C. IONESCU-TULCEA
R. MAHER

A note on almost strong liftings


<http://www.numdam.org/item?id=AIF_1971__21_2_35_0>
A NOTE ON ALMOST STRONG LIFTINGS (1)
By C. IONESCU TULCEA (2) and R. MAHER

1.

We denote below by \( X \) a locally compact space and by \( \mathcal{M}_R(X) \) the vector space of Radon measures on \( X \), endowed with the usual order relation. Let \( \mu \neq 0 \) be a positive Radon measure on \( X \). We say that a lifting \( \varphi \) of \( \mathcal{M}_R(X, \mu) \) is almost strong (see [7], Chap. viii) if there is a \( \mu^* \)-negligible (that is, locally \( \mu \)-negligible) set \( A \subset X \) such that

\[
\varphi(f)|_A = f|_A
\]

for all \( f \in C^b(X) \).

We say that the couple \( (X, \mu) \) has the almost strong lifting property (a.s. lifting property) if there exists an almost strong lifting of \( \mathcal{M}_R(X, \mu) \).

To shorten some of the statements below we also say that \( (X, \mu) \) has the a.s. lifting property whenever \( \mu = 0 \).

The problem as to whether or not every \( (X, \mu) \) (where \( X \) is a locally compact space and \( \mu \) a positive Radon measure on \( X \)) has the a.s. lifting property is open (see [5] and [7], Chap. viii). However there are many important examples of couples \( (X, \mu) \) having the a.s. lifting property (see [5], [6], [7], Chap. viii and [8]). Recently, K. Bichteler (see [1] and [2]) has noticed the interesting fact that the set of all Radon measures \( \mu \) on \( X \) such that \( (X, |\mu|) \) has the a.s. lifting property is a band of \( \mathcal{M}_R(X) \). In this paper we present a short proof of this result by a method different from that of K. Bichteler.

(1) We use the notations and terminology introduced in [7].

(2) Supported by contract DAHCOH 68 C0005, U.S. Army Research Office, Durham, N.C.
2.

For any positive Radon measure \( \mu \) on \( X \) we denote by \( \mathcal{C}(X, \mu) \) the set of all locally countable families \( (K_j)_{j \in J} \) having the following properties:

a) \( K_j \) is compact and \( \mu(K_j) > 0 \) for each \( j \in J \).

b) \( K_j \cap K_{j'} = \emptyset \) if \( j \neq j' \).

c) The set \( X - \bigcup_{j \in J} K_j \) is \( \mu \)-negligible.

The following result will be often used below:

**Theorem 1.** — Let \( \mu \) be a positive Radon measure on \( X \).

1.1) If \( (X, \mu) \) has the a.s. lifting property and \( K \subset X \) is compact, then \( (K, \mu_K) \) has the a.s. lifting property.

1.2) Conversely, let \( (K_j)_{j \in J} \in \mathcal{C}(X, \mu) \) be such that, for each \( j \in J \), \( (K_j, \mu_{K_j}) \) has the a.s. lifting property. Then \( (X, \mu) \) has the a.s. lifting property.

**Proof.** — 1.1) It is enough to consider the case \( \mu_K \neq 0 \). Let \( \rho \) be an almost strong lifting of \( M_R^X(X, \mu) \) and let \( A \subset X \) be a \( \mu \)-negligible set such that the relations \( \rho(f)|_{CA} = f|_{CA} \) are satisfied for all \( f \in C_b(K) \). Let \( \chi \) be a character of \( L^*_R(K, \mu_K) \). For \( f \in M_R^X(K, \mu_K) \) define \( f' : X \to \mathbb{R} \) by \( f'(t) = \begin{cases} f(t) & \text{if } t \in K \\ 0 & \text{if } t \notin K \end{cases} \). Then \( f \mapsto f' \) is a representation of \( M_R^X(K, \mu_K) \) into \( M_R^X(X, \mu) \). Define now \( \rho'(f) \), for \( f \in M_R^X(K, \mu_K) \), by

\[
\rho'(f)(t) = \begin{cases} \rho(f'(t)) & \text{if } t \in K \cap \rho(K) \\ \chi(f) & \text{if } t \in K - \rho(K). \end{cases}
\]

It is easy to see that \( \rho' \) is a lifting of \( M_R^K(K, \mu_K) \) and that \( \rho'(f)(x) = f(x) \) if \( f \in C_b(K) \) and \( t \in K \cap (A \cup (K - \rho(K))) \). Hence \( \rho' \) is an almost strong lifting of \( M_R^K(K, \mu_K) \) and hence the couple \( (K, \mu_K) \) has the a.s. lifting property.

