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## ON A ABSTRACT STIELTJES MEASURE

by James E. HUNEYCUTT, Jr <sup>(2)</sup>

### 1. Introduction.

In 1955, A. Revuz [4] considered a type of Stieltjes measure defined on analogues of half-open, half-closed intervals in a partially ordered topological space. He states that these functions are finitely additive but his proof has an error. We shall furnish a new proof and extend some of his results to "measures" taking values in a topological abelian group.

### 2. Preliminaries.

If  $X$  is a set and  $\mathfrak{S}$  is a non-void collection of subsets of  $X$ , then  $\mathfrak{S}$  is called a semi-ring provided

$$\text{i) } A, B \in \mathfrak{S} \Rightarrow A \cap B \in \mathfrak{S},$$

$$\text{ii) } A, B \in \mathfrak{S}, A \subseteq B \Rightarrow \exists \{C_i\}_{i=0}^n \subseteq \mathfrak{S} \text{ such that}$$

$$A = C_0 \subseteq C_1 \subseteq \dots \subseteq C_n = B \text{ and } C_i \setminus C_{i-1} \in \mathfrak{S} \text{ for } 1 \leq i \leq n.$$

$\mathfrak{S}$  is a weak semi-ring provided that, in place of ii) we require

$$\text{iii) } A, B \in \mathfrak{S}, A \subseteq B \Rightarrow \exists \{C_i\}_{i=1}^n \text{ such that}$$

$$B \setminus A = \bigcup_1^n C_i \text{ and } C_i \cap C_j = \emptyset \text{ if } i \neq j.$$

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(<sup>1</sup>) The results presented in this paper are a part of the author's Ph.D. dissertation, written at the University of North Carolina at Chapel Hill under the direction of Professor B. J. Pettis.

DEFINITION. — Let  $\mathfrak{S} \subseteq 2^X$  and let  $\mathcal{J}$  be a topological abelian group. If  $\mu : \mathfrak{S} \rightarrow \mathcal{J}$  then

i)  $\mu$  is 2-additive if  $A, B, A \cup B \in \mathfrak{S}$ ,

$$A \cap B = \Phi \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$$

ii)  $\mu$  is finitely additive if whenever  $\{A_i\}_1^n$  is any finite, pairwise disjoint sequence in  $\mathfrak{S}$  such that  $\bigcup_1^n A_i \in \mathfrak{S}$ , then  $\mu\left(\bigcup_1^n A_i\right) = \sum_1^n \mu(A_i)$ .

iii)  $\mu$  is countably additive if whenever  $\{A_i\}_1^\infty$  is any pairwise disjoint sequence in  $\mathfrak{S}$  such that  $\bigcup_1^\infty A_i \in \mathfrak{S}$ , then  $\sum_{i=1}^\infty \mu(A_i) \rightarrow \mu\left(\bigcup_1^\infty A_i\right)$ .

Von Neumann [3, p. 94] has shown that if  $\mathfrak{S}$  is a semi-ring and  $\mu : \mathfrak{S} \rightarrow \mathcal{J}$  is 2-additive, then  $\mu$  is finitely additive. This does not hold in general for weak semi-rings. The smallest ring  $\mathcal{R}(\mathfrak{S})$  containing the semi-ring  $\mathfrak{S}$  is the collection of all unions of finite pairwise disjoint sets of members of  $\mathfrak{S}$ ; Von Neumann showed that a finitely (respectively countably) additive function on  $\mathfrak{S}$  has a unique finitely (respectively countably) additive extension defined on  $\mathcal{R}(\mathfrak{S})$ .

