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A NOTE ON ALMOST STRONG LIFTINGS ⁽¹⁾

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1.

We denote below by X a *locally compact space* and by $\mathcal{M}(X)$ the vector space of Radon measures on X , endowed with the usual order relation. Let $\mu \neq 0$ be a positive Radon measure on X . We say that a *lifting* ρ of $M_{\mathbb{R}}^{\infty}(X, \mu)$ is *almost strong* (see [7], Chap. VIII) if there is a μ^* -negligible (that is, locally μ -negligible) set $A \subset X$ such that

$$\rho(f)|_{CA} = f|_{CA}$$

for all $f \in C_{\mathbb{R}}^b(X)$.

We say that the couple (X, μ) has the *almost strong lifting property* (*a.s. lifting property*) if there exists an almost strong lifting of $M_{\mathbb{R}}^{\infty}(X, \mu)$.

To shorten some of the statements below we also say that (X, μ) has the *a.s. lifting property* whenever $\mu = 0$.

The problem as to whether or not every (X, μ) (where X is a locally compact space and μ a positive Radon measure on X) has the *a.s. lifting property* is open (see [5] and [7], Chap. VIII). However there are many important examples of couples (X, μ) having the *a.s. lifting property* (see [5], [6], [7], Chap. VIII and [8]). Recently, K. Bichteler (see [1] and [2]) has noticed the interesting fact that the set of all Radon measures μ on X such that $(X, |\mu|)$ has the *a.s. lifting property* is a *band* of $\mathcal{M}(X)$. In this paper we present a short proof of this result by a method different from that of K. Bichteler.

⁽¹⁾ We use the notations and terminology introduced in [7].

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2.

For any positive Radon measure μ on X we denote by $\mathcal{C}(X, \mu)$ the set of all locally countable families $(K_j)_{j \in J}$ having the following properties :

- a) K_j is compact and $\mu(K_j) > 0$ for each $j \in J$.
- b) $K_{j'} \cap K_{j''} = \emptyset$ if $j' \neq j''$.
- c) The set $X - \bigcup_{j \in J} K_j$ is μ^* -negligible.

The following result will be often used below :

THEOREM 1. — *Let μ be a positive Radon measure on X . 1.1) If (X, μ) has the a.s. lifting property and $K \subset X$ is compact, then (K, μ_K) has the a.s. lifting property. 1.2) Conversely, let $(K_j)_{j \in J} \in \mathcal{C}(X, \mu)$ be such that, for each $j \in J$, (K_j, μ_{K_j}) has the a.s. lifting property. Then (X, μ) has the a.s. lifting property.*

Proof. — 1.1) It is enough to consider the case $\mu_K \neq 0$. Let ρ be an almost strong lifting of $M_{\mathbb{R}}^{\infty}(X, \mu)$ and let $A \subset X$ be a μ^* -negligible set such that the relations $\rho(f)|_{\mathbf{C}A} = f|_{\mathbf{C}A}$ are satisfied for all $f \in C_{\mathbb{R}}^b(X)$. Let χ be a character of $L_{\mathbb{R}}^{\infty}(K, \mu_K)$. For $f \in M_{\mathbb{R}}^{\infty}(K, \mu_K)$ define $f' : X \rightarrow \mathbb{R}$ by $f'(t) = f(t)$ if $t \in K$ and $f'(t) = 0$ if $t \notin K$. Then $f \mapsto f'$ is a representation of $M_{\mathbb{R}}^{\infty}(K, \mu_K)$ into $M_{\mathbb{R}}^{\infty}(X, \mu)$. Define now $\rho'(f)$, for $f \in M_{\mathbb{R}}^{\infty}(K, \mu_K)$, by

$$\rho'(f)(t) = \begin{cases} \rho(f')(t) & \text{if } t \in K \cap \rho(K) \\ \chi(f) & \text{if } t \in K - \rho(K). \end{cases}$$

It is easy to see that ρ' is a lifting of $M_{\mathbb{R}}^{\infty}(K, \mu_K)$ and that $\rho'(f)(x) = f(x)$ if $f \in C_{\mathbb{R}}^b(K)$ and $t \notin K \cap (A \cup (K - \rho(K)))$. Hence ρ' is an almost strong lifting of $M_{\mathbb{R}}^{\infty}(K, \mu_K)$ and hence the couple (K, μ_K) has the a.s. lifting property.

