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INFINITELY DIVISIBLE PROCESSES AND THEIR POTENTIAL THEORY (First Part) (1)

by Sidney C. PORT and Charles J. STONE (2)

1. Introduction.

Let $\mathfrak{G}$ be a locally compact, non-compact, second countable Abelian group. An infinitely divisible (i.d.) process X_t on $\mathfrak{G}$ is a spatially homogeneous standard Markov process having states in $\mathfrak{G}$. We will show that associated with every such process is a corresponding potential theory that yields definitive results on the asymptotic behavior of the process in both space and time.

Our results are stated and proved in the general context of an i.d. process on an arbitrary second countable locally compact Abelian group. Most of these results are new when applied to and i.d. process on Euclidean space.

The potential theory we develop for i.d. processes, when applied to Brownian processes (a particular family of i.d. processes), yields that of classical Newtonian potentials for Brownian motion processes on R^d, $d \geqslant 3$ and that of logarithmic potentials for planar Brownian motion. We may therefore view our potential theoretic results as an extension of these classical results to the more general setting of i.d. processes. In our development, both probabilistic and potential

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theoretic, we have been guided on the one hand by the known facts due to Doob, Hunt and Kac about Brownian motion and to Port about stable processes and on the other hand by our previous general results on random walks, which were based in part on earlier work of Spitzer, Kesten, and Ornstein.

Basic notation and concepts used throughout this paper are listed in § 2. The reader should refer to this section while reading the introduction as the need arises.

Given any continuous convolution semi-group μ^t of probability measures on $\mathfrak{G}$ a fundamental theorem (see [2] Chapter I, § 9) on the construction of Markov processes assures us that there is an i.d. process X_t such that

$$P_x(X_t \in A) = \mu^t(A - x).$$

An i.d. process is called non-singular if for some $t > 0$, μ^t has a non-trivial absolutely continuous (with respect to Haar measure) component. Otherwise the process is called singular. As we shall see, the strongest possible results are usually valid for non-singular processes.

A point $x \in \mathfrak{G}$ is called possible if for each open neighborhood N of 0 there is a $t > 0$ such that $\mu^t(N + x) > 0$. The collection Σ of all possible points is a closed sub semigroup of $\mathfrak{G}$. Throughout this paper we assume that the closed group generated by Σ is $\mathfrak{G}$. This assumption entails no loss in generality and is essential to the proper formulation of our results.

The process is called recurrent if

$$G(x, A) = \int_0^\infty P^s(x, A)\, ds = \infty$$

for all non-empty open sets A and all points $x \in \mathfrak{G}$. Otherwise the process is called transient. For transient processes $G(x, A) < \infty$ for all $x \in \mathfrak{G}$ and all relatively compact sets A. Every i.d. process is either transient of recurrent, and for any recurrent process $\Sigma = \mathfrak{G}$. These details can be found in § 4.

The i.d. process $\tilde{X}_t = - X_t$ is called the dual process (to X_t). Quantities referring to this process are prefixed with co-.

In § 3 we gather together various facts of a technical nature that are used throughout the remainder of the paper. Some of these are of intrinsic interest.

Section 4, as mentioned above, gives the details of the classification of an i.d. process as transient or recurrent.

In § 5 we discuss the periodicities of the process and prove some ratio limit theorems. These theorems take their nicest form when the process satisfies

Condition 1. — For some compact set C

$$\limsup_{t\to\infty} (\mu^t(C))^{1/t} = 1.$$

Condition 1 is necessarily satisfied for recurrent processes. It is convenient to let Φ^* denote the functions in Φ (bounded, measurable functions having compact support) if the process is non-singular and the functions in C_c (continuous with compact support) otherwise. In Theorem 5.3 we suppose that Condition 1 holds and let $f \in \Phi^*$ and $g \in \Phi^*$ with $0 \neq J(g) = \int_{\mathfrak{G}} g(x)\, dx$. We show that

$$\lim_{t\to\infty} \frac{\int_t^\infty P^s f(x)\, ds}{\int_t^\infty P^s g(x)\, ds} = \frac{J(f)}{J(g)}$$

or

$$\lim_{t\to\infty} \frac{\int_0^t P^s f(x)\, ds}{\int_0^t P^s g(x)\, dx} = \frac{J(f)}{J(g)}$$

according as the process is transient or recurrent.

