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## BEHAVIOR OF BIHARMONIC FUNCTIONS ON WIENER'S AND ROYDEN'S COMPACTIFICATIONS

by Y.K. KWON, L. SARIO, and B. WALSH

In the theory of bending of thin plates the biharmonic functions play an important role ; their local properties have been studied by several authors (cf. Bergman-Schiffer [1], Vekua [9], Garabedian [3]). The main purpose of the present paper is to establish some global properties of biharmonic functions in terms of Wiener's and Royden's compactifications of a smooth Riemannian manifold (see also Nakai-Sario [4], [5]). For notation and terminology we refer the reader to the monograph Sario-Nakai [7].

1. On a smooth Riemannian manifold  $R$  of dimension  $n \geq 2$ , consider the Laplace-Beltrami operator

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \right)$$

where  $x = (x^1, \dots, x^n)$  is the local coordinate,  $(g^{ij})$  the inverse matrix of the fundamental metric tensor  $(g_{ij})$ , and  $g$  the determinant of  $(g_{ij})$ .

A  $C^4$ -function  $u$  on  $R$  satisfying the equation

$$\Delta^2 u = \Delta \Delta u = 0$$

is called a *biharmonic function* on  $R$ . In view of a theorem of de Rham [6, p. 149], every biharmonic function is smooth. We denote by  $W(R)$  the family of all biharmonic functions on  $R$ .

As an example of a simple biharmonic function we give the function  $\nu$  of the following

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LEMMA. — *If the volume of R is finite and R is hyperbolic, then the function*

$$v(x) = \int_R g(x, y) dy$$

*is biharmonic on R. Here  $g(x, y)$  is the harmonic Green's function on R with singularity at  $y$ , and  $dy$  the volume element of R.*

*Proof.* — For an arbitrary  $x \in R$  choose a real  $\alpha = \alpha_x > 0$  so large that the set

$$A = \{y \in R \mid g(x, y) > \alpha\}$$

is relatively compact in R. If  $g_A(x, y)$  is the Green's function of A, then

$$0 \leq v(x) - \int_A g_A(x, y) dy = \int_A [g(x, y) - g_A(x, y)] dy + \int_{R-A} g(x, y) dy \leq \alpha \text{vol}(A) + \alpha \text{vol}(R - A) = \alpha \text{vol}(R) .$$

Since  $\int_A g_A(x, y) dy \leq \int_A g(x, y) dy < \infty$ ,  $v(x)$  is well-defined on R. In view of the fact that  $\Delta v = -1$  (see Theorem 3 below), we can draw the desired conclusion.

We remark in passing that the finiteness of the volume of R is not necessary for  $v(x)$  to be defined on R. In fact, take

$$R = \{x = (x^1, x^2) \mid |x|^2 = (x^1)^2 + (x^2)^2 < 1\}$$

with the metric tensor  $g_{ij} = (1 - r)^{-1} \delta_{ij}$ , where  $r = \sqrt{(x^1)^2 + (x^2)^2}$ . It is easy to see that  $\text{vol}(R) = \infty$  and

$$v(x) \leq \frac{1}{\varepsilon} \log \frac{\varepsilon + 2}{\varepsilon} + \frac{1}{2} \varepsilon \log(2 + \varepsilon) + 1 - \log \varepsilon < \infty$$

for any  $x \in R$  and  $0 < 2\varepsilon < 1 - |x|$ .

2. Throughout the following discussion we assume that the manifold R has finite volume. Let  $N(R)$  be the Wiener algebra, which consists of bounded continuous harmonizable functions on R, and  $N_\delta(R)$  the Wiener potential subalgebra, i.e. the subfamily of functions  $f \in N(R)$  whose harmonic projections  $\pi f$  on R vanish identically (cf. Sario-Nakai [7]).

**THEOREM.** — *Let  $v$  be an element of the Wiener potential subalgebra  $N_\delta(\mathbb{R})$  such that  $\Delta v$  is bounded. Then*

$$v(x) = - \int_{\mathbb{R}} g(y, x) \Delta v(y) dy$$

on  $\mathbb{R}$ .