The topology for the topological abelian group  $\mathcal{J}$  is determined by a family  $\{\|\cdot\|_p : p \in P\}$  of semi-norms

$$(\| -g \|_p = \|g\|_p, \|g+h\|_p \leq \|g\|_p + \|h\|_p, \|g\|_p \geq 0)$$

Suppose  $\mu : \mathfrak{S} \rightarrow \mathcal{J}$ ; then for each  $p$  in  $P$  and each subset  $B$  of  $X$ , we define

$$1) (\mu_R)_p(B) = \sup \{ \|\mu(A)\|_p : A \in \mathfrak{S}, A \subseteq B \}$$

$$2) (\mu_D)_p(B) = \sup \{ \|\sum_1^n \mu(A_i)\|_p \}$$

$$3) |\mu|_p(B) = \sup \{ \|\sum_1^n \mu(A_i)\|_p \}$$

where the supremum in 2) and 3) is taken over all finite, pairwise disjoint sequences in  $\mathfrak{S}$  whose union is a subset of  $B$ .

Let  $X$  be a topological space and  $\mathfrak{S}$  a weak semi-ring of subsets of  $X$  and let  $\mu : \mathfrak{S} \rightarrow \mathcal{J}$  be finitely additive.

DEFINITION. —  $\mu$  is  $\mu_p$ -regular on  $\mathfrak{S}$  provided that for all  $p \in P$ ,  $A \in \mathfrak{S}$ , and  $\epsilon > 0$ , there exist  $C$  countably compact,  $o$  open,  $A' \in \mathfrak{S}$  such that  $A' \subseteq C \subseteq A \subseteq o \subseteq A''$  and  $(\mu_R)_p(A'' \setminus A') < \epsilon$ .

Similar definitions are made for  $\mu_D$  and  $|\mu|$ -regularity. In a previous paper [1], we have shown that a  $\mu_D$ -regular, finitely additive function on a weak semi-ring is countably additive. We note that  $|\mu|$ -regularity  $\Rightarrow \mu_D$ -regularity  $\Rightarrow \mu_R$ -regularity and that, for a ring of sets,  $\mu_D$ -regularity is the same as  $\mu_R$ -regularity.

$$(\mu, \nu) \geq \mu \geq \nu \geq \mu - \nu \geq \mu - \nu$$

3. The main theorems.

Revuz considered the problem of obtaining countable additivity from finite additivity and derived a suitable regularity condition to obtain countable additivity for non-negative real valued functions ([4], p. 208). The work of this paper generalizes the regularity condition of Revuz so that countable additivity may be obtained from finite additivity in the case of a function with values in a topological abelian group. We also show that an argument of Revuz concerning finite additivity is wrong (Example 3.1) and we give an alternate argument (Theorem 3.2).

Let  $X$  be a non-void set and  $\leq$  a binary relation on  $X$ . We shall say that  $(X, \leq)$  is a *conditional lower semilattice* provided that

- i)  $\leq$  is reflexive, transitive, and antisymmetric.
- ii) If  $x$  and  $y$  are members of  $X$ , and there is some member  $z$  in  $X$  such that  $z \leq x$  and  $z \leq y$ , then there is a largest (relative to  $\leq$ ) such member of  $X$ ; we shall denote such a member of  $X$  by " $\inf xy$ ".

We now form our "intervals" in this set. For any  $x$  in  $X$ ,  $C_-(x)$  will denote the set of all members  $y$  of  $X$  such that  $y \leq x$ , and  $C_+(x)$  will denote the set of all members  $y$  of  $X$  such that  $x \leq y$ . For each positive integer  $n$  and each  $x, u_1, u_2, \dots, u_n$  in  $X$ , let

$$S(x; u_1, u_2, \dots, u_n) = C_-(x) \setminus \bigcup_{i=1}^n C_-(u_i) \\ = \{y \in X : y \leq x \text{ but } y \not\leq u_i \\ \text{for any } i = 1, 2, \dots, n\}.$$

Revuz ([4], p. 195) has shown that each non-empty set of the form above has a unique representation in which each  $u_i \leq x$  but  $u_i \not\leq u_j$  for  $i \neq j$ . This form will be called the *canonical form*. In

particular when  $(X, \leq)$  is the real line with the usual ordering, the  $S$ 's are simply intervals of the form  $(a, b]$ .