For a measurable set B let $T_B = \inf\{t \geqslant 0 : X_t \in B\}$ ($= \infty$ if no such t) denote the first hitting time of B. In § 6 we show that to each B and $\lambda > 0$ there is a unique Radon measure μ_B^λ supported on the closure $\overline{B}$ of B such that

$$E_x(e^{-\lambda T_B})\, dx = \mu_B^\lambda G^\lambda\, (dx).$$

The measure μ_B^λ is called the λ-capacitory measure of B and its total mass $C^\lambda(B)$ is called the λ-capacity of B. The corresponding quantities $\tilde{\mu}_B^\lambda$ and $\tilde{C}^\lambda(B)$ are called the co-λ-capacitory measure and co-λ-capacity of B. The quantity $C^\lambda(\cdot)$ is a Choquet capacity on the Borel sets having the additional properties $C^\lambda(B + x) = C^\lambda(B)$ and $C^\lambda(-B) = C^\lambda(B)$.

For any Borel set B, $C^\lambda(B) = \tilde{C}^\lambda(B)$. We call a Borel set B essentially polar if $P_x(T_B < \infty) = 0$ a.e. and essentially co-polar if $P_x(\tilde{T}_B < \infty) = 0$ a.e. The set B is essentially polar if and only if $C^\lambda(B) = 0$ for some (and hence all) $\lambda > 0$. On the other hand if $C^\lambda(B) > 0$ and $\Sigma = \mathfrak{C}$, then

$$P_x(T_B < \infty) > 0$$

a.e. x and, in the non-singular case, for all x.

The λ-capacity theory developed in § 6 is applied in § 7 to investigate when a one point set is essentially polar. We show that one point sets are not essentially polar if and only if $G^\lambda(0, dx)$ has a bounded density and moreover if this is the case then a point a is regular for $\{a\}$ if and only if $G^\lambda(0, dx)$ has a bounded continuous density. We apply these results to processes on R^d and prove the result (due to Kesten) that one point sets are essentially polar whenever $d \geqslant 2$. We also show that continuous paths having bounded variation are also essentially polar when $d \geqslant 3$.

In § 8 we show that if B is a Borel set, then either $P_x(T_B < \infty) = 1$ a.e. x or $\lim_{t\to\infty} P_x(X_\tau \in B \text{ for some } \tau \geqslant t) = 0$ a.e. Sets of the first type are called recurrent sets while those of the latter type are called transient sets. For a recurrent process every non-essentially polar set is a recurrent set. For transient processes a set can be of either type but $B \in \mathbf{B}$ (the relatively compact Borel sets) is both transient and co-transient. One of the most important results about co-transient sets is that associated with each such set is a unique Radon measure μ_B supported on $\bar{B}$ such that

$$P_x(\tilde{T}_B < \infty)\, dx = \mu_B G\, (dx).$$

The measure μ_B is called the equilibrium measure or capacitory measure of B; its total mass C(B) is called the capacity of B and $C(B) < \infty$ whenever $B \in \mathbf{B}$. In addition, $C(B) = 0$ if and only if B is essentially polar. The measure μ_B can be obtained as the vague limit of the measures μ_B^λ as $\lambda \downarrow 0$ and also as the vague limit of the measures

$$\mu_h\,(dx) = \frac{1}{h}\,[\tilde{\Phi}_B(x) - \tilde{P}^h\tilde{\Phi}_B(x)]\, dx,$$

where $\tilde{\Phi}_B(x) = P_x(\tilde{T}_B < \infty)$. The measures μ_h have the common mass $C(B)$. The set function $C(\bullet)$ is a Choquet capacity on the relatively compact sets and $C(B) = \tilde{C}(B)$ for such sets. For non-relatively compact sets $\tilde{C}(B) = C(B)$ whenever B is both transient and co-transient.

Section 9 is of a technical nature. The class $\mathcal{B}_4$ consists of those $B \in \mathcal{B}$ with non-empty interior having the property $P_x(T_{\bar{B}} = T_{\mathring{B}}) = 1$ a.e. x. In this section we show that functions such as

$$H_B f(x) = E_x[f(X_{T_B}); T_B < \infty] \quad \text{and} \quad G_B f(x) = E_x\left[\int_0^{T_B} f(X_t)\, dt\right]$$

are continuous for a.e. x when $B \in \mathcal{B}_4$ and $f \in C_c$. These results are needed for the work in later sections. We also show that sets with a nice boundary in R^d are in $\mathcal{B}_4$. For an arbitrary $\mathfrak{G}$ we show that given any relatively compact set B we can find $K \supset B$, K compact and $K \in \mathcal{B}_4$.