*Proof.* — For a regular exhaustion  $\{R_m\}$  of  $\mathbb{R}$ , let  $g_m(y, x)$  be the Green's function on  $R_m$  and  $\{B_l\}$  a sequence of parametric balls about  $x \in R_1$  such that  $\bar{B}_l \subset R_1$  and the sequence  $\{B_l\}$  shrinks down to  $x$  as  $l \rightarrow \infty$ . Then the Green's formula yields

$$\begin{aligned} \int_{\partial(R_m - B_l)} v(y) * dg_m(y, x) - g_m(y, x) * dv(y) &= \\ &= - \int_{R_m - B_l} g_m(y, x) \Delta v(y) dy . \end{aligned}$$

On the other hand

$$\int_{\partial(R_m - B_l)} g_m(y, x) * dv(y) = - \int_{\partial B_l} g_m(y, x) * dv(y) \rightarrow 0$$

as  $l \rightarrow \infty$ ,

and

$$\int_{R_m - B_l} g_m(y, x) \Delta v(y) dy \rightarrow \int_{R_m} g_m(y, x) \Delta v(y) dy \text{ as } l \rightarrow \infty .$$

In view of  $\int_{\partial B_l} v(y) * dg_m(y, x) \rightarrow -v(x)$  as  $l \rightarrow \infty$ , we obtain

$$v(x) = - \int_{\partial R_m} v(y) * dg_m(y, x) - \int_{R_m} g_m(y, x) \Delta v(y) dy$$

for all  $x \in R_m$ .

Consider  $v_m \in H(R_m)$  such that  $v_m \equiv v$  on  $\mathbb{R} - R_m$ . Then we may assume that the sequence  $\{v_m\}$  converges to zero uniformly on compact subsets of  $\mathbb{R}$  (cf. Sario-Nakai [7]), and

$$v_m(x) = - \int_{\partial R_m} v(y) * dg_m(y, x)$$

on  $R_m$ . Thus Lebesgue's dominated convergence theorem yields

$$v(x) = - \int_{\mathbb{R}} g(y, x) \Delta v(y) dy$$

on  $\mathbb{R}$  as desired.

3. For the sake of completeness we include the proof of the following well-known theorem, which establishes a right inverse of the Laplace-Beltrami operator on bounded smooth functions :

THEOREM. — For any function  $f \in C^\infty(\mathbb{R}) \cap B(\mathbb{R})$  ,

$$\Delta_x \int_{\mathbb{R}} g(x, y) f(y) dy = -f(x)$$

for all  $x \in \mathbb{R}$ .

*Proof.* — Fix a point  $x_0 \in \mathbb{R}$ , and construct a function  $h$  defined on a neighborhood  $U$  of  $x_0$  in  $\mathbb{R}$  such that  $\Delta h = f$  on  $U$  (de Rham [6, p. 151]). Choose an open neighborhood  $V$  of  $x_0$  with  $\bar{V} \subset U$ , and a function  $\varphi \in C_0^\infty(\mathbb{R})$  with  $\varphi|_V \equiv 1$ ,  $\varphi|_{\mathbb{R} - U} \equiv 0$ , and  $0 \leq \varphi \leq 1$ . Then  $\Delta(h\varphi) = f$  on  $V$ , and  $h\varphi \in N_\delta(\mathbb{R})$  with its obvious extension to  $\mathbb{R}$ .

By Theorem 2 we have

$$(h\varphi)(x) = - \int_{\mathbb{R}} g(x, y) \Delta(h\varphi)(y) dy$$

on  $\mathbb{R}$ . In particular on  $V$ ,

$$\begin{aligned} h(x) &= (h\varphi)(x) = - \int_{\mathbb{R}} g(x, y) \Delta(h\varphi)(y) dy \\ &= - \int_{\mathbb{R}} g(x, y) [\Delta(h\varphi)(y) - f(y)] dy - \int_{\mathbb{R}} g(x, y) f(y) dy . \end{aligned}$$

Moreover

$$f(x_0) = (\Delta h)(x_0) = - [\Delta_x \int_{\mathbb{R}} g(x, y) f(y) dy]_{x=x_0}$$

as asserted, because the first integral on the right is harmonic on  $V$  (cf. Constantinescu-Cornea [2, p. 15]).

*Remark.* — The boundedness of  $\Delta v$  in Theorem 2 and of  $f$  in Theorem 3 was used only to assure the existence of their Green's potentials. Although these potentials do exist under milder conditions we do not intend to seek the most general statements.