Let  $\mathfrak{S}$  denote the collection of all sets of the form  $S(x; u_1, \dots, u_n)$ . We note that  $\Phi \in \mathfrak{S}$  since for any  $x$  in  $X$ ,  $S(x; x) = \Phi$ .

In the case of the real line with the usual ordering,

$$\{(a, b] : -\infty < a \leq b < \infty\}$$

forms a semi-ring. Revuz has shown that  $\mathfrak{S}$  is a weak semi-ring ([4], p. 199) ; we shall show that  $\mathfrak{S}$  is actually a semi-ring.

LEMMA. — If  $S_1 = S(x; v_1, v_2, \dots, v_n)$  and

$$S_2 = S(x; v_2, \dots, v_n), \text{ then}$$

- i)  $S_1 \subseteq S_2$  and in particular,  $S_1 = S_2 \cap \widetilde{C}_-(v_1)$ .  
 ii)  $S_2 \setminus S_1 = S(\inf xv_1; v_2, v_3, \dots, v_n)$  or  $\Phi$  if  $\inf xv_1$  does not exist.

*Proof :*

$$\text{i) } S_1 = C_-(x) \setminus \bigcup_1^n C_-(v_i) = C_-(x) \cap \left( \bigcap_1^n \widetilde{C}_-(v_i) \right)$$

$$= C_-(x) \cap \left( \bigcap_2^n \widetilde{C}_-(v_i) \right) \cap \widetilde{C}_-(v_1)$$

$$= S(x; v_2, \dots, v_n) \cap \widetilde{C}_-(v_1) = S_2 \cap \widetilde{C}_-(v_1)$$

$$\text{ii) } S_2 \setminus S_1 = S_2 \setminus [S_2 \cap \widetilde{C}_-(v_1)] = S_2 \setminus (\widetilde{C}_-(v_1)) = S_2 \cap C_-(v_1)$$

$$= (C_-(x) \setminus \bigcup_2^n C_-(v_i)) \cap C_-(v_1)$$

$$= (C_-(x) \cap C_-(v_1)) \setminus \bigcup_2^n C_-(v_i)$$

$$= \Phi \text{ or } S(\inf xv_1; v_2, v_3, \dots, v_n). \quad \square$$

We shall use the preceding lemma to prove

THEOREM 3.1. —  $\mathfrak{S}$  is a semi-ring

*Proof.* — By a result of Revuz  $\mathfrak{S}$  is closed under finite inter-

sections. We must show that if  $S$  and  $S^*$  are in  $\mathfrak{F}$  with  $S \subseteq S^*$ , then there is a finite sequence  $S_1, S_2, \dots, S_m$  in  $\mathfrak{F}$  with

$$S = S_0 \subseteq S_1 \subseteq \dots \subseteq S_{m+1} = S^* \quad \text{and} \quad S_i \setminus S_{i-1} \in \mathfrak{F}$$

for  $1 \leq i \leq m + 1$ . Suppose  $S = S(x; \nu_1, \dots, \nu_m)$  and

$$S^* = S(x^*; u_1, \dots, u_n)$$

with  $S \subseteq S^*$  and  $S^*$  is in canonical form ( $u_i \leq x^*$  for  $1 \leq i \leq n$ , but  $u_i \leq u_j$  if  $i \neq j$ ). If  $y \in S$ , then  $y \in S^*$  so  $y$  not  $\leq u_i$  for any

$$i = 1, 2, \dots, n ;$$

thus  $S$  can be put into the (not necessarily canonical) form

$$S = S(x; \nu_1, \nu_2, \dots, \nu_m, u_1, \dots, u_n) .$$

Now let

$$\begin{aligned} S_0 &= S = S(x; \nu_1, \nu_2, \dots, \nu_m, u_1, \dots, u_n) \\ S_1 &= S(x; \nu_2, \dots, \nu_m, u_1, \dots, u_n) \\ &\vdots \\ S_i &= S(x; \nu_{i+1}, \dots, \nu_m, u_1, \dots, u_n) \\ &\vdots \\ S_m &= S(x; u_1, \dots, u_n) . \end{aligned}$$

