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ON THE MEAN VALUES OF AN ENTIRE FUNCTION REPRESENTED BY DIRICHLET SERIES

by S. K. BAJPAI

1. Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined everywhere in the complex plane by absolutely convergent Dirichlet series

$$f(s) = \sum_{n=0}^{\infty} a_n e^{\lambda_n s}$$

where $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$ as $n \to \infty$.

As usual, the symbols $M(o, r)$, $\mu(o, r)$ and $\nu(o, r)$ denote the maximum modulus, the maximum term and the rank of the maximum term respectively for $f(s)$, and can be found in [8]. We define

$$A_{k}(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^k dt$$

and

$$W_{\lambda, \delta, k}(\sigma) = \lim_{T \to \infty} \frac{e^{-\delta \sigma}}{2T} \int_{-T}^{T} \int_{-T}^{T} |f(x + it)|^k e^{\delta x} dx dt; \quad \delta > 0.$$ 

The mean values for $k = 2$ and $y = 0$ were defined by Hadamard [2] and also by Kamthan ([3], [4]) who has obtained the following main results for $k = 2$ and $y = 0$

$$\lim_{\sigma \to -\infty} \sup_{\sigma} \log \log A_{k}(\sigma) = \rho$$

$$\lim_{\sigma \to -\infty} \sup_{\sigma} \log \log W_{\lambda, \delta, k}(\sigma) = \rho$$

Juneja [6] also proved (1.3) by using a different technique for $k = 2$ and $y = 0$. In [7] another extension of (1.3)
has been obtained for \( k = 2 \) and \( y = -\infty \). Kamthan [5] has further attempted to establish (1.2) for every \( k > 0 \), but his arguments of the proof are vague. Thus, in the present paper, we restrict ourselves to functions of finite Ritt-order \( \rho \), establish (1.2) and extend (1.3) for every \( k > 0 \) by a different technique.

2. We have

**Theorem 1.** — If \( f(s) \), defined by Dirichlet series (1.1), be an entire function of finite Ritt-order \( \rho \), lower order \( \lambda \), type \( T \) and lower type \( t \), then, for \( 0 < k < \infty \)

\[
\rho = \lim_{\sigma \to \infty} \sup_{\sigma} \log \log A_k(\sigma) = \lim_{\sigma \to \infty} \sup_{\sigma} \log \log W_{Y, \delta, k}(\sigma)
\]

and

\[
kT = \lim_{\sigma \to \infty} \sup_{\sigma} \log A_k(\sigma) = \lim_{\sigma \to \infty} \sup_{\sigma} \log W_{Y, \delta, k}(\sigma)
\]

First we prove the following:

**Lemma.** — Let \( f(s) \), given by (1.1), be an entire function of finite Ritt-order \( \rho \), then, for \( 0 < k < \infty \)

\[
\log A_k(\sigma) \sim \log W_{Y, \delta, k}(\sigma) \sim k \log M(\sigma, f).
\]

Proof. — It is well known

\[
|a_n|e^{\sigma \lambda_n} = \left| \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma + it)e^{-\lambda_n t} \, dt \right|
\]

First, let \( k \geq 1 \), then by Holder’s inequality, we have

\[
|a_n|e^{\sigma \lambda_n} \leq C_k \left[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^k \, dt \right]^{1/k}
\]

where

\[
C_k = \begin{cases} 
1 & \text{if } k = 1 \\
4 \left[ \Gamma \left( \frac{1}{2} + \frac{k}{2(k-1)} \right) \right]^{k-1} & \text{if } k > 1.
\end{cases}
\]

\[\sqrt{\pi} \Gamma \left( 1 + \frac{k}{2(k-1)} \right) \]
Further, from (2.2) we have
\[ \{(a_n e^{\lambda_n})^k \} \leq C_k \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(x + it)|^k dt \]
Hence, for \( y \leq 0 \)
\[
(2.3) \quad \frac{\{(a_n e^{\lambda_n})^k \}}{k\lambda_n + \delta} - \frac{e^{-\delta} |a_n| k e^{(k\lambda_n + \delta)y}}{k\lambda_n + \delta} \leq C_k \mathcal{W}_{y, \delta, k}(\sigma) \leq \frac{C_k}{\delta} M^k(\sigma, f)
\]
The case when \( 0 < k < 1 \) may be treated as following:
\[
(2.4) \quad |a_n e^{\lambda_n} \leq C_{k+1} \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^{k+1} dt \right\}^{\frac{1}{k+1}} \leq C_{k+1} \{ M(\sigma, f) \Lambda_k(\sigma) \}^{\frac{1}{k+1}}
\]
Also, from (2.1), for \( 0 < \alpha < 1 \), we have
\[
(2.5) \quad \frac{\{(a_n e^{\lambda_n})^{k+1} \}}{(k+1)\lambda_n + \delta} - \frac{e^{-\delta} |a_n| (k+1)e^{(k+1)(\lambda_n + \delta)y}}{(k+1)\lambda_n + \delta} \leq C_{k+1} \lim_{T \to \infty} \frac{e^{-\delta} \int_{-T}^{T} e^{x\delta} |f(x + it)|^{k+1} dt dx}{2T} \leq C_{k+1}^{k+1} M(\sigma, f) \mathcal{W}_{y, \delta, k}(\sigma) \leq \frac{1}{\delta} C_{k+1}^{k+1} M^{k+1}(\sigma, f)
\]
Since \( f(s) \) is of finite Ritt-order \( \rho \), it follows that \([1], \text{p. } 719), ([9], \text{p. } 265)
\[
(2.6) \quad \log \mu(\sigma, f) \sim \log M(\sigma, f)
\]
and
\[
\log \log \mu(\sigma, f) \sim \log \lambda(\sigma, f)
\]
Hence (2.2), (2.3), (2.4), (2.5) and (2.6) together imply the lemma. Proof of theorem 1, follows immediately from the above lemma.

**Theorem 2.** — If \( f(s) \) is an entire function of finite Ritt-order \( \rho \) and is defined by (1.1) then
\[
\lim_{\sigma \to \infty} \sup \frac{\log \{ W_{x, \delta, k}(\sigma, f')/W_{x, \delta, k}(\sigma, f) \}}{\inf \sigma} = k \rho
\]
provided \( k \) is an even integer.
For \( k = 2 \), this result is due to Juneja [6] and for \( k \) even it follows with the use of Minkowski’s inequality and the
fact that \( W_{r,\delta,k}(\sigma) \) is an increasing convex function of \( \sigma \) for \( k \) even.

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