4. For a bounded measurable function  $f$  on  $\mathbb{R}$ , set

$$(\Gamma f)(x) = - \int_{\mathbb{R}} g(x, y) f(y) dy$$

on  $R$ . It is easy to see that  $\Gamma f$  is harmonizable on  $R$ . If  $f$  belongs to the family  $B(R)$  of bounded continuous functions, then  $\Gamma f$  is continuous and therefore the operator

$$\Gamma : B(R) \rightarrow N(R)$$

is well-defined whenever  $\Gamma 1$  is bounded.

Set  $WBB_{\Delta}(R) = \{u \in W(R) \mid u, \Delta u \text{ are bounded}\}$ , and denote by  $HB(R)$  the class of bounded harmonic functions on  $R$ .

**THEOREM.** — *Let  $\Gamma 1$  be bounded on  $R$ . Then the decomposition*

$$WBB_{\Delta}(R) = HB(R) \oplus \Gamma HB(R)$$

*is valid.*

*Proof.* — In view of Theorem 3 it is easily seen that

$$HB(R) + \Gamma HB(R) \subseteq WBB_{\Delta}(R)$$

when  $\Gamma 1$  is bounded. Since every function in  $\Gamma HB(R)$  vanishes on the Wiener harmonic boundary  $\Delta_N$ , the maximum principle for HB-functions yields

$$HB(R) \cap \Gamma HB(R) = \{0\}.$$

Thus it remains to show that every  $u \in WBB_{\Delta}(R)$  has the desired decomposition.

Let  $\pi : N(R) \rightarrow HB(R)$  be the harmonic projection (cf. Sario-Nakai [7]). By Theorem 3, the function

$$u(x) - (\pi u)(x) - [\Gamma \Delta(u - \pi u)](x)$$

is a bounded harmonic function on  $R$ . Furthermore it vanishes on the Wiener harmonic boundary  $\Delta_N$  and therefore on  $R$ . Thus

$$u = \pi u + \Gamma \Delta u$$

on  $R$  as desired.

**COROLLARY.** — *Suppose  $\Gamma 1$  is bounded on  $R$ . Then for any  $m \geq 1$ ,  $\dim WBB_{\Delta}(R) = 2m$  if and only if the cardinality of the Wiener harmonic boundary  $\Delta_N$  of  $R$  is  $m$ .*

*Proof.* — It is known that the cardinality of  $\Delta_N$  is  $m$  if and only if  $\dim HB(R) = m$ .

Let  $\{u_1, \dots, u_m\}$  be a basis of the space  $HB(R)$ . In view of Theorems 3 and 4, it is seen that the set  $\{u_1, \dots, u_m; \Gamma u_1, \dots, \Gamma u_m\}$  forms a basis for the space  $WBB_\Delta(R)$ .

*Throughout the rest of our discussion we shall assume that the function  $\Gamma 1$  is bounded on  $R$ .*

5. Denote by  $NN_\Delta(R)$  the family of functions  $f$  on  $R$  with  $f, \Delta f \in N(R)$ , and by  $N_\delta N_{\delta\Delta}(R)$  the family of functions  $g$  on  $R$  with  $g, \Delta g \in N_\delta(R)$ .

**THEOREM.** — *The following biharmonic decomposition of the class  $NN_\Delta(R)$  is valid :*

$$NN_\Delta(R) = WBB_\Delta(R) \oplus N_\delta N_{\delta\Delta}(R) .$$

*Proof.* — Note that for any  $v \in N_\delta(R)$  with  $\Delta v$  bounded,

$$v(x) = - \int_R g(x, y) \Delta v(y) dy = (\Gamma \Delta v)(x)$$

on  $R$  (Theorem 2).

Let  $f \in NN_\Delta(R)$ . By the direct sum decomposition

$$N(R) = HB(R) \oplus N_\delta(R) ,$$

$f = u_1 + v_1$  for  $u_1 \in HB(R)$  and  $v_1 \in N_\delta(R)$ . Thus the above remark yields  $v_1 = \Gamma \Delta v_1$  on  $R$ . Since  $\Delta v_1 = \Delta f = u_2 + v_2$  for some  $u_2 \in HB(R)$  and  $v_2 \in N_\delta(R)$ ,  $v_1 = \Gamma u_2 + \Gamma v_2$  on  $R$  and therefore

$$f = (u_1 + \Gamma u_2) + \Gamma v_2$$

on  $R$ .