By Lemma 6.2, we have that  $S_i \setminus S_{i-1} \in \mathfrak{F}$  for  $1 \leq i \leq m$  ; and we also have  $S = S_0 \subseteq S_1 \subseteq \dots \subseteq S_m$  so we need only show that if  $S^* = S_{m+1}$ , then  $S_m \subseteq S_{m+1}$  and  $S_{m+1} \setminus S_m \in \mathfrak{F}$ . If  $y \in S_m$ , then  $y \leq x$  ; since  $S \subseteq S^*$  then  $x \in S^*$  and  $x \leq x^*$  ; thus  $y \leq x^*$ . By definition of  $S_m$ , if  $y \in S_m$  then  $y$  not  $\leq u_i$  for

$$1 \leq i \leq n \quad \text{and so} \quad S_m \subseteq S_{m+1} = S^* .$$

We also note that  $S_m = \{y \in X : y \leq x \text{ but } y \text{ not } \leq u_i \text{ for } 1 \leq i \leq n$  and  $S_{m+1} = \{y \in X : y < x^* \text{ but } y \text{ not } \leq u_i \text{ for } 1 \leq i \leq n\}$ .

$$\begin{aligned} \text{Thus, } S_{m+1} \setminus S_m &= \{y \in X : y \leq x^* \text{ but } y \text{ not } \leq x \text{ and } y \text{ not } \leq u_i \\ &\quad \text{for } 1 \leq i \leq n\} \end{aligned}$$

$$= S(x^*; x, u_1, \dots, u_n) \in \mathfrak{F} .$$

and  $\mathfrak{F}$  is a semi-ring.  $\square$

We recall from Chapter II, that one property that a semi-ring has but a weak semi-ring lacks is that a two-additive function is necessarily

In a previous paper [2], we generated such functions on  $\{(a, b) : -\infty < a \leq b < \infty\}$  from abelian group valued functions on the reals. We perform a similar feat in our more abstract setting. Suppose  $F$  is a function on  $X$  with values in an abelian group  $\mathcal{G}$ ; we define  $\mu$  from  $\mathcal{S}$  to  $\mathcal{G}$  as follows:

- 1) if  $S = \Phi$ ,  $\mu(S) = 0$
  - 2) if  $S \neq \Phi$  and  $S = S(x; u_1, \dots, u_n)$  in some (not necessarily canonical) representation, define
- $$(*) \quad \mu(S) = F(x) - \sum_1 F(\inf xu_{i_1}) + \sum_2 F(\inf xu_{i_1} u_{i_2}) - \dots$$

where  $\sum_m F(\inf xu_{i_1} \dots u_{i_m})$  represents the sum over all distinct sets of  $m$  indices  $\{i_1, i_2, \dots, i_m\}$  and  $F(\inf xu_{i_1} \dots u_{i_m}) = 0$  if that inf does not exist.

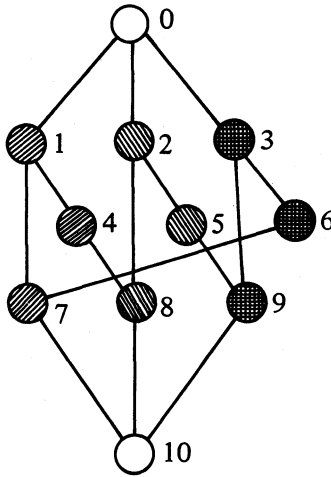
Revuz ([4], p. 197) has shown that any such real valued function  $\mu$  is well-defined; exactly the same proof carries over for the case in which  $\mu$  takes values in an abelian group. We note that (\*) is simply an extension of the usual modularity law :