A transient i.d. process is said to be type I if $\lim_{x\to\infty} Gf(x) = 0$ for every bounded measurable function f having compact support. It is called type II otherwise. In § 10 we first establish the renewal theorem. According to this theorem, the process is type II only if $\mathfrak{G} \cong R \oplus H$ or $\mathfrak{G} \cong Z \oplus H$, where H is compact. We suppose that $\mathfrak{G} = R \oplus H$ or $\mathfrak{G} = Z \oplus H$, Haar measure on $\mathfrak{G}$ being chosen as the direct product of normalized Haar measure on H and Lebesgue measure on R or counting measure on Z. We let ψ denote the projection from $\mathfrak{G}$ to R or Z. We say that $x \to +\infty$ or $x \to -\infty$ according as $\psi(x) \to +\infty$ or $\psi(x) \to -\infty$. With this description, the process is type II transient if and only if $\psi(X_t)$ has finite non-zero mean m. In the type II case if, say, $m > 0$, then for $f \in \Phi^*$

$$\lim_{x\to-\infty} Gf(x) = 0 \qquad \text{and} \qquad \lim_{x\to+\infty} Gf(x) = \frac{J(f)}{m}.$$

Most of Section 10 is devoted to establishing the asymptotic behavior of $H_B f(x)$ for a type II process. Suppose $m > 0$. Then $C(B) \leqslant m$ for any co-transient set B and $C(B) = m$ for any co-transient recurrent set B. In addition, for any $\varphi \in C_c$, the smoothed hitting measure $\int_{\mathfrak{G}} dy\, \varphi(y) H_B(x + y, \bullet)$

converges strongly to the measure $m^{-1}J(\varphi)\mu_B$ as $x \to -\infty$. These smoothed results are the best possible for arbitrary Borel sets and arbitrary transient type II processes. Sharper, unsmoothed, results are obtained for special sets. For example, if $B \in \mathscr{B}_4$, then $H_B(x, \bullet)$ converges weakly to $m^{-1}\mu_B$ as $x \to -\infty$ and in the non-singular case the measure $H_B(x, \bullet)$ converges strongly to $m^{-1}\mu_B$ for any $B \in \mathscr{B}$.

For a Borel set B set

$$E_B(t, A) = \int_{\mathfrak{G}} P_x(T_B \leqslant t, X_{T_B} \in A)\, dx$$

and $E_B(t) = E_B(t, \bar{B})$. In Section 11 we show that for a transient process

$$\lim_{t \to \infty} [E_B(t+h, A) - E_B(t, A)] = h\mu_B(A)$$

for any $A \in \mathscr{B}$ and co-transient set B. In addition, for such sets,

$$\int_{\mathfrak{G}} P_x(T_B < \infty, X_{T_B} \in A) P_x(\tilde{T}_B = \infty)\, dx = t\mu_B(A).$$

For a transient set B and any Borel set A

$$\int_{\mathfrak{G}} P_x(W_B \leqslant t, X_{W_B -} \in A)\, dx = t\tilde{\mu}_B(A),$$

where W_B, the last hitting time of B, is undefined on $[T_B = \infty]$ and defined on $[T_B < \infty]$ by

$$W_B = \sup \{t \geqslant 0 : X_t \in B\}.$$

Sections 12-14 are concerned with the asymptotic behavior of

$$P^t H_B f,\ E_x[f(X_{W_B -}); W_B > t, T_B < \infty]$$

and

$$E_x[f(X_{T_B}); t < T_B < \infty]$$

for large t when X_t is a transient process satisfying Condition 1 and B is a relatively compact set. Let $g \in C_c$, $J(g) = 1$, and set $r(t) = \int_t^\infty (g, P^s g)\, ds$. We show that if $f \in C_c$ and $B \in \mathscr{B}_4$ then

$$P^t H_B f(x) \sim r(t)(\mu_B, f)$$

and

$$E_x[f(X_{W_B -}); W_B > t, T_B < \infty] \sim r(t)(\tilde{\mu}_B, f).$$

For non-singular processes these results hold for any $B \in \mathcal{B}$ and any $f \in \Phi$. For singular processes we obtain results of this type for sets in $\mathcal{B}$ and functions $f \in \Phi$ if we first smooth out on x.