To show the uniqueness of the decomposition, suppose that  $u \in WBB_\Delta(R) \cap N_\delta N_{\delta\Delta}(R)$ . Since  $\Delta u \in HB(R) \cap N_\delta(R)$ ,  $\Delta u \equiv 0$  on  $R$  and therefore  $u \in HB(R) \cap N_\delta(R) = \{0\}$  as desired.

This completes the proof of the theorem.

6. We turn to the integral representation of the  $WBB_\Delta$ -functions on  $R$ .

Let  $P(x, t)$  be the harmonic kernel on  $R \times \Delta_N$  with  $P(x_0, t) \equiv 1$ , and  $\mu$  the harmonic measure on  $\Delta_N$  centered at the fixed point  $x_0 \in R$ .

As immediate consequences of Theorem 4 we state the following results.

**THEOREM.** — *Every  $u \in WBB_{\Delta}(\mathbb{R})$  has the integral representation*

$$u(x) = \int_{\Delta_N} P(x, t) f(t) d\mu(t) - \iint_{\mathbb{R} \times \Delta_N} g(x, y) P(y, t) \Delta u(t) d\mu(t) dy$$

on  $\mathbb{R}$ .

**THEOREM.** — *Let  $f$  and  $h$  be bounded  $\mu$ -measurable functions on  $\Delta_N$ . Then the function*

$$u(x) = \int_{\Delta_N} P(x, t) f(t) d\mu(t) - \iint_{\mathbb{R} \times \Delta_N} g(x, y) P(y, t) h(t) d\mu(t) dy$$

*is biharmonic on  $\mathbb{R}$ . If  $f$  and  $h$  are continuous at  $t_0 \in \Delta_N$ , then  $\lim_{x \in \mathbb{R}, x \rightarrow t_0} u(x) = f(t_0)$  and  $\lim_{x \in \mathbb{R}, x \rightarrow t_0} \Delta u(x) = h(t_0)$ .*

7. A function  $u \in WBB_{\Delta}(\mathbb{R})$  is called *WBB $_{\Delta}$ -minimal* on  $\mathbb{R}$  if  $u \neq 0$ ,  $u \geq 0$ ,  $\Delta u \leq 0$ , and for any  $v \in WBB_{\Delta}(\mathbb{R})$  with  $0 \leq v \leq u$ , there exists a constant  $c_v$  with  $v = c_v u$  on  $\mathbb{R}$ .

The *WBB $_{\Delta}$ -minimal* functions have the following characterization in terms of  $\Delta_N$ .

**THEOREM.** — *If  $u$  is WBB $_{\Delta}$ -minimal on  $\mathbb{R}$ , then there exists an isolated point  $t \in \Delta$  such that either  $u_1(x) = P(x, t) \mu(t)$ ,  $u_2 = 0$ , or  $u_1 = 0$ ,  $u_2(x) = P(x, t) \mu(t)$  where  $u_1 = \pi u$  and  $u_2 = -\Delta u$ .*

*Proof.* — By the proof of Theorem 4, we have  $u = u_1 - \Gamma u_2$  on  $\mathbb{R}$ . Since  $0 \leq -\Gamma u_2 \leq u$ , the *WBB $_{\Delta}$ -minimality* of  $u$  yields  $-\Gamma u_2 = c_1 u = c_1 u_1 - c_1 \Gamma u_2$  for some constant  $c_1$ . Since  $\Gamma u_2 = 0$  on  $\Delta_N$ ,  $c_1 u_1 = 0$  on  $\Delta_N$  and therefore on  $\mathbb{R}$  in view of the maximum principle for HB-functions. If  $c_1 = 0$ ,  $u_2 = 0$  and  $u_1$  is a HB-minimal function. If  $c_1 \neq 0$ , then  $u_1 = 0$  and  $-\Gamma u_2$  is *WBB $_{\Delta}$ -minimal*.

It remains to show that  $u_2$  is HB-minimal whenever  $-\Gamma u_2$  is *WBB $_{\Delta}$ -minimal*. Let  $w \in \text{HB}(\mathbb{R})$  be such that  $0 \leq w \leq u_2$ . Since  $-\Gamma$  is a positive operator, we have  $0 \leq -\Gamma w \leq -\Gamma u_2$  and therefore

$$\Gamma(w - cu_2) = 0$$



on  $R$  for some constant  $c$ . The assertion follows by virtue of Theorem 3.

8. Finally we turn to the study of biharmonic functions in connection with the Dirichlet integrals of these functions and their Laplacians.