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

Revuz attempts to show as follows that such a function  $\mu$  is finitely additive. If  $S = S(x; u_1, \dots, u_n)$  then  $x$  is called the summit of  $S$ . Revuz ([4], p. 201) defines a relation  $\ll$  on arbitrary collections of pairwise disjoint members of  $\mathcal{S}$  by setting  $S_1 \ll S_2$  if and only if there is some  $x$  in  $S_1$  with  $x \leq$  the summit of  $S_2$ . He considered  $S_1, S_2, \dots, S_n$  in  $\mathcal{S}$  with  $S = \cup S_i \in \mathcal{S}$  and sought to pick a minimal  $S_i$  relative to the ordering  $\ll$ . He asserted ([4], p. 201) that this is possible since  $\ll$  is reflexive, antisymmetric, and transitive. However, the transitive property does not hold in general and a situation in which we cannot pick such a minimal element is shown in

*Example 3.1.* — Let  $(X, \leq)$  be the partially ordered set determined by the following Hasse diagram where, as usual,  $n \leq m$  provided  $n$  is no higher than  $m$  and there is an ascending path from  $n$  to  $m$ . Note that  $(X, \leq)$  is a conditional lower semilattice

We recall from Chapter II that one property that a semi-lattice necessarily lacks is that a two-additive function is necessarily



Now we pick  $S_0, S_1, S_2, S_3$  as follows :

$$S_0 = \{0\} = S(0 ; 1, 2, 3)$$

$$S_1 = \{1, 4, 7\} = S(1 ; 8, 10) \text{ denoted by } \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$S_2 = \{2, 5, 8\} = S(2 ; 9, 10) \text{ denoted by } \begin{array}{c} \diagdown \\ \diagup \end{array}$$

$$S_3 = \{3, 6, 9\} = S(3 ; 7, 10) \text{ denoted by } \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array}$$

$S_1 \ll S_3$  since  $7 \in S_1$  and  $7 \leq 3 = \text{summit of } S_3$ ,  $S_3 \ll S_2$  since  $9 \in S_3$  and  $9 \leq 2 = \text{summit of } S_2$ , and  $S_2 \ll S_1$  since  $8 \in S_2$  and  $8 \leq 1 = \text{summit of } S_1$ . Thus, there is no minimal member relative to the order  $\ll$ .

Even though Revuz's proof is incorrect, we do get finite additivity for such a function  $\mu$ . In view of Von Neumann's work and the fact that  $\mathfrak{S}$  is a semi-ring, we need only prove that  $\mu$  is 2-additive and a relatively trivial modification of Revuz's proof accomplishes this.

For the remainder of the chapter, we shall assume that  $X$  is a topological space and  $(X, \leq)$  is a conditional lower semilattice and we shall be interested in the following relationships between the order and the topology :

$X_a$  : Each  $C_-(x)$  is closed and the closure of each member of  $\mathfrak{S}$  is countably compact.



$X_b$  : *inf is continuous from the right* in the sense that one of the following must hold for each  $x$  and  $y$  in  $X$  :

i) If  $w = \inf xy$  and  $V$  is a neighborhood of  $w$ , then there exists  $V_x$  and  $V_y$  neighborhoods of  $x$  and  $y$  respectively such that if  $x' \in V_x$  with  $x \leq x'$  and  $y' \in V_y$  with  $y \leq y'$ , then  $\inf x'y' \in V$ .

ii) If  $\inf xy$  does not exist, then there exist neighborhoods  $V_x$  and  $V_y$  of  $x$  and  $y$  respectively such that if  $x' \in V_x$  with  $x \leq x'$  and  $y' \in V_y$  with  $y \leq y'$  then  $\inf x'y'$  does not exist.

$X_c$  : *If  $x \in C_-(y)$  then for each neighborhood  $V_x$  of  $x$ , there exists  $z$  in  $V_x \cap C_+(x) \cap C_-(y)$  such that  $C_-(x) \subseteq \{C_-(z)\}^{int}$  where the interior is relative to the subspace topology of  $C_-(y)$ .*

The meanings of  $X_a$  and  $X_b$  are clear, but  $X_c$  may require an illustration. Let  $(X, \leq)$  be the real line with the usual topology and the usual ordering. Let  $x \leq Y$  and  $\varepsilon > 0$ . If  $x = y$  then

$$(-\infty, x] = (-\infty, y]$$

and the interior of  $(-\infty, x]$  relative to the subspace topology of  $C_-(y)$  is  $(-\infty, x]$ . Thus the  $z$  whose existence is asserted in  $X_c$  is just  $x$ . Now if  $x < y$ , then there is some  $z$  strictly between  $x$  and  $y$  so  $z \in C_+(x) \cap C_-(y)$  and  $C_-(x) - (-\infty, x] - (-\infty, z) = C_-(z)^{int}$ .

