ROBERT KAUFMAN

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ANALYSIS ON SOME LINEAR SETS
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0.

Let $F$ be a compact subset of $(-\infty, \infty)$ and for each integer $N \geq 1$ let $v_N = v(N; F)$ be the number of intervals $[kN^{-1}, (k + 1)N^{-1}]$ meeting $F$; $F$ is called small provided $\log v_N = o(\log N)$. The existence of small sets of « multiplicity » ($M_0$-sets in [61, p. 344]) was proved in 1942 by Salem and used by Rudin [4, VIII]; a program somewhat analogous for locally compact abelian groups was completed by Varopoulos [5].

Does there exist a small set $F$ with the property that both $F$ and (say) $F^2 = \{x^2 : x \in F\}$ are $M_0$-sets? The construction of these sets doesn’t seem accessible by the method of Rudin and Salem [4], nor by the Brownian motion [3]. In this note an affirmative answer is given to a more general problem.

**Theorem 1.** — Let $(h_n)$ be a sequence of real functions of class $C^1(-\infty, \infty)$ with derivatives $h'_n > 0$. Then there is a small set $F$ with the property that each $h_n(F)$ is an $M_0$-set.

Small sets occur naturally in the construction of independent sets [3, 4, 5]; after the metrical theory of Diophantine approximation a set $F$ is called metrically independent if to each integer $N \geq 1$ and each $\varepsilon$ in $(0, 1)$ there is a $U_0$ so that the simultaneous inequalities

$$\left| \sum_{j=1}^{N} u_j x_j - v \right| < U^{-N-\varepsilon}, \ U = \max(|u_1|, \ldots, |u_N|) > U_0$$

$$|x_i - x_j| \geq \varepsilon \quad \text{for} \quad 1 \leq i < j \leq N$$
have no solution in integers \( u_1, \ldots, u_N, v \) and members \( x_1, \ldots, x_N \) of \( F \). Compare [1, VII].

Uncountable metrically independent subsets could perhaps be constructed by classical arguments, for example that of Perron [1, p. 79] or Davenport [2].

**Theorem 2.** — The set \( F \) determined in Theorem 1 can be required to have the property that each \( h_n(F) \) be metrically independent.

**Theorems 1a, 2a.** — Theorems 1 and 2 remain true provided each \( h_n \) is monotone-continuous and \( h_n' > 0 \) almost everywhere.

1.

In the proof of Theorem 1 we require two arrays of independent random variables \((Y_{k,m})\) and \((\xi_{k,m})\) defined on a space \((\Omega, P)\) for \( 1 \leq k < \infty, 1 \leq m \leq k^2 \). Each \( Y_k \) is uniformly distributed upon \([0, 1]\) while

\[
P(\xi_{k,m} = 1) = \pi_k = k^{-1} = 1 - P(\xi_{k,m} = 0).
\]

Suppose that \( f \) is a measurable function on \( (-\infty, \infty) \) and \(-1 \leq f \leq 1\), and let \( \mu = \pi_k E(f(Y)) \); elementary calculations show that

\[
E(e^{k \xi_k(Y_k)e^{-tu}}) \leq \exp \left( \frac{1}{2} \pi_k t^2 \exp 0(\pi_k t^3) \right)
\]

with an ' 0 ' uniform for \(-1 \leq f \leq 1\), \(-1 \leq t \leq 1\), \(0 \leq \pi_k \leq 1\). Hence for any \( z > 0 \) and \( 1 > t > 0 \)

\[
P(\left| \sum_m \xi_{k,m} - k^5 \right| > zk^5) \leq 2 \exp - zk^5 t \exp \frac{1}{2} k^8 \pi_k t^2 \exp 0(\pi_k k^3).
\]

Choosing \( z = t = k^{-2} \) and using \( \pi_k = k^{-1} \) we obtain

\[
P(\left| \sum_m \xi_{k,m} - k^5 \right| \geq k^3) \leq o(1) \exp - \frac{1}{2} k.
\]