A transient process is called strongly transient if

$$\int_0^\infty r(t)\, dt < \infty.$$

It is called weakly transient if $\int_0^\infty r(t)\, dt = \infty$. For strongly transient processes

$$\int^\infty \mathrm{E}_x[f(\mathrm{X}_{\mathrm{T}_\mathrm{B}});\, t < \mathrm{T}_\mathrm{B} < \infty]\, dt = \mathrm{G}_\mathrm{B}\mathrm{H}_\mathrm{B}f(x)$$

and

$$\lim_{t\to\infty} [\mathrm{E}_\mathrm{B}(t, \mathrm{A}) - t\mu_\mathrm{B}(\mathrm{A})] = \int_{\mathfrak{G}} \mathrm{P}_x(\tilde{\mathrm{T}}_\mathrm{B} < \infty)\mathrm{H}_\mathrm{B}(x, \mathrm{A})\, dx$$

for any $B \in \mathcal{B}$, $A \in \mathcal{B}$ and $f \in \Phi$. For weakly transient processes

$$\int_0^t \mathrm{E}_x[f(\mathrm{X}_{\mathrm{T}_\mathrm{B}});\, \tau < \mathrm{T}_\mathrm{B} < \infty]\, d\tau \sim \left(\int_0^t r(\tau)\, d\tau\right) (\mu_\mathrm{B}, f)$$

and

$$[\mathrm{E}_\mathrm{B}(t;\, \mathrm{A}) - t\mu_\mathrm{B}(\mathrm{A})] \sim \left(\int_0^t r(\tau)\, d\tau\right) \mathrm{C}(\mathrm{B})\mu_\mathrm{B}(\mathrm{A})$$

for sets B in $\mathcal{B}_4$, functions $f \in \mathrm{C}_c$, and sets A such that $\mu_\mathrm{B}(|\partial \mathrm{A}|) = 0$. For non-singular processes we may enlarge the class of sets and functions for which these results are valid. If the process satisfies Condition 1 and also

$$\sup_t (r(t)/r(2t)) < \infty,$$

then these results can be strengthened (by omitting the integration on t in the first result). Examples show however that in general these stronger results need not be true.

For an arbitrary transient i.d. process examples show that in general for f, $g \in \mathrm{C}_c$, and $\mathrm{J}(g) \neq 0$, the ratios $\mathrm{G}f(x)/\mathrm{G}g(x)$ need not have a limit as $x \to \infty$. In Section 15 we first show that these ratio's do have a limit if one goes to infinity along the path of the dual process. More precisely we show that for all $x \in \mathfrak{G}$,

$$\mathrm{P}_x\left[\lim_{t\to\infty} \frac{\mathrm{G}f(-\mathrm{X}_t)}{\mathrm{G}g(-\mathrm{X}_t)} = \frac{\mathrm{J}(f)}{\mathrm{J}(g)}\right] = 1.$$

This result is used to show that for any $B \in \mathscr{B}_4$ and $f \in C_c$

$$P_x\left[\lim_{t\to\infty} E_{-X_t}[f(X_{T_B})|T_B < \infty] = \frac{(\mu_B, f)}{C(B)}\right] = 1$$

and for $g \in C_c$, $J(g) \neq 0$,

$$P_x\left[\lim_{t\to\infty} \frac{G_B f(-X_t)}{Gg(-X_t)} = \frac{1}{J(g)}\int_{\mathfrak{G}} \tilde{P}_x(T_B = \infty)\, f(x)\, dx\right] = 1.$$

We also show that results of this type for arbitrary relatively compact sets hold provided that we first smooth out on the initial point $-X_t$. We also show that for any $B \in \mathbf{B}$ and sets A and C in $\mathbf{B}$ such that $|\partial A| = |\partial C| = 0$ and $|C| > 0$,

$$P_x\left[\lim_{t\to\infty} \frac{G_B(x, A + X_t)}{G(0, C + X_t)} = \frac{|A|}{|C|} P_x(T_B = \infty)\right] = 1.$$

Let $\hat{\mu}^t(\theta)$ denote the characteristic function of the distribution of X_t when $X_0 = 0$. In Section 16 we will show that there is a continuous function $\log \hat{\mu}(\theta)$ which vanishes only at $\theta = 0$ and is such that

$$\hat{\mu}^t(\theta) = e^{t \log \hat{\mu}(\theta)}, \qquad t \geqslant 0 \quad \text{and} \quad \theta \in \hat{\mathfrak{G}}.$$

We will show that the process is transient or recurrent according as

$$\int_Q \Re\left(\frac{1}{\log \hat{\mu}(\theta)}\right) d\theta$$

converges or diverges for a compact neighborhood Q of the origin of $\hat{\mathfrak{G}}$.

