x on $[T_B < \infty]$ *and for some constant* α $f(X_{\tau_n}) \to \alpha$ a.s. P_x *on* $[T_B = \infty]$. *A harmonic function on* B' *is said to be regular if* $\lim_{x \to r} f(x) = \varphi(r)$ *for* $r \in B^r$ *and in the case of transient processes* $\lim_{x \to \infty} f(x) = f(\infty)$ *exists if the process is type* I *transient and* $\lim_{x \to +\infty} f(x) = f(+\infty)$ *exists if the process is type* II *transient with* $m > 0$, *or* $\lim_{x \to -\infty} f(x) = f(-\infty)$ *exists if the process is type* II *transient with* $m < 0$.

PROPOSITION 24.10. *Let* B *be a closed set and let* φ *be a Borel function that is bounded on* B. *Then every function* f *of the form* $f(x) = H_B\varphi(x) + \alpha P_x(T_B = \infty)$ *is a bounded stochastically regular harmonic function on* B' *and conversely every bounded stochastically regular harmonic function on* B' *is of this form.*

Proof. — Let U be an open set having compact closure contained in B'. Then $H_B = H_{U'}H_B$ and thus $H_B\varphi$ is a harmonic function on B'. In particular, for $\varphi = 1$ we see that $P_x(T_B < \infty)$ and thus also $P_x(T_B = \infty)$ are harmonic functions on B'. The regularity of these functions is the contents of Proposition 24.8. Now suppose f is a stochastically regular bounded harmonic function on B'. Let U_n be open $\overline{U}_n$ compact, $\overline{U}_n \subset B'$ and $U_n \uparrow B'$. Then $T_{U'_n} \uparrow T \leqslant T_B$ and by quasi-left continuity $X_{T_{U'_n}} \to X_T$ a.s. P_x on $[T < \infty]$. Since B is closed $X_T \in B$ and thus $T = T_B$ a.s. P_x on $[T \leqslant \infty]$. If $T = \infty$ then $T_B = \infty$ so $P_x(T = T_B) = 1$. Now

$$f(x) = E_x[f(X_{T_{U'_n}})] = E_x[f(X_{T_{U'_n}}); T_B < \infty] + E_x[f(X_{T_{U'_n}}); T_B = \infty]$$

and using the stochastic regularity of f and letting $n \to \infty$ we see that f is of the desired form.

The stochastic Dirichlet problem for a closed set B with boundary function φ is as follows. Given a Borel function φ that is bounded on B find a bounded stochastically regular harmonic function f such that $f = \varphi$ on B. It follows at once from Proposition 24.10 that the functions

$$f = H_B\varphi + \alpha P_x(T_B = \infty)$$

are the only solutions of this problem. From Proposition 24.6 we see that if X_t is a transient process that is non-singular the Dirichlet problem is equivalent to finding a bounded f such that $\Delta_B f = 0$, $f = \varphi$ on B. This is also true for a singular transient process if B is compact or B′ is relatively compact. For recurrent processes the two problems are equivalent whenever B is such that $P_x(T_B = \infty) \equiv 0$. In particular, for any i.d. process, if f is bounded, then $\Delta_B f = 0$, $f = \varphi$ on B if and only if f is a solution of the stochastic Dirichlet problem for B with boundary function φ whenever B is compact or B′ is compact.

The Dirichlet problem for a closed set B with boundary function φ is as follows. Given a Borel function that is continuous on B find a bounded regular harmonic function f such that $f = \varphi$ on B. In general this problem has no solution. For strong Feller i.d. process there are solutions for closed sets having the property that $P_x(X_{T_B} \in B^r | T_B < \infty) = 1$ for all $x \in B'$ and are such that $P_x(T_B = \infty)$ has the correct limiting behavior at ∞. The next result is typical of what can be proved. The proof uses continuity properties of $H_B\varphi(x)$ that will be established in the next section.

Proposition 24.11. — *Suppose X_t is a strong Feller process. Let B be a compact set such that $P_x(X_{T_B} \in B^r | T_B < \infty) = 1$ for all $x \in B'$. Let φ be a bounded function that is continuous at each point of B^r. Then every function of the form $f(x) = H_B\varphi(x) + \alpha P_x(T_B = \infty)$ is a solution to the Dirichlet problem with boundary function φ and conversely every solution of this problem is of this form.*

Proof. — If f is of the stated form then f is harmonic and the needed continuity properties follow from Theorem 25.1 and Proposition 3.6. Conversely if f is a solution of the Dirichlet problem, then using the continuity properties of f and the fact that $P_x(X_{T_B} \in B^r | T_B < \infty) = 1$ for all $x \in B^r$ it easily follows that f is a stochastically regular harmonic function on B′, and thus by Proposition 24.10 f must be of the stated form.