Thus

**Lemma 1.** — \( \sum_{m=1}^{k^4} \xi_{k,m} = k^5 + 0(k^3) \) almost surely in \( \Omega \).

A sequence of random measures \( \lambda_k \) is now determined as
follows: for any function $g$ on $(-\infty, \infty)$
\[
\int g \, d\lambda_k = k^{-2}g(0) + k^{-5} \sum_m \xi_{m,g}(e^{-k \log^2 k} \mathcal{Y}_{k,m}).
\]
Thus in every instance $\lambda_k \geq 0$ and $\|\lambda_k\| \geq k^{-2}$; moreover $\|\lambda_k\| = 1 + O(k^{-2})$ almost surely. Because $\Sigma e^{-k \log^2 k} < \infty$ the convolution $\lambda = \pi * \lambda_k$ converges, and $F$ is defined to be its closed support. $F$ is contained in at most
\[
\prod_{j=1}^{k} [j^5 + O(j^3)] = e^{O(k \log k)}
\]
intervals of length $e^{-k \log^2 k}$.
Because $(k + 1) \log^2 (k + 1)/k \log^2 k \to 1$, this is sufficient to obtain

\textbf{Lemma 2.} — $F$ is almost surely a small set.

\textbf{Lemma 3.} — Let $h \in C^1(-\infty, \infty)$ and $h' > 0$; let $(c_m)$, $(u_m)$, $(v_m)$ be sequences of real numbers such that
\[
|c_m| + |v_m| = O(1) \quad \text{and} \quad |u_m v_m| \to \infty.
\]
Then
\[
\lim_{m \to \infty} \int_0^a \exp \left( iu_m h(c_m + v_m t) \right) dt = 0.
\]

\textbf{Proof.} — Let $g$ denote the $C^1$ function inverse to $h$, and let $v_m > 0$. The integral is transformed to
\[
J = \int_{x_m}^{\beta_m} g'(y) \exp iu_m y \cdot v^{-1}_m \, dy,
\]
where $\alpha_m = h(c_m)$, $\beta_m = h(v_m + c_m)$. A further substitution $y = y_1 + \pi u^{-1}_m$ yields
\[
J = \frac{1}{2} \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \cdot v^{-1}_m \, dy
\]
\[
- \frac{1}{2} \int_{\alpha_m - \pi u^{-1}_m}^{\beta_m - \pi u^{-1}_m} g'(y + \pi u^{-1}_m) \exp iu_m y \cdot v^{-1}_m \, dy.
\]
This tends to 0 because $\beta_m - \alpha_m = O(v_m)$ and $v^{-1}_m u^{-1}_m = o(1)$.

\textbf{Proof of Theorem 1.} — We show that for each function $h_n$
\[
\lim_{n \to \infty} \int \exp iuh_n(s) \lambda \, (ds) = 0,
\]
almost surely. Then $h_n(F)$ is an
Mo-set; because $h_n(F)$ is compact it is enough to prove
\[
\lim_{r \to \infty} \int \exp \frac{1}{r} h_n(s) \lambda (ds) = 0, \quad r = 1, 2, 3, \ldots.
\]

To each integer $r \geq 3$ we attach the integer $k(r)$ defined by $k(r) \leq \log^3 r < k(r) + 1$ and write $\lambda'_k = \prod_{j \neq k} * \lambda_j$. Then
\[
\int \exp \frac{1}{r} h_n(s) \lambda (ds) = \int \int \exp \frac{1}{r} h_n(s + \omega) \lambda_k (ds) \lambda'_k (d\omega).
\]
For each real number $\omega$ in the support of $\lambda'_k$ let $m(\omega)$ be the expected value of $\int \exp \frac{1}{r} h_n(s + \omega) \lambda_k (ds)$. Then
\[
\left| \int \exp \frac{1}{r} h_n(s) \lambda (ds) \right| \leq \left| \int \int \exp \frac{1}{r} h_n(s + \omega) \lambda_k (ds) \right| - m(\omega) | \lambda'_k (d\omega) + \| \lambda'_k \| \max |m(\omega)|.
\]