In Section 17-22 the process X_t is assumed to be recurrent. In Section 17 we define a collection $\mathscr{F}$ of integrable functions whose Fourier transforms have compact support and which satisfy certain other conditions (described at the beginning of Section 17). Properties of this family of functions were developed in [7]. We let $\mathscr{F}^* = \mathscr{F}$ in general and $\mathscr{F}^* = \Phi$ in the non-singular case. For suitable positive constants $c^\lambda, \lambda > 0$, operators A^λ are defined by

$$A^\lambda f = c^\lambda J(f) - G^\lambda f.$$

We will show that, for $f \in \mathcal{F}^*$, $A^\lambda f$ has a finite limit as $\lambda \downarrow 0$. This limit defines the recurrent potential operator A acting on $\mathcal{F}^*$. Various properties of the operators A^λ and A are obtained in this section. In stating and proving these results we must distinguish between type I and type II recurrent processes. A recurrent process can be type II only if

$$\mathfrak{G} \cong R \oplus H \qquad \text{or} \qquad \mathfrak{G} \cong Z \oplus H.$$

Suppose $\mathfrak{G} = R \oplus H$ or $\mathfrak{G} = Z \oplus H$, Haar measure and ψ being chosen as indicated above in our discussion of type II transient processes investigated in Section 10. Then the recurrent process is type II if and only if $\psi(X_1)$ has mean 0 and finite variance σ^2.

In Section 18 we introduce a classification of the sets in $\mathcal{B}$ corresponding to a recurrent process. $\mathcal{B}_1$ denotes the sets in $\mathcal{B}$ which are not essentially polar. $\mathcal{B}_2$ denotes the sets in $\mathcal{B}$ such that $G_B(x, A)$ is locally integrable for all compact sets A. $\mathcal{B}_3$ denotes those sets in $\mathcal{B}$ such that $G_B(x, A)$ is bounded in x for all compact sets A. Finally $\mathcal{B}_4$ denotes, as discussed above, those sets in $\mathcal{B}$ having a non-empty interior and such that $P_x(T_{\bar{B}} = T_{\mathring{B}}) = 1$ for almost all $x \in \mathfrak{G}$. Then $\mathcal{B} \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3 \supseteq \mathcal{B}_4$. In the non-singular case $\mathcal{B}_1 = \mathcal{B}_3$. We construct an example of a process such that some set in $\mathcal{B}$ having positive measure is not in $\mathcal{B}_2$. Such a set is not essentially polar so that in general $\mathcal{B}_1$ need not equal $\mathcal{B}_2$. We obtain a basic identity for sets $B \in \mathcal{B}_2$:

$$Af(x) - H_B Af(x) = - G_B f(x) + L_B(x) J(f)$$

for $f \in \mathcal{F}^*$ and $x \in \mathfrak{G}$. Here L_B is non-negative, vanishes on B, and is locally integrable. If $B \in \mathcal{B}_3$, then L_B is locally bounded. In the type II case we set

$$L_B^{\pm} = L_B \pm \sigma^2(\psi - H_B\psi).$$

Using the above basic identity we determine the asymptotic behavior of $G_B f_y(x)$ as $y \to \infty$. For $B \in \mathcal{B}_3$ and $f \in \Phi^*$

$$\lim_{y \to \infty} G_B f_y(x) = L_B(x) J(f), \qquad x \in \mathfrak{G},$$

or

$$\lim_{y \to \pm\infty} G_B f_y(x) = L_B^{\pm}(x) J(f), \qquad x \in \mathfrak{G},$$

according as the process is type I or type II. For $B \in {}_2\mathcal{B}$ similar results hold if we smooth out on x.