In Chapter II, we have seen that an additive set function which is  $\mu_D$ -regular on  $\{(a, b) : a, b \in \mathbb{R}\}$  is countably additive. Now for each  $(a, b]$  with  $a < b$ , let  $c$  and  $d$  be numbers such that  $a < c \leq b < d$ . Then  $(c, b] \subseteq [c, b] \subseteq (a, b] \subseteq (a, d] \subseteq (a, d]$  and for regularity, it is sufficient that  $(a, d] \setminus (c, b]$  be "small". Now

$$(a, d] \setminus (c, b] = (a, c] \cup (b, d] = (a, b] \Delta (c, d]$$

where  $\Delta$  denotes the symmetric difference. Thus, for regularity, we may require that each member  $(a, b]$  be "approximated from the right" by some member  $(c, d]$ . To the end of generalizing this regularity for use in our more abstract setting we first of all obtain an approximation notion and then consider "approximation from the right".

Recalling the definitions of Chapter II, we define

$$(V_F)_p(S(x; u_1, \dots, u_m)) = \sup \left\{ \sum_1^n \|\mu(S_i)\|_p \right\}$$

$$(DV_F)_p(S(x; u_1, \dots, u_m)) = \sup \left\{ \left\| \sum_1^n \mu(S_i) \right\|_p \right\}$$

where in each case the supremum is taken over all finite, pairwise-disjoint sequences of members of whose union is in  $S(x; u_1, \dots, u_n)$ . These will be called the variation and the Dunford variation, respectively, of  $F$ . We note that these definitions are made so that

$$|\mu|_p(s) = (V_F)_p(S) \quad \text{and} \quad (\mu_D)_p(S) = (DV_F)_p(S)$$

for each  $S$  in and each  $p$  in  $P$ .

For a regularity condition, we shall consider

$X_d$  : If  $S = S(x; u_1, \dots, u_n) \in \mathfrak{S}$ ,  $\varepsilon > 0$ , and  $p \in P$ , then there exist neighborhoods  $V_x$  of  $x$  and  $V_i$  of  $u_i$  ( $1 < i < n$ ) such that whenever  $x' \in V_x \cap C_+(x)$ ,  $u'_i \in V_i \cap C_+(u_i)$  ( $1 \leq i \leq n$ ), then we have  $(\mu_D)_p(S \Delta S') < \varepsilon$  where  $S' = S(x'; u'_1, \dots, u'_n)$ .

$X'_d$  : Same as  $X_d$  but with  $\mu_D$  replaced by  $|\mu|$ .

We note that  $X'_d$  implies  $X_d$ .

LEMMA. — Let  $X = C_-(y)$  for some  $y$  and suppose  $X_a, X_c$ , and  $X_d$  are satisfied by  $(X, <)$ . Then  $\mu$  is countably additive on  $\mathfrak{S}$ .

*Proof.* — Let  $\mathcal{U}$  be the collection of open sets of  $X$  and  $\mathcal{C}$ , the collection of closed countably compact sets of  $X$ . We shall show that  $\mu$  is  $\mu_D$ -regular and thus  $\mu$  will be countably additive.