First we establish :

THEOREM. — For a function  $f \in C^\infty(R) \cap B(R)$ ,

$$D(\Gamma f) = \iint_{R \times R} g(x, y) f(x) f(y) dx dy .$$

*Proof.* — Let  $\{R_m\}$  be a regular exhaustion of  $R$  and  $g_m(x, y)$  the Green's function for  $R_m$ . Define  $g_m(x, y) = 0$  on

$$(R - R_m) \times R_m \cup R_m \times (R - R_m) .$$

Set

$$v_m(x) = - \int_R g_m(x, y) f(y) dy .$$

Then  $v_m = 0$  on  $R - R_m$  and the Green's formula yields

$$0 = \int_{\partial R_m} v_m(x) * dv_m(x) = D_R(v_m) + \int_R v_m(x) \Delta v_m(x) dx .$$

Since  $\Delta v_m(x) = f(x)$  on  $R_m$ , we have

$$\begin{aligned} D_R(v_m) &= - \int_R v_m(x) f(x) dx \\ &= \iint_{R \times R} g_m(x, y) f(y) f(x) dx dy \leq \|f\|_\infty^2 \iint_{R \times R} g(x, y) dx dy . \end{aligned}$$

Therefore we may assume that the sequence  $\{D_R(v_m)\}_m$  converges. By Fatou's lemma and Lebesgue's convergence theorem we obtain

$$D_R(\Gamma f) \leq \varliminf_{m \rightarrow \infty} D_R(v_m) = \iint_{R \times R} g(x, y) f(x) f(y) dx dy < \infty .$$

On the other hand  $\Delta(\Gamma f - v_m) = 0$  on  $R_m$  and  $v_m = 0$  on  $\partial R_m$ . Let  $h_m \in H(R_m)$  be such that  $h_m|_{\partial R_m} = \Gamma f \in M_\delta(R)$ . Here  $M_\delta(R)$  is the Royden potential subalgebra which consists of limits of uniformly bounded functions in the Royden algebra  $M(R)$  with compact supports which converge uniformly in compact subsets and in the

Dirichlet norm. The sequence  $\{h_m\}$  converges to zero uniformly on compact subsets of  $R$  and  $D_R(h_m) \rightarrow 0$  as  $m \rightarrow \infty$  (cf. Sario-Schiffer-Glasner [8]). Since

$$D_R(\Gamma f - h_m) = D_R(v_m) = \iint_{R \times R} g_m(x, y) f(x) f(y) dx dy ,$$

we have

$$\lim_{m \rightarrow \infty} D_R(\Gamma f - h_m) = \iint_{R \times R} g(x, y) f(x) f(y) dx dy .$$

In view of

$$|\sqrt{D_R(\Gamma f)} - \sqrt{D_R(\Gamma f - h_m)}| \leq \sqrt{D_R(h_m)} \rightarrow 0 ,$$

we conclude that

$$D_R(\Gamma f) = \iint_{R \times R} g(x, y) f(x) f(y) dx dy .$$

9. Let  $WCC_\Delta(R)$  be the family of all biharmonic functions  $u \in WBB_\Delta(R)$  such that  $u$  and  $\Delta u$  are Dirichlet-finite.

By virtue of the above theorem we have a counterpart of Theorem 4 :

**THEOREM.** — *The decomposition*

$$WCC_\Delta(R) = HBD(R) \oplus \Gamma HBD(R)$$

*is valid.*

**COROLLARY.** — *For any  $m \geq 1$ ,  $\dim WCC_\Delta(R) = 2m$  if and only if the cardinality of the Royden harmonic boundary  $\Delta_M$  of  $R$  is  $m$ .*

Let  $MM_\Delta(R) = \{f \in M(R) \mid \Delta f \in M(R)\}$  and

$$M_\delta M_{\delta\Delta}(R) = \{g \in M_\delta(R) \mid \Delta g \in M_\delta(R)\} .$$

As in Theorem 5 we have :

**THEOREM.** —  $MM_\Delta(R) = WCC_\Delta(R) + M_\delta M_{\delta\Delta}(R)$  .

We remark that the integral representation of  $WCC_\Delta$ -functions along  $\Delta_M$  is also valid, and that a characterization of  $WCC_\Delta$ -minimal functions, similar to that in Theorem 7, can be given in terms of the Royden harmonic boundary.

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