In Section 19 we investigate the asymptotic behavior of $G_B f(x)$ and $H_B f(x)$ as $x \to \infty$. In stating these results it is convenient to let $\mathcal{B}_1^* = \mathcal{B}_1$ in the non-singular case and $\mathcal{B}_4$ otherwise. If $B \in \mathcal{B}_1^*$ and $f \in \Phi^*$, then

$$\lim_{x \to \infty} G_B f(x) = (f, \tilde{L}_B)$$

or

$$\lim_{x \to \pm\infty} G_B f(x) = (f, \tilde{L}_B^{\pm})$$

according as the process is type I or type II. Similar results hold for $B \in \mathcal{B}_2$ if we smooth out on x. There is an equilibrium probability measure μ_B supported by $\overline{B}$ associated with every $B \in \mathcal{B}_1$. In the type II case there are also two auxiliary probability measures μ_B^+ and μ_B^- such that $\mu_B = (\mu_B^+ + \mu_B^-)/2$. If $B \in \mathcal{B}_1^*$ and $f \in \Phi^*$, then

$$\lim_{x \to \infty} H_B f(x) = (f, \mu_B)$$

or

$$\lim_{x \to \pm\infty} H_B f(x) = (f, \mu_B^{\pm})$$

according as the process is type I or type II. Similar results hold for $B \in \mathcal{B}_1$ if we smooth out on x.

In Section 20 we show that there is a real-valued « Robin's constant » $k(B) < \infty$ associated with all sets in $\mathcal{B}$. Moreover $k(B) > -\infty$ if and only if $B \in \mathcal{B}_2$. In particular, in the non-singular case $k(B) > -\infty$ if and only if B is not essentially polar. The construction described above shows that there are singular processes having sets $B \in \mathcal{B}$ which are not essentially polar but have $-\infty$ for their Robin's constant. The Robin's constant is related to the other potential theoretic quantities. For instance, in the type I case for $B \in \mathcal{B}_1^*$ and $f \in \mathcal{F}^*$

$$\lim_{x \to \infty} (Af(x) - L_B(x)J(f)) = k(B)J(f)$$

and

$$\lim_{x \to \infty} G_B f(x) = (f, \tilde{L}_B) = -k(B)J(f) + (Af, \mu_B).$$

The Robin's constant $k(B)$ depends on B in a nice way. Let

$B \in \mathcal{B}$ and let $B_n \in \mathcal{B}$, $n \geqslant 1$, be such that $B_n \downarrow$ and

$$P_x(T_{B_n} \uparrow T_B \quad \text{as} \quad n \to \infty) = 1 \quad \text{a.e.} \quad x \in \mathfrak{G}.$$

Then

$$\lim_{n\to\infty} k(B_n) = k(B).$$

We also show that $k(B)$ defines a Choquet capacity on $\mathcal{B}$ which is translation invariant and such that

$$\tilde{k}(B) = k(-B) = k(B).$$

In Section 21 we investigate the time dependent behavior of the process (some of the results stated here in this introduction are not proved until Section 22 in the type II case). We show that, for $B \in \mathcal{B}_1$, $E_B(t+s)/E_B(t) \to 1$ as $t \to \infty$ and for B and C both in $\mathcal{B}_1$, $E_B(t)/E_C(t) \to 1$ as $t \to \infty$. If $B \in \mathcal{B}_3$, then

$$\lim_{t\to\infty} \int_0^t P_x(T_B \geqslant s)\, ds/E_B(t) = L_B(x)$$

and if $B \in \mathcal{B}_2$ similar results hold if we smooth out on x. In the type I case we show that for $B \in \mathcal{B}_1^*$ and $f \in \Phi^*$

$$\lim_{t\to\infty} \int_0^t E_x(f(X_{T_B});\, T_B > s)\, ds/E_B(t) = L_B(x)(f, \mu_B).$$

(No corresponding results are obtained in general for sets B in $\mathcal{B}_2$ or even in $\mathcal{B}_3$). We obtain a formula for μ_B, $B \in \mathcal{B}_3$, namely

$$\int_{\mathfrak{G}} \tilde{L}_B(y) P_y(T_B \leqslant t,\, X_{T_B} \in A)\, dy = t\mu_B(A), \quad A \in \mathcal{B}.$$

We show that for $B \in \mathcal{B}_1$

$$\lim_{t\to\infty} \frac{E_B(t, A)}{E_B(t)} = \mu_B(A), \quad A \in \mathcal{B}.$$

Finally we show that for a suitable positive function $g(t)$, $t \geqslant 0$,

$$\lim_{t\to\infty} \int_0^t (E_C(s) - E_B(s))\, ds/g(t) = k(C) - k(B)$$

whenever B and C are both in $\mathcal{B}$ and $k(B)$ and $k(C)$ are not both $-\infty$.