1.2) It is enough to consider the case $\mu \neq 0$. For each $j \in J$ let ρ_j be an almost strong lifting of $M_{\mathbb{R}}^{\infty}(K_j, \mu_{K_j})$ and $A_j \subset K_j$ a μ_K^* -negligible set such that $\rho_j(f)|_{\mathbf{C}A_j} = f|_{\mathbf{C}A_j}$ for $f \in C_{\mathbb{R}}^b(K_j)$. Let χ be a character of $L_{\mathbb{R}}^{\infty}(X, \mu)$. If $f \in M_{\mathbb{R}}^{\infty}(X, \mu)$, then $f|_{K_j} \in M_{\mathbb{R}}^{\infty}(K_j, \mu_{K_j})$ for each $j \in J$ and

hence we may define

$$\rho(f)(t) = \begin{cases} \rho_j(f|K_j)(t) & \text{if } t \in K_j \\ \chi(f) & \text{if } t \in X - \bigcup_{j \in J} K_j. \end{cases}$$

It is easy to see that ρ is a lifting of $M_{\mathbb{R}}^{\infty}(X, \mu)$ and that $\rho(f)(x) = f(x)$ if $f \in C_{\mathbb{R}}^b(X)$ and $t \in \left(\bigcup_{j \in J} A_j\right) \cup \left(X - \bigcup_{j \in J} K_j\right)$. Hence ρ is an almost strong lifting of $M_{\mathbb{R}}^{\infty}(X, \mu)$ and hence the couple (X, μ) has the a.s. lifting property.

Remarks. — Theorem 1 is similar to Proposition 2, [7], Chap. VIII (in fact it can be easily deduced from this proposition).

3.

If μ and ν are two positive Radon measures on X we write $\mu \prec \nu$ if μ is absolutely continuous with respect to ν (that is, if $\mu = \varphi \cdot \nu$ with $\varphi : X \rightarrow \mathbb{R}_+$, locally ν -integrable). We say that μ and ν are equivalent if $\mu \prec \nu$ and $\nu \prec \mu$. If μ and ν are equivalent, then (X, μ) has the a.s. lifting property if and only if (X, ν) has the a.s. lifting property.

Notice that if $\mu \prec \nu$ then there is $(K_j)_{j \in J} \in \mathcal{C}(X, \mu)$ such that, for each $j \in J$, μ_{K_j} and ν_{K_j} are equivalent.

In fact if $\mu \prec \nu$ then $\mu = \varphi \cdot \nu$ with $\varphi : X \rightarrow \mathbb{R}_+$ locally ν -integrable. Let $A = \{x | \varphi(x) > 0\}$ and consider a partition of A consisting of a μ -negligible set N and a locally countable family of compact sets $(K_j)_{j \in L}$ such that $\varphi|K_j$ is continuous for each $j \in L$ (see Corollary 1, Chap. IV, § 5 [3]). If $J = \{j \in L | \mu(K_j) > 0\}$, then $(K_j)_{j \in J} \in \mathcal{C}(X, \mu)$. Since for each $j \in J$, $\mu_{K_j} = (\varphi|K_j) \cdot \nu_{K_j}$ and since

$$0 < \inf_{x \in K_j} (\varphi|K_j)(x) \leq \sup_{x \in K_j} (\varphi|K_j)(x) < + \infty,$$

we deduce that μ_{K_j} and ν_{K_j} are equivalent.

THEOREM 2. — Let μ and ν be two positive Radon measures on X . If (X, ν) has the a.s. lifting property and $\mu \prec \nu$ then (X, μ) has the a.s. lifting property ⁽³⁾.

⁽³⁾ See [1].

Proof. — We have noticed above that there is

$$(K_j)_{j \in J} \in \mathcal{C}(X, \mu)$$

such that, for each $j \in J$, μ_{K_j} and ν_{K_j} are equivalent. By Theorem 1, for each $j \in J$, (K_j, ν_{K_j}) has the a.s. lifting property, whence (K_j, μ_{K_j}) has the a.s. lifting property. Using again Theorem 1 we deduce that (X, μ) has the a.s. lifting property.