i) “inner regularity” : Let  $S = S(x; u_1, \dots, u_n)$ ,  $p \in P$  and  $\varepsilon > 0$ ; then since  $X$  satisfies  $X_c$ , for each neighborhood  $V_i$  of  $u_i$ , we can find  $v_i$  such that  $v_i \in V_i \cap C_+(u_i)$  and  $C_-(u_i) \subseteq (C_-(v_i))^{\text{int}}$ . Now we have that  $\overline{C_-(x) \setminus C_-(v_i)} \subseteq C_-(x) \setminus (C_-(v_i))^{\text{int}} \subseteq C_-(x) \setminus C_-(u_i)$ . Thus,  $S_1 = S(x; v_1, \dots, v_n) = C_-(x) \setminus \bigcup_1^n C_-(v_i) = \bigcap_1^n (C_-(x) \setminus C_-(v_i))$ , then

$$\overline{S_1} = \bigcap_1^n \overline{(C_-(x) \setminus C_-(v_i))} \subseteq \bigcap_1^n \overline{(C_-(x) \setminus C_-(v_i))} \subseteq \bigcap_1^n (C_-(x) \setminus C_-(u_i)) = S.$$

Therefore  $S_1 \subseteq \overline{S_1} \subseteq S$  and  $\overline{S_1}$  is countably compact by  $X_a$ . Now by  $X_d$ , we may pick the neighborhoods  $V_i$  ( $1 \leq i \leq n$ ) such that  $(\mu_D)_p(S_1 \Delta S) < \varepsilon/2$ . Since  $S_1 \Delta S = S/S_1$  whenever  $S_1 \subseteq S$ , we have that  $(\mu_D)_p(S \setminus S_1) < \varepsilon/2$ .

ii) "outer regularity" : Let  $S = S(x ; u_1, \dots, u_n)$  be in canonical form,  $p \in P$  and  $\varepsilon > 0$  ; since  $X$  satisfies  $X_c$ , for each neighborhood  $V_x$  of  $x$ , there is a  $z$  in  $V_x \cap C_+(x)$  such that  $C_-(x) \subseteq (C_-(z))^{int}$ . Let  $S_2 = S(z ; u_1, \dots, u_n)$ . Now  $S \subseteq S_2^{int} \subseteq S_2$  since

$$S_2^{int} = (C_-(z) \setminus \bigcup_1^n C_-(u_i))^{int} = (C_-(z))^{int} \setminus \bigcup_1^n C_-(u_i) \supseteq C_-(x) \setminus \bigcup_1^n C_-(u_i).$$

By  $X_d$ ,  $V_x$  can be picked so that  $(\mu_D)_p(S_2 \Delta S) < \varepsilon/2$  ; since

$$S_2 \Delta S = S_2 \setminus S$$

whenever  $S \subseteq S_2$ , we have that  $(\mu_D)_p(S_2 \setminus S) < \varepsilon/2$ .

Now from i) and ii) we may conclude that for each  $S$  in  $\mathfrak{S}$ , each  $p \in P$ , and each  $\varepsilon > 0$ , there exist  $S_1, S_2$  in  $\mathfrak{S}$ ,  $C(= S_1)$  countably compact, and  $U(= S_2^{int})$  open such that  $S_1 \subseteq \bar{S}_1 \subseteq S \subseteq S_2^{int} \subseteq S_2$  and  $(\mu_D)_p(S_2 \setminus S_1) \leq (\mu_D)_p(S_2 \setminus S) + (\mu_D)_p(S \setminus S_1) < \varepsilon$ . Thus  $\mu$  is  $\mu_D$ -regular and so  $\mu$  is countably additive.  $\square$

**THEOREM 3.2.** — *Let  $X$  be a topological space and  $(X, \leq)$  a conditional lower semilattice. If  $X_a, X_c$ , and  $X_d$  are satisfied, then  $\mu$  is countably additive on  $\mathfrak{F}$ .*

*Proof.* — Let  $\{S_i\}_1^\infty$  be a pairwise disjoint sequence of members of  $\mathfrak{S}$  such that  $\bigcup_1^\infty S_i = S \in \mathfrak{S}$ . Let  $S = S(x ; u_1, \dots, u_n)$ , then  $S_i \subseteq S \subseteq C_-(x)$  for  $1 \leq i < \infty$ . If  $X' = C_-(x)$ , then  $(X', \leq)$  satisfies  $X_a, X_c$ , and  $X_d$ . Thus, by the preceding lemma,  $\mu$  is countably additive on  $\mathfrak{S} \cap 2^{X'}$  and so  $\sum_1^n \mu(S_i) \rightarrow \mu(S)$ . Therefore,  $\mu$  is countably additive on  $\mathfrak{S}$ .  $\square$