The second integral, say $I$, can be handled by Jensen's inequality and the estimates at the beginning of 1. Let $-1 < t < 1$ and $\Phi(x) = e^{tx}$. Then
\[
E(\Phi(\| \lambda'_k \|^{-1} k^4 \text{Re} I)) \leq 2 \exp \frac{1}{2} k^4 \log 2 (\exp k^4).
\]
Choosing $t = k^{-\frac{1}{2}}$ we observe
\[
P(\{ |\text{Re} I| > \| \lambda'_k \| \frac{1}{k^2} \}) = P(\{ \Phi(\| \lambda'_k \|^{-1} k^4 \text{Re} I) > \exp k^4 \}) \leq 2 \exp \frac{1}{2} k^4 \exp 0(k^7) \exp - k^4.
\]

This is the general term of a convergent series, inasmuch as $k = k(r) > -1 + \log^3 r$. Thus, almost surely in $\Omega$, for $r > r_0$
\[
|\text{Re} \int \exp \frac{1}{r} h_n(s) \lambda (ds)| \leq k^{-\frac{1}{2}} \| \lambda'_k \| + \| \lambda'_k \| \max |m(\omega)|
\]
and of course a similar statement holds for the imaginary part of the integral. Now\[
|m(\omega)| \leq k^{-2} + \left| \int_0^1 \exp \frac{1}{r} h_n(e^{-k \log^3 t} + \omega) dt \right|
\]
with $\omega = 0(1)$ and $k = k(r)$. To apply Lemma 3 we must
verify \( r^2 e^{-k \log^k k} \to \infty \) but this is plain from \( k(r) < \log^\frac{1}{3} r \). Because \( \max_k \| \lambda_k \| < \infty \) almost surely, the proof of Theorem 1 is complete.

2.

Theorem 2 requires the construction of a random function \( \varphi \) in \( C^\infty(-\infty, \infty) \). Let \( \psi \) be a function in \( C^\infty(-\infty, \infty) \) with the properties

(i) \( \psi = 0 \) on \([-\infty, -2]\), \( \psi = 3 \) on \([2, \infty]\),
(ii) \( \psi' > 0 \), and \( \psi' > 1 \) on \((-1, 1)\).

Let \( (a_p) \) be a sequence of real numbers such that every real number belongs to infinitely many of the intervals \( (a_p - p^{-1}, a_p + p^{-1}) \). Finally, let \( (Z_p) \) be a sequence of independent random variables on \((\Omega, \mathbb{P})\), uniformly distributed upon \([0, 1]\). We define

\[
\varphi(x) = \sum_{p=1}^\infty e^{-\psi(p^{-1}Z_p + p^{\frac{1}{2}}(x - a_p))} + x.
\]

To each compact set \( F \) and number \( \delta > 0 \) there are numbers \( q_1 \) and \( q_2 \) so that

\[
q_1 \geq 4, \quad q_1^{\frac{1}{2}} \delta > 5, \quad \bigcup_{p=1}^{q_1} (a_p - p^{-1}, a_p + p^{-1}) \supseteq F.
\]

**Theorem 3.** — Let \( F \) be a small set and \( h \in C^1(-\infty, \infty) \), \( h' > 0 \); then \( h\varphi(F) \) is almost surely metrically independent.

For each integer \( U \geq 1 \) we can choose a subset \( S(N, U) \) of \( \mathbb{R}^n \) so that every point in \( F^N \) has distance \(< U^{-3N} \) from some point in \( S(N, U) \), while \( \text{card} \ S(N, U) \leq \nu^N(NU^N; F) \).