THEOREM 3. — *Let μ and ν be two positive Radon measures on X such that (X, μ) and (X, ν) have the a.s. lifting property. Then $(X, \mu + \nu)$ has the a.s. lifting property.*

Proof. — Let $\mu = \mu_a + \mu_s$, where μ_a is the absolutely continuous part of μ with respect to ν and μ_s the singular part of μ with respect to ν . Then

$$\mu + \nu = (\mu_a + \nu) + \mu_s.$$

Since $\mu_a + \nu \prec \nu$, the couple $(X, \mu_a + \nu)$ has the a.s. lifting property; since $\mu_s \prec \mu$, the couple (X, μ_s) has the a.s. lifting property. Moreover, there are two disjoint universally measurable parts of X , X' and X'' , the union of which is X , such that $\mu_a + \nu$ is concentrated on X' and μ_s is concentrated on X'' .

Let now $(K_j)_{j \in J} \in \mathcal{C}(X, \mu + \nu)$ such that for each $j \in J$ we have either $K_j \subset X'$ or $K_j \subset X''$ and let

$$J' = \{j | K_j \subset X'\} \quad \text{and} \quad J'' = \{j | K_j \subset X''\}.$$

If $j \in J'$ then $(\mu + \nu)_{K_j} = (\mu_a + \nu)_{K_j}$ so that $(K_j, (\mu + \nu)_{K_j})$ has the a.s. lifting property; if $j \in J''$ then $(\mu + \nu)_{K_j} = (\mu_s)_{K_j}$, so that $(K_j, (\mu + \nu)_{K_j})$ has again the a.s. lifting property. By Theorem 1, $(X, \mu + \nu)$ has the a.s. lifting property.

COROLLARY 1. — *Let μ and ν be as in the statement of Theorem 3. Then $(X, \inf \{\mu, \nu\})$ and $(X, \sup \{\mu, \nu\})$ have the a.s. lifting property.*

Proof. — It is enough to notice that

$$\inf \{\mu, \nu\} \prec \mu + \nu \quad \text{and} \quad \sup \{\mu, \nu\} \prec \mu + \nu.$$

We note before proceeding further that if \mathcal{F} is a filtering set of positive Radon measures on a compact space X , bounded above, then there is an increasing sequence $(\mu_n)_{n \in \mathbf{N}}$ of measures belonging to \mathcal{F} such that

$$\sup \mathcal{F} = \sup_{n \in \mathbf{N}} \mu_n$$

(use Theorem 4, Chap. I, [7]).

If $\lambda = \sup \mathcal{F}$ then $\lambda^*(A) = 0$ if and only if $\mu_n^*(A) = 0$ for every $n \in \mathbf{N}$ (use Proposition 11, Chap. v, § 1, [3]). We also notice that if $(B_n)_{n \in \mathbf{N}}$ is a sequence of parts of X such that $\mu_n^*(B_n) = 0$ for every $n \in \mathbf{N}$, then

$$\lambda^*(\limsup_{n \in \mathbf{N}} B_n) = 0.$$

In fact it is enough to observe that, for each $p \in \mathbf{N}$

$$\limsup_{n \in \mathbf{N}} B_n \subset \bigcup_{n=p}^{+\infty} B_n$$

and

$$\mu_p^*\left(\bigcup_{n=p}^{+\infty} B_n\right) \leq \sum_{n=p}^{+\infty} \mu_p^*(B_n) \leq \sum_{n=p}^{+\infty} \mu_n^*(B_n) = 0.$$

THEOREM 4. — *Let \mathcal{F} be a set of positive Radon measures on (the locally compact space) X , bounded above and let $\lambda = \sup \mathcal{F}$. Suppose that (X, μ) has the a.s. lifting property for every $\mu \in \mathcal{F}$. Then (X, λ) has the a.s. lifting property.*

Proof. — By Corollary 1, we may suppose that \mathcal{F} is filtering. On the basis of Theorem 1 and the fact that for every compact $K \subset X$,

$$\lambda_K = \sup \{\mu_K | \mu \in \mathcal{F}\}$$

(see Proposition 5, Chap. v, § 5, [3]). It is enough to establish that (X, λ) has the a.s. lifting property when X is compact.