1.2) It is enough to consider the case \( \mu \neq 0 \). For each \( j \in J \) let \( \rho_j \) be an almost strong lifting of \( M_R^K(K_j, \mu_{K_j}) \) and \( A_j \subset K \) a \( \mu_{K_j} \)-negligible set such that \( \rho_j(f)|_{CA_j} = f|_{CA_j} \) for \( f \in C_b(K_j) \). Let \( \chi \) be a character of \( L^*_R(X, \mu) \). If \( f \in M_R^X(X, \mu) \), then \( f|_{K_j} \in M_R^K(K_j, \mu_{K_j}) \) for each \( j \in J \) and
hence we may define
\[ \rho(f)(t) = \begin{cases} \rho_j(f|K_j)(t) & \text{if } t \in K_j \\ \chi(f) & \text{if } t \in X - \bigcup_{j \in J} K_j. \end{cases} \]

It is easy to see that \( \rho \) is a lifting of \( M_\#(X, \mu) \) and that
\[ \rho(f)(x) = f(x) \text{ if } f \in C_0(X) \text{ and } t \in \left( \bigcup_{j \in J} A_j \right) \cup \left( X - \bigcup_{j \in J} K_j \right). \]
Hence \( \rho \) is an almost strong lifting of \( M_\#(X, \mu) \) and hence the couple \( (X, \mu) \) has the a.s. lifting property.

Remarks. — Theorem 1 is similar to Proposition 2, [7], Chap. viii (in fact it can be easily deduced from this proposition).

3.

If \( \mu \) and \( \nu \) are two positive Radon measures on \( X \) we write \( \mu \prec \nu \) if \( \mu \) is absolutely continuous with respect to \( \nu \) (that is, if \( \mu = \phi \cdot \nu \) with \( \phi : X \to \mathbb{R}_+ \), locally \( \nu \)-integrable). We say that \( \mu \) and \( \nu \) are equivalent if \( \mu \prec \nu \) and \( \nu \prec \mu \). If \( \mu \) and \( \nu \) are equivalent, then \( (X, \mu) \) has the a.s. lifting property if and only if \( (X, \nu) \) has the a.s. lifting property.

Notice that if \( \mu \prec \nu \) then there is \( (K_j)_{j \in J} \in \mathcal{C}(X, \mu) \) such that, for each \( j \in J \), \( \mu_{\#} = (\phi|K_j) \cdot \nu_{K_j} \) and \( \nu_{K_j} \) are equivalent.

In fact if \( \mu \prec \nu \) then \( \mu = \phi \cdot \nu \) with \( \phi : X \to \mathbb{R}_+ \) locally \( \nu \)-integrable. Let \( \Lambda = \{ x|\phi(x) > 0 \} \) and consider a partition of \( \Lambda \) consisting of a \( \mu \)-negligible set \( N \) and a locally countable family of compact sets \( (K_j)_{j \in J} \) such that \( \phi|K_j \) is continuous for each \( j \in J \) (see Corollary 1, Chap. iv, § 5 [3]). If \( J = \{ j \in L|\mu(K_j) > 0 \} \), then \( (K_j)_{j \in J} \in \mathcal{C}(X, \mu) \). Since for each \( j \in J \), \( \mu_{K_j} = (\phi|K_j) \cdot \nu_{K_j} \) and since
\[ 0 < \inf_{x \in K_j} (\phi|K_j)(x) \leq \sup_{x \in K_j} (\phi|K_j)(x) < + \infty, \]
we deduce that \( \mu_{K_j} \) and \( \nu_{K_j} \) are equivalent.

Theorem 2. — Let \( \mu \) and \( \nu \) be two positive Radon measures on \( X \). If \( (X, \nu) \) has the a.s. lifting property and \( \mu \prec \nu \) then \( (X, \mu) \) has the a.s. lifting property \(^3\).

\(^3\) See [1].
Proof. — We have noticed above that there is 

\[(K_j)_{j \in J} \in \mathcal{E}(X, \mu)\]

such that, for each \(j \in J\), \(\mu_{K_j}\) and \(\nu_{K_j}\) are equivalent. By Theorem 1, for each \(j \in J\), \((K_j, \nu_{K_j})\) has the a.s. lifting property, whence \((K_j, \mu_{K_j})\) has the a.s. lifting property. Using again Theorem 1 we deduce that \((X, \mu)\) has the a.s. lifting property.

