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A HEREDITARY PROPERTY IN LOCALLY CONVEX SPACES ⁽¹⁾

by Manuel VALDIVIA

J. Dieudonné has given in [1] the two following theorems :

1) *If F is a subspace, of finite codimension, of a barrelled space E , then F is a barrelled space.*

2) *If F is a subspace, of finite codimension, of a bornological space, then F is a bornological space.*

In this paper we give a theorem analogous to the previous ones, but using infrabarrelled spaces instead of barrelled or bornological spaces. So we shall prove the following theorem :
If F is a subspace, of finite codimension, of an infrabarrelled space E , then F is an infrabarrelled space.

Let K be the field of real or complex numbers. Let E be a locally convex topological vector space over the field K . If \mathfrak{B} is the family of all the absolutely convex, bounded and closed sets of E , we denote with E_B , $B \in \mathfrak{B}$, the linear hull of E with the seminorm associated to B . Let \mathfrak{C} be the topology on E , so that $E[\mathfrak{C}]$ is the inductive limit of the family $\{E_B : B \in \mathfrak{B}\}$.

THEOREM. — *Let F be a subspace of E , with finite codimension. If U is a closed, bornivorous and absolutely convex set of F , then there exists in E an U' , closed, bornivorous and absolutely convex set, such that $U' \cap F = U$.*

In particular, if E is an infrabarrelled space, then F is also an infrabarrelled space.

Proof. — Clearly, the \mathfrak{C} -topology is finer than the initial one on E . On the other hand, for every bounded set A , there exists a set $B \in \mathfrak{B}$, such that $A \subset B$. Hence A is a bounded

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set of E_B , therefore A is a bounded set of $E[\mathcal{C}]$. That is, the bounded sets of E and those of $E[\mathcal{C}]$ are the same.

We denote with $F[\mathcal{C}]$ the subspace F , equipped with the topology induced by \mathcal{C} . Since $E[\mathcal{C}]$ is the inductive limit of seminormed spaces, it is a bornological space and, according to theorem 2), $F[\mathcal{C}]$ is a bornological space. Hence, U is a closed neighborhood of 0 in $F[\mathcal{C}]$.

Clearly, it is sufficient to prove the theorem in the case of F being a vector subspace of E , with codimension one. So that we suppose that F is so.

Two cases are possible :

1° $F[\mathcal{C}]$ being dense in $E[\mathcal{C}]$. Let \bar{U} and \bar{U}^* be the closures of U in E and $E[\mathcal{C}]$ respectively. Since U is a neighborhood of 0 in $F[\mathcal{C}]$, then \bar{U}^* is a neighborhood of 0 in $E[\mathcal{C}]$, hence \bar{U}^* is a bornivorous set in the same space.

Furthermore, $\bar{U} \supset \bar{U}^*$, then \bar{U} is a bornivorous set in E . We can take $U' = \bar{U}$, then U' is a closed, bornivorous and absolutely convex set of E , such that $U' \cap F = U$.

2° $F[\mathcal{C}]$ being closed in $E[\mathcal{C}]$. If $U = \bar{U}$, we take a vector x such that $x \in E$ and $x \notin F$. Let C be the balanced hull of the set $\{x\}$, then $U + C$ is a closed set in E and $U + C$ is a neighborhood of 0 in $E[\mathcal{C}]$, therefore, $U + C$ is bornivorous in E . If we take $U' = U + C$ the theorem is satisfied.

If $U \neq \bar{U}$, \bar{U} is absorbing in E , hence there exists an element $z \in \bar{U}$ such as $z \notin F$. Let D be the balanced hull of $\{z\}$. $U + D$ is a neighbourhood of 0 in $E[\mathcal{C}]$, hence it is bornivorous in E . Furthermore $\bar{U} \supset U$ and $\bar{U} \supset D$, then $2\bar{U} \supset U + D$, hence \bar{U} is bornivorous in E . If we take $\bar{U} = U'$ the theorem is satisfied.

BIBLIOGRAPHY

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