We may also assume $\lambda \neq 0$. Let then $(\mu_n)_{n \in \mathbf{N}}$ be an increasing sequence of strictly positive measures belonging to \mathcal{F} , such that $\lambda = \sup_{n \in \mathbf{N}} \mu_n$. For each $n \in \mathbf{N}$ let ρ_n be an almost strong lifting of $M_{\mathbf{R}}^{\infty}(X, \mu_n)$ and $A(n)$ a μ_n^* -negligible set such that $\rho_n(f)|_{\mathbf{C}A(n)} = f|_{\mathbf{C}A(n)}$ for all $f \in C_{\mathbf{R}}^b(X)$.

Let \mathcal{U} be an *ultrafilter* on \mathbf{N} finer than the Fréchet filter associated with \mathbf{N} . For every $f \in M_{\mathbf{R}}^{\infty}(X, \lambda)$ define ⁽⁴⁾

$$\rho(f) = \lim_{n, \mathcal{U}} \rho_n(f).$$

Then ρ is a representation of the algebra $M_{\mathbf{R}}^{\infty}(X, \lambda)$ into the algebra $B_{\mathbf{R}}^{\infty}(X)$ of all bounded functions on X to \mathbf{R} , such that $\rho(1) = 1$. Moreover $f \equiv g (\lambda)$ implies $f \equiv g (\mu_n)$, that is, $\rho_n(f) = \rho_n(g)$ for all $n \in \mathbf{N}$, whence $\rho(f) = \rho(g)$. Let now $f \in M_{\mathbf{R}}^{\infty}(X, \lambda)$ and for each $n \in \mathbf{N}$ let

$$B(n) = \{x | \rho_n(f)(x) \neq f(x)\}.$$

Clearly $\rho(f)(x) = f(x)$ for

$$x \notin \limsup_{n \in \mathbf{N}} B(n).$$

Since $\limsup_{n \in \mathbf{N}} B(n)$ is λ^* -negligible, we deduce $\rho(f) \in M_{\mathbf{R}}^{\infty}(X, \lambda)$ and $\rho(f) \equiv f$. Hence ρ is a *lifting* of $M_{\mathbf{R}}^{\infty}(X, \lambda)$. In the same way we see that for every $f \in C_{\mathbf{R}}^b(X)$, $\rho(f)(x) = f(x)$ if $x \notin \limsup_{n \in \mathbf{N}} A(n)$. Since $\limsup_{n \in \mathbf{N}} A(n)$ is λ^* -negligible we conclude that ρ is an almost strong lifting of $M_{\mathbf{R}}^{\infty}(X, \mu)$.

Hence (X, μ) has the a.s. lifting property.

Remark. — By the same method we can prove the following :
Let $(\mu_n)_{n \in \mathbf{N}}$ be a sequence of positive Radon measures on X and λ a positive Radon measure on X . Suppose that :

i) $\mu_n \prec \mu_{n+1}$ for all $n \in \mathbf{N}$;

ii) $\lambda^*(A) = 0$ if and only if $\mu_n^*(A) = 0$ for all $n \in \mathbf{N}$.
Then (X, λ) has the a.s. lifting property if and only if (X, μ_n) has the a.s. lifting property for every $n \in \mathbf{N}$.

We shall say that (X, μ) , where $\mu \in \mathfrak{M}(X)$, has the a.s. lifting property if and only if $(X, |\mu|)$ has the a.s. lifting property. Denote by \mathbf{U} the set of all $\mu \in \mathfrak{M}(X)$ such that (X, μ) has the a.s. lifting property. Then :

THEOREM 5 (Bichteler). — The set \mathbf{U} is a band of $\mathfrak{M}(X)$.

Proof. — The assertion follows from Theorems 2, 3 and 4.

(4) See also [4].

Let \mathbf{V} be the set of all positive Radon measures μ on X such that (X, μ) has the *strong lifting property* (see Definition 1, Chap. VIII [7]). Clearly $\mathbf{V} \subset \mathbf{U}$.

COROLLARY 2. — *The set \mathbf{V} is a cone of $\mathcal{M}(X)$ having the properties :*

- j) if μ and ν belong to \mathbf{V} , then $\sup \{\mu, \nu\} \in \mathbf{V}$;
- jj) if $\mathcal{F} \subset \mathbf{V}$ is bounded above, in $\mathcal{M}(X)$, then $\sup \mathcal{F} \in \mathbf{V}$.

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