**Theorem 3.** — Let \(\mu\) and \(\nu\) be two positive Radon measures on \(X\) such that \((X, \mu)\) and \((X, \nu)\) have the a.s. lifting property. Then \((X, \mu + \nu)\) has the a.s. lifting property.

Proof. — Let \(\mu = \mu_a + \mu_s\), where \(\mu_a\) is the absolutely continuous part of \(\mu\) with respect to \(\nu\) and \(\mu_s\) the singular part of \(\mu\) with respect to \(\nu\). Then \n
\[\mu + \nu = (\mu_a + \nu) + \mu_s.\]

Since \(\mu_a + \nu \prec \nu\), the couple \((X, \mu_a + \nu)\) has the a.s. lifting property; since \(\mu_s \prec \mu\), the couple \((X, \mu_s)\) has the a.s. lifting property. Moreover, there are two disjoint universally measurable parts of \(X\), \(X'\) and \(X''\), the union of which is \(X\), such that \(\mu_a + \nu\) is concentrated on \(X'\) and \(\mu_s\) is concentrated on \(X''\).

Let now \((K_j)_{j \in J} \in \mathcal{E}(X, \mu + \nu)\) such that for each \(j \in J\) we have either \(K_j \subset X'\) or \(K_j \subset X''\) and let

\[J' = \{j | K_j \subset X'\}\]
\[J'' = \{j | K_j \subset X''\}.\]

If \(j \in J'\) then \((\mu + \nu)_{K_j} = (\mu_a + \nu)_{K_j}\) so that \((K_j, (\mu + \nu)_{K_j})\) has the a.s. lifting property; if \(j \in J''\) then \((\mu + \nu)_{K_j} = (\mu_s)_{K_j}\) so that \((K_j, (\mu + \nu)_{K_j})\) has again the a.s. lifting property. By Theorem 1, \((X, \mu + \nu)\) has the a.s. lifting property.

**Corollary 1.** — Let \(\mu\) and \(\nu\) be as in the statement of Theorem 3. Then \((X, \inf \{\mu, \nu\})\) and \((X, \sup \{\mu, \nu\})\) have the a.s. lifting property.

Proof. — It is enough to notice that

\[\inf \{\mu, \nu\} \prec \mu + \nu\]
\[\sup \{\mu, \nu\} \prec \mu + \nu.\]
We note before proceeding further that if $\mathcal{F}$ is a filtering set of positive Radon measures on a compact space $X$, bounded above, then there is an increasing sequence $(\mu_n)_{n \in \mathbb{N}}$ of measures belonging to $\mathcal{F}$ such that

$$\sup \mathcal{F} = \sup_{n \in \mathbb{N}} \mu_n$$

(use Theorem 4, Chap. 1, [7]).

If $\lambda = \sup \mathcal{F}$ then $\lambda^*(A) = 0$ if and only if $\mu^*_n(A) = 0$ for every $n \in \mathbb{N}$ (use Proposition 11, Chap. v, § 1, [3]). We also notice that if $(B_n)_{n \in \mathbb{N}}$ is a sequence of parts of $X$ such that $\mu^*_n(B_n) = 0$ for every $n \in \mathbb{N}$, then

$$\lambda^* \left(\limsup_{n \in \mathbb{N}} B_n \right) = 0.$$

In fact it is enough to observe that, for each $p \in \mathbb{N}$

$$\limsup_{n \in \mathbb{N}} B_n \subset \bigcup_{n=p}^{+\infty} B_n$$

and

$$\mu^*_p \left(\bigcup_{n=p}^{+\infty} B_n \right) \leq \sum_{n=p}^{+\infty} \mu^*_n(B_n) \leq \sum_{n=p}^{+\infty} \mu^*_n(B_n) = 0.$$

**Theorem 4.** — Let $\mathcal{F}$ be a set of positive Radon measures on (the locally compact space) $X$, bounded above and let $\lambda = \sup \mathcal{F}$. Suppose that $(X, \mu)$ has the a.s. lifting property for every $\mu \in \mathcal{F}$. Then $(X, \lambda)$ has the a.s. lifting property.

**Proof.** — By Corollary 1, we may suppose that $\mathcal{F}$ is filtering. On the basis of Theorem 1 and the fact that for every compact $K \subset X$,

$$\lambda_K = \sup \{\mu_K | \mu \in \mathcal{F}\}$$

(see Proposition 5, Chap. v, § 5, [3]). It is enough to establish that $(X, \lambda)$ has the a.s. lifting property when $X$ is compact.