25. Continuity Properties of Strong Feller Processes.

If an i.d. process has the strong Feller property i.e. $P^t\varphi(x)$ is continuous in x whenever φ is bounded and measurable, then the various functions associated with the process have desirable continuity properties. These details will be spelled out in the following propositions. The strong Feller property holds whenever X_t has a density p_t for all t. Indeed, in this situation $P^t\varphi(x)$ is uniformly continuous because

$$|P^t\varphi(x + x_0) - P^t\varphi(x_0)| \leqslant \|\varphi\|_\infty \int |p(t, x + z) + p(t, z)|\, dz$$

and the desired continuity follows from the continuity of translation in the L_1 norm. Throughout this section we will assume that X_t is a strong Feller i.d. process.

Proposition 25.1. — *Let* B *be a Borel set and let* φ *be a bounded measurable function. Then for* $t > 0$ *the functions* $Q^t_B\varphi(x)$, $E_x[\varphi(X_t); T_B \leqslant t]$, $E_x[\varphi(X_{T_B}); T_B \leqslant t]$, and $H_B\varphi(x)$ *are continuous for* $x \in (\overline{B})'$.

Proof. — Let $s \in (0, t)$. Then

$$(25.1) \quad Q^t_B\varphi(x) = Q^s_B Q^{t-s}_B\varphi(x) = P^s Q^{t-s}_B\varphi(x) - E_x[Q^{t-s}_B\varphi(X_s); T_B \leqslant s].$$

The first term on the right is continuous by the strong Feller property. As to the second term we see that

$$|E_x[Q^{t-s}_B\varphi(X_s); T_B \leqslant s]| = \|\varphi\|_\infty P_x(T_B \leqslant s).$$

If $x_0 \in (\overline{B})'$ then there is a neighborhood U of 0 such that $U + x_0 \subset (\overline{B})'$. Let N be a neighborhood of 0 such that $N + N \subset U$. Given $\varepsilon > 0$ we can choose s such that $P_0(X_u \in N$ for all $u \leqslant s) \geqslant 1 - \varepsilon$, and thus for $x \in N$, $P_x(X_u \in U$ for all $u \leqslant s) \geqslant 1 - \varepsilon$. Consequently, for all $x \in N + x_0 P_x(T_B \leqslant s) < \varepsilon$. The continuity of $Q^t_B\varphi(x)$ for $x \in (\overline{B})'$ now follows from these facts. Since

$$E_x[\varphi(X_t); T_B \leqslant t] = P^t\varphi(x) - Q^t_B\varphi(x)$$

it follows that $E_x(\varphi(X_t); T_B \leqslant t)$ is continuous on $(\overline{B})'$.

The continuity of $H_B\varphi$ on $(\overline{B})'$ follows from the fact that $P^s H_B\varphi$ is continuous and the estimate

$$|P^s H_B\varphi(x) - H_B\varphi(x)| \leqslant \|\varphi\|_\infty P_x(T_B \leqslant s)$$

Since

$$E_x[\varphi(X_{T_B}); T_B \leqslant t] = H_B\varphi - Q_B^t H_B\varphi$$

we see that it too is continuous on $(\overline{B})'$.

Corollary 25.1. — *If the process is transient then* $G_B\varphi(x)$ *is continuous on* $(\overline{B})'$ *whenever* φ *is bounded with compact support.*

Proof. — This follows at once from Proposition 25.1, the fact that

$$G_B\varphi = G\varphi - H_B G\varphi,$$

and the fact that $G\varphi$ is continuous.

Recall that a point r is regular for B if $P_r(V_B = 0) = 1$.

Proposition 25.2. — *If* r *is a regular point of* B *and* X_t *is a strong Feller process then* $P_x(V_B \leqslant t)$ *and* $P_x(T_B \leqslant t)$ *are continuous at* r.

Proof. — Observe that

$$P_x(X_s \in B \quad \text{for some} \quad s \in (\tau, t]) = \int_{\mathfrak{S}} P^\tau(x, dy) P_y(T_B \leqslant t - \tau)$$

is continuous in x and thus as

$$P_x(X_s \in B \quad \text{for some} \quad s \in (\tau, t]) \uparrow P_x(V_B \leqslant t) \quad \text{as} \quad \tau \downarrow 0$$

we see that $P_x(V_B \leqslant t)$ is a lower-semi-continuous function. Thus

$$\varliminf_{x \to r} P_x(V_B \leqslant t) \geqslant P_r(V_B \leqslant t) = 1.$$

Since $P_x(V_B \leqslant t) \leqslant 1$ we see that

$$\lim_{x \to r} P_x(V_B \leqslant t) = 1.$$

Now $P_x(T_B \leqslant t) \geqslant P_x(V_B \leqslant t)$ and thus

$$\varliminf_{x \to r} P_x(T_B \leqslant t) = 1.$$

Also, $P_r(T_B \leqslant t) = 1$ because for $r \in B$ this is obviously true while for $r \notin B$, $P_r(T_B \leqslant t) = P_r(V_B \leqslant t)$. Thus $P_x(T_B \leqslant t)$ is continuous at r.