A recurrent process satisfies Condition 2 if there is a $g \in \mathcal{F}^*$, $J(g) = 1$, such that for some α, $1 \leqslant \alpha \leqslant 2$, and some slowly varying function H, $G^\lambda g(x) \sim \lambda^{-1+1/\alpha}H(1/\lambda)$, uniformly in x on compacts. This condition is satisfied for every type II process with $\alpha = 2$ and H the constant function $(2/\pi)^{1/2}\sigma$. On R or Z this condition is satisfied for any process in the domain of attraction of a stable law with exponent α. In Section 22 we show that considerable strengthenings of the results in Section 21 are possible for processes satisfying this condition. As examples of these we show that for every type II process

$$\lim_{t\to\infty} [E_B(t+h, A) - E_B(t, A)]\sqrt{t} = h(2/\pi)^{1/2}\sigma\mu_B(A)$$

for any $B \in \mathcal{B}_1$ and any Borel set A, and that for any set $B \in \mathcal{B}_1^*$ and $f \in \Phi_1^*$

$$\lim_{t\to\infty} E_x[f(X_{T_B}); T_B > t]\sqrt{t} = (2/\pi)^{1/2}(\sigma/2)[\mu_B^+(A)L_B^+(x) + \mu_B^-(A)L_B^-(x)].$$

For any process satisfying Condition 2

$$[E_C(t) - E_B(t)] \sim [k(C) - k(B)]\frac{t^{-1+2/\alpha}}{H(t)^2\Gamma(2/\alpha)}$$

for $C \in \mathcal{B}_2$ and $B \in \mathcal{B}$. In particular, for every type II process,

$$\lim_{t\to\infty} [E_C(t) - E_B(t)] = 2\sigma^2[k(C) - k(B)].$$

We also show in this section that if the process is type II, then for $B \in \mathcal{B}_1$

$$\int_{\mathfrak{G}^\pm} P_x(t < T_B \leqslant t+h,\ X_{T_B} \in A)\,dx \sim h(\sigma/2)(2/\pi)^{1/2}\mu_B^\pm(A)t^{-1/2}.$$

Let $Q_B^t f(x) = E_x[f(X_{T_B}); T_B > t]$. A function f is said to be essentially Q_B^t invariant if for each t, $Q_B^t f = f$ a.e. If $Q_B^t f(x) = f(x)$ for all x, then f is said to be Q_B^t invariant. In Section 23 we first show that every bounded essentially Q_B^t invariant function is of the form $\alpha P_x(T_B = \infty)$ for some constant α. For recurrent processes we show that L_B (and $L_B^\pm$ in the type II case) are essentially Q_B^t invariant functions for sets $B \in \mathcal{B}_2$. For sets in $\mathcal{B}_3$ the only Q_B^t invariant functions that are locally bounded and bounded from below are multiples of L_B (and linear combinations of L_B^+ and L_B^- in the type II case).

Let B be a closed set, not necessarily relatively compact. The process stopped on B has transition operator ${}_B P^t$ given by

$$ {}_B P^t f(x) = Q_B^t f(x) + Ex[f(X_{T_B}); T_B \leqslant t]. $$

We define the operator Δ_B as follows. The domain $D(\Delta_B)$ consists of all measurable functions f such that

$$ \sup_{0 \leqslant t \leqslant 1} \sup_{x \in \mathfrak{G}} \frac{|{}_B P^t f(x) - f(x)|}{t} \leqslant 1 $$

and

$$ \lim_{t \downarrow 0} \frac{{}_B P^t f(x) - f(x)}{t} \quad \text{exists.} $$

For $f \in D(\Delta_B)$ set

$$ \Delta_B f(x) = \lim_{t \downarrow 0} \frac{{}_B P^t f(x) - f(x)}{t}. $$

In particular ${}_\varphi P^t = P^t$, and we set $\Delta_\varphi = \Delta$. In Section 24 we investigate Poisson's equation for Δ and Δ_B. Let $h \in C_C$. Then for transient processes the only