We may also assume $\lambda \neq 0$. Let then $(\mu_n)_{n \in \mathbb{N}}$ be an increasing sequence of strictly positive measures belonging to $\mathcal{F}$, such that $\lambda = \sup \mu_n$. For each $n \in \mathbb{N}$ let $\rho_n$ be an almost strong lifting of $M^\omega_\mathbb{R}(X, \mu_n)$ and $A(n)$ a $\mu^*_n$-negligible set such that $\rho_n(f)\mathcal{C}A(n) = f\mathcal{C}A(n)$ for all $f \in C^b_\mathbb{R}(X)$. 

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ finer than the Fréchet filter associated with $\mathbb{N}$. For every $f \in M^\infty_\mathbb{R}(X, \lambda)$ define (\(4\))
\[ \rho(f) = \lim_{n, \mathcal{U}} \rho_n(f). \]
Then $\rho$ is a representation of the algebra $M^\infty_\mathbb{R}(X, \lambda)$ into the algebra $B^\infty_\mathbb{R}(X)$ of all bounded functions on $X$ to $\mathbb{R}$, such that $\rho(1) = 1$. Moreover $f \equiv g(\lambda)$ implies $f \equiv g(\mu_n)$, that is, $\rho_n(f) = \rho_n(g)$ for all $n \in \mathbb{N}$, whence $\rho(f) = \rho(g)$.

Let now $f \in M^\infty_\mathbb{R}(X, \lambda)$ and for each $n \in \mathbb{N}$ let
\[ B(n) = \{ x | \rho_n(f)(x) \neq f(x) \}. \]
Clearly $\rho(f)(x) = f(x)$ for $x \in \limsup_{n \in \mathbb{N}} B(n)$.

Since $\limsup_{n \in \mathbb{N}} B(n)$ is $\lambda^*$-negligible, we deduce $\rho(f) \in M^\infty_\mathbb{R}(X, \lambda)$ and $\rho(f) \equiv f$. Hence $\rho$ is a lifting of $M^\infty_\mathbb{R}(X, \lambda)$. In the same way we see that for every $f \in C^\infty_\mathbb{R}(X)$, $\rho(f)(x) = f(x)$ if $x \in \limsup_{n \in \mathbb{N}} A(n)$. Since $\limsup_{n \in \mathbb{N}} A(n)$ is $\lambda^*$-negligible we conclude that $\rho$ is an almost strong lifting of $M^\infty_\mathbb{R}(X, \mu)$.

Hence $(X, \mu)$ has the a.s. lifting property.

Remark. — By the same method we can prove the following:
Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of positive Radon measures on $X$ and $\lambda$ a positive Radon measure on $X$. Suppose that:

i) $\mu_n \prec \mu_{n+1}$ for all $n \in \mathbb{N}$;

ii) $\lambda^*(A) = 0$ if and only if $\mu^*_n(A) = 0$ for all $n \in \mathbb{N}$.

Then $(X, \lambda)$ has the a.s. lifting property if and only if $(X, \mu_n)$ has the a.s. lifting property for every $n \in \mathbb{N}$.

We shall say that $(X, \mu)$, where $\mu \in \mathcal{M}_b(X)$, has the a.s. lifting property if and only if $(X, |\mu|)$ has the a.s. lifting property. Denote by $\mathcal{U}$ the set of all $\mu \in \mathcal{M}_b(X)$ such that $(X, \mu)$ has the a.s. lifting property. Then:

Theorem 5 (Bichteler). — The set $\mathcal{U}$ is a band of $\mathcal{M}_b(X)$.

Proof. — The assertion follows from Theorems 2, 3 and 4.

(\(4\)) See also [4].
Let $V$ be the set of all positive Radon measures $\mu$ on $X$ such that $(X, \mu)$ has the \textit{strong lifting property} (see Definition 1, Chap. VIII [7]). Clearly $V \subset U$.

\textbf{Corollary 2.} — The set $V$ is a cone of $\mathcal{M}(X)$ having the properties:

j) if $\mu$ and $\nu$ belong to $V$, then $\sup \{\mu, \nu\} \in V$;

jj) if $F \subset V$ is bounded above, in $\mathcal{M}(X)$, then $\sup F \in V$.

\textbf{BIBLIOGRAPHY}


Manuscrit reçu le 28 septembre 1970.

C. Ionescu Tulcea
Northwestern University
Department of Mathematics
Evanston, Illinois 60201 (U.S.A.).

and R. Maher,
Loyola University
Department of Mathematics
Chicago, Illinois 60626 (U.S.A.).