Corollary 25.2. — *If* r *is a regular point of* B *then for any* $t > 0$ *and any bounded measurable* φ,

$$Q_B^t\varphi(r) = E_r[\varphi(X_t); V_B > t] = 0$$

and $Q_B^t\varphi(x)$, $E_x[\varphi(X_t); V_B > t]$ *are continuous at* r. *Also*

$$E_r[\varphi(X_t); V_B \leqslant t] = E_r[\varphi(X_t); T_B \leqslant t] = P^t\varphi(r)$$

and $E_x[\varphi(X_t); V_B \leqslant t]$, $E_x[\varphi(X_t); T_B \leqslant t]$ *are continuous at* r.

Proof. — These follow at once from the facts that $P_r(T_B = 0) = P_r(V_B = 0) = 1$, Proposition 25.2, and the relations

$$P^t\varphi(x) = E_x[\varphi(X_t); T_B \leqslant t] + Q_B^t\varphi(x),$$
$$P^t\varphi(x) = E_x[\varphi(X_t); V_B \leqslant t] + E_x[\varphi(X_t); V_B > t].$$

Proposition 25.3. — *Let* r *be a regular point of* B. *Then*

$$\lim_{x \to r} H_B\varphi(x) = \lim_{x \to r} \Pi_B\varphi(x) = \varphi(r) \tag{25.2}$$

whenever r *is also a point of continuity of* φ. *In addition for such a point* r *for any* $t \geqslant 0$,

$$\lim_{x \to r} E_x[\varphi(X_{T_B}); T_B \leqslant t] = \lim_{x \to r} E_x[\varphi(X_t); V_B \leqslant t] = \varphi(r) \tag{25.3}$$

Proof. — We will prove the Proposition for $H_B\varphi$ and $E_x(\varphi(X_{T_B}); T_B \leqslant t]$. The same argument also yields the desired result for the other functions. Let U be any neighborhood of 0, and let $\varepsilon > 0$ be given. Then there is a neighborhood N of 0 and a time τ such that $P_x(X_t \in U$ for $t \leqslant \tau) > 1 - \varepsilon$ for $x \in N$. Hence for all $x \in N + r$, $P_x(T_B \leqslant \tau,\ X_{T_B} \notin U + r) < \varepsilon$. By Proposition 25.2 we can find a neighborhood $N_1 \subset N$ such that for $x \in N_1 + r$, $P_x(T_B \leqslant t) > 1 - \varepsilon$. Thus for $x \in N_1 + r$, $P_x(T_B \leqslant \tau$, $X_{T_B} \in U + r) > 1 - 2\varepsilon$. Hence

$$\lim_{x \to r} P_x(T_B \leqslant \tau,\ X_{T_B} \in U + r) = 1.$$

Now suppose φ is continuous at r. Then for some neighborhood $U + r$ of r, $|\varphi(x) - \varphi(r)| < \varepsilon$, $x \in r$. Thus

$$\begin{aligned}|H_B\varphi(x) - \varphi(r)| \leqslant \int_{\mathfrak{S}} H_B(x, dz)[\varphi(z) - \varphi(r)] \\ - \varphi(r)P_x(T_B = \infty) \leqslant \varepsilon\, H_B(x, U + r) \\ + 2\|\varphi\|P_x(X_{T_B} \notin U + r;\ T_B < \infty) \\ + \varphi(r)P_x(T_B = \infty) \leqslant \varepsilon + 4\|\varphi\|\varepsilon + \varphi(r)\varepsilon.\end{aligned}$$

Thus

$$\lim_{x \to r} H_B\varphi(x) = \varphi(r).$$

Since

$$E_x[\varphi(X_{T_B});\ T_B \leqslant t] = H_B\varphi(x) - Q_B^t\varphi(x)$$

it now follows from what has just geen proved and Corollary 25.2 that

$$\lim_{x \to r} E_x[\varphi(X_{T_B});\ T_B \leqslant t] = \varphi(r).$$

This establishes the theorem.

Corollary 25.3. — *If* φ *is bounded and with compact support and the process is transient then for* r *a regular point of* B,

$$\lim_{x \to r} G_B\varphi(x) = G_B\varphi(r) = 0.$$

Proof. — This follows at once from the fact that $G\varphi$ is continuous, the first passage relation and Proposition 25.3.

The preceding results can be summed up as follows.