Since  $X'_d$  implies  $X_d$ , we obtain the following

**COROLLARY 3.2.1.** — *Let  $X$  be a topological space and  $(X, \leq)$  a conditional lower semilattice. If  $X_a, X_c$ , and  $X'_d$  are satisfied, then  $\mu$  is countably additive on  $\mathfrak{S}$ .*

As an additional corollary, we also obtain Revuz' original result below in Theorem 3.3 ([4], p. 208).

We shall say that a function  $F$  defined on  $X$  and taking values in the topological space  $Y$  is *continuous from the right* at  $x$  in  $X$

provided that for each neighborhood  $U$  of  $F(x)$ , there is a neighborhood  $V$  of  $x$  such that  $F(x') \in U$  whenever  $x' \in V \cap C_+(x)$ .

LEMMA. — Suppose  $X$  is a topological space and  $(X, \leq)$  is a conditional lower semilattice. If  $(X, \leq)$  satisfies  $X_a$  and  $X_c$  and if  $F : X \rightarrow \mathbb{R}$  is such that

- i)  $\mu$  is non-negative.
  - ii)  $F$  is continuous from the right on  $X$ ,
- then if  $X_b$  is satisfied, so is  $X_d$ .

Proof. — Notice that under the condition that  $\mu$  is non-negative we have  $\mu_D(S_1 \setminus S_2) = \mu(S_1) - \mu(S_2)$  if  $S_2 \subseteq S_1$ . Let

$$S = S(x ; u_1, \dots, u_n)$$

and  $\varepsilon < 0$ . Now  $F$  and  $\inf$  are continuous from the right on  $X$ , so for each  $u_i$ , there is a neighborhood  $V_i$  of  $u_i$  such that if

$$v_i \in V_i \cap C_+(u_i) \cap C_-(x)$$

then  $0 \leq F(u_i) - F(v_i) \leq \varepsilon/2n$ . Also, there is a neighborhood  $V$  of  $x$  such that if  $x' \in V \cap C_+(x)$ , then  $0 \leq F(x') - F(x) < \varepsilon/2$ . Now let  $S^* = S(x ; v_1, \dots, v_n)$ ; then  $S \setminus S^* \subseteq \bigcup_1^n S(v_i ; u_i)$  and so

$$\begin{aligned} \mu_D(S \setminus S^*) &\leq \sum_1^n \mu_D(S(v_i ; u_i)) = \sum_1^n \mu(S(v_i ; u_i)) = \\ &= \sum_1^n F(v_i) - F(u_i) < \varepsilon/2 . \end{aligned}$$

Also,  $S^* \setminus S = S(x' ; x)$  and

$$\mu_D(S^* \setminus S) = \mu_D(S(x' ; x)) = \mu(S(x' ; x)) = F(x') - F(x) < \varepsilon/2 .$$

Thus  $\mu_D(S \Delta S^*) \leq \mu_D(S \setminus S^*) + \mu_D(S^* \setminus S) < \varepsilon/2 + \varepsilon/2 = \varepsilon$  and  $X_d$  is satisfied.  $\square$

Combining the preceding lemma with Theorem 3.2, we obtain the main result of Revuz in this area.

THEOREM 3.3. — Let  $X$  be a topological space and  $(X, \leq)$  be a conditional lower semilattice. If  $(X, \leq)$  satisfies  $X_a$ ,  $X_b$ , and  $X_c$  and

$F : X \rightarrow \mathbb{R}$  is such that  $\mu$  is non-negative and  $F$  is continuous from the right on  $X$ , then  $\mu$  is countably additive on  $\mathfrak{S}$ .

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