Beginning with an inequality

\[
\left| \sum_{j=1}^N u_j h\varphi(y_j) - \nu \right| < U^{-n-\varepsilon}, \quad |h\varphi(y_i) - h\varphi(y_i)| > \varepsilon \quad (i \neq j)
\]

we conclude first that \( |y_i - y_j| > \eta \) for some fixed \( \eta > 0 \). Let \( (z_1, \ldots, z_n) \) be the member of \( S(N, U) \) associated to
For large $U$ we can find $\delta < \eta - 2U^{3N}$ and corresponding numbers $q_1, q_2$. Let $q_1 \leq p \leq q_2$, $|z_i - a_p| < p^{-1}$.

\[
\begin{align*}
|p^{-1}Z_p + p^\frac{1}{2}(z_i - a_p)| &< p^{-1} + p^{-\frac{1}{2}} < 1, \\
|p^{-1}Z_p + p^\frac{1}{2}(Z_j - a_p)| &> p^2 \delta - p^{-1} - p^{-\frac{3}{2}} > 4, \text{ when } j \neq i.
\end{align*}
\]

Therefore \[\frac{\partial}{\partial Z_p} \sum_{j=1}^{N} u_j \varphi(Z_j) = u_1 \frac{\partial}{\partial Z_p} \varphi(Z_1)\] exceeds $\alpha|u_i|$ in modulus, with an $\alpha > 0$ independent of $u_1, \ldots, u_N$. Hence the probability of the inequality (1) is $0(U^{-1}.U^{-N-1})$ for each $(z_1, \ldots, z_N)$. The requirement $U = \max(|u_1|, \ldots, |u_N|)$ determines $0(U^{N-1})$ N-tuples and plainly $\nu = 0(U)$. Because $F$ is a small set $\nu^N(NU^{3N}; F) = U^{0(1)}$ as $U \to \infty$. Theorem 3 follows from this and $\sum U^{-1-t}U^{0(1)} < \infty$.

**Proof of Theorem 2.** — Here we use the fact that $F$ and $\varphi$ depend on independent $\sigma$-fields. $F$ is almost surely small, whence each $h_n\varphi(F)$ is almost surely metrically independent, by Theorem 3. By Theorem 1, each $h_n\varphi(F)$ is almost surely an $M_0$-set and Theorem 2 is proved.

**3.**

**Proof of Theorems 1a and 2a.** — According to a theorem of Marcinkiewicz [611, pp. 73-77], to each $\delta > 0$ there exist functions $g_n$ in $C^1(-\infty, \infty)$ so that

\[m(h_n \neq g_n) < \delta n^{-2}, \quad n = 1, 2, 3, \ldots.\]

At almost all points of density of the set $(h_n = g_n), g_n' = h_n' > 0$. Passing to a perfect subset of the set $(g_n' > 0, g_n' = h_n', g_n = h_n)$, we can find a $\tilde{g}_n$ in $C^1(-\infty, \infty)$ such that

\[m(h_n \neq \tilde{g}_n) < 2\delta n^{-2}, \quad n = 1, 2, 3, \ldots,\]

$\tilde{g}_n' > 0$ everywhere.
We observe next that to each $\varepsilon > 0$ there is a constant $B(\varepsilon)$ so that for all Borel sets $S$

$$\int_{\Omega} \lambda(S) \, dP \leq \varepsilon + B(\varepsilon) m(S).$$

Thus to each $\varepsilon > 0$ we can choose functions $\tilde{g}_n$ by Marcin-kiewicz' theorem, so that

$$P\{\lambda(x: \tilde{g}_n \varphi(x) \neq h_n \varphi(x) \text{ for some } n) > \varepsilon \} < \varepsilon.$$

In proving this implication it must be observed that $\varphi$ and $\lambda$ are stochastically independent and $\varphi' > 1$. Writing $G$ for the inner set in the last inequality, we know that $h_n \varphi(G' \cap F) = \tilde{g}_n \varphi(G' \cap F)$ is almost surely metrically independent and that $h_n \varphi(G' \cap F)$ is almost surely an $M_0$-set, if only $\lambda(G' \cap F) > 0$; and this holds for $\|\lambda\| > \varepsilon$ excepting an event of probability $< \varepsilon$. Thus Theorems 1a and 2a are derived from Theorems 1 and 2.

BIBLIOGRAPHY


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Robert Kaufman
Altgeld Hall,
Department of Mathematics,
University of Illinois,
Urbana (Illinois).