Theorem 25.1. — *Let* φ *be a bounded measurable function and let* B *be a Borel set. Then the functions* $Q_B^t\varphi(x)$, *and* $E_x[\varphi(X_t);\ T_B \leqslant t]$ *are continuous on* $B^r \cup (\overline{B})'$. *If* φ *has compact support then* $G_B\varphi$ *is continuous on this set. Denote the continuity points of* φ *by* C_φ. *Then* $H_B\varphi$ *and* $E_x[\varphi(X_{T_B});\ T_B \leqslant t)$ *are continuous on* $(B^r \cap C_\varphi) \cup (\overline{B})'$.

For recurrent processes we have the following

Theorem 25.2. — *Let* $B \in \mathcal{B}_1$ *and let* $\varphi \in \Phi$. *Then* $A\varphi$ *is continuous and* $G_B\varphi$ *and* L_B *are continuous on* $B^r \cup (\overline{B})'$. *In the type* II *case* $L_B^{\pm}$ *are continuous on this set.*

Proof. — It is clear that $A^\lambda\varphi$ is continuous and since $A^\lambda\varphi \to A\varphi$ uniformly on compacts so is $A\varphi$. Likewise by Theorem 25.1, $G_B^\lambda\varphi$ is continuous on $B^r \cup (\overline{B})'$ and as $G_B^\lambda\varphi \to G_B\varphi$ uniformly on compacts $G_B\varphi$ is continuous on this set. By Proposition $H_BA\varphi$ is continuous on $B^r \cup (\overline{B})'$ and thus as

$$J(\varphi)L_B = A\varphi - H_BA\varphi + G_B\varphi$$

we see that L_B is also continuous on $B^r \cup (\overline{B})'$. Finally the fact that in the type II case $L_B^\pm$ are continuous on $B^r \cup (\overline{B})'$ follows from Theorem 25.1 and the fact that L_B is continuous on the set. This establishes the proposition.

From Theorems 24.1 and 24.2 we obtain the following

Corollary 25.4. — *Let* B *be a closed set, let* $h \in C_c(B')$ *and let* φ *be a measurable function that is bounded on* B. *Then the only bounded solutions of* $\Delta_B f = -h$, $f = \varphi$ *on* B *are* $f(x) = G_Bh(x) + H_B\varphi(x) + \alpha P_x(T_B = \infty)$ *for* α *a constant. These solutions are continuous on* $B' \cup (B^r) \cap C_\varphi)$ $(C_\varphi$ *is the set of continuity points of* $\varphi)$.

Using our results on Q_B^t invariant functions from § 24 plus Theorem 25.2 we obtain the following.

Corollary 25.5. — *Let* X_t *be a recurrent process and let* $B \in \mathcal{B}_1$. *Then in the type* I *case the only* Q_B^t *invariant functions* $f \geqslant 0$ *that are continuous on* $(\overline{B})'$ *coincide with* αL_B *on* $(\overline{B})'$. *In the type* II *case every such function coincides with* $\alpha_1 L_B^+ + \alpha_2 L_B^-$ *on* $(\overline{B})'$.

BIBLIOGRAPHY

[6] B. Belkin, Cornell University Thesis,

[2] R. Blumenthal and R. Getoor, Markov Processes and Potential Theory, *Academic Press*, N.Y. (1968).

[3] N. Dunford and J. T. Schwartz, Linear Operators I, *Academic Press*, N. Y. (1958).

[4] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis I, *Academic Press*, N. Y. (1963).

[5] G. A. Hunt, Markov Processes and Potentials I, *Illinois J. Math.*, 1, 294-319 (1956).

[6] G. A. HUNT, Markoff Chains and Martin Boundaries, *Illinois J. Math.* 4, 313-340 (1960).
[7] S. C. PORT and C. J. STONE, Potential Theory of Random Walks on Abelian Groups. *Acta Math.*, 122, 19-114 (1969).
[8] S. C. PORT and C. J. STONE, Hitting Times for Transient Random Walks, *J. Math. Mech.*, 17, 1117-1130 (1968).
[9] S. C. PORT and C. J. STONE, Hitting Times and Hitting Places for non-lattice Recurrent Random Walks, *J. Math. Mech.*, 17, 35-58, (1968).
[10] C. J. STONE, Ratio Limit Theorems for Random Walks on Groups, *Trans. Amer. Math. Soc.*, 125, 86-100 (1966).
[11] C. J. STONE, Applications of Unsmoothing and Fourier Analysis to Random Walks. Markov Processes and Potential Theory, Ed. by C. Chover. J. Wiley et Sons, N. Y. (1967).
[12] C. J. STONE, On Local and Ratio Limit Theorems. Proceedings of the Fifth Berkeley Symposium on Probability and Statistics. University of California Press, Berkeley, California (1968).
[13] C. J. STONE, Infinite Particle Systems and Multi-Dimensional Renewal Theory, *J. Math. Mech.*, 18, 201-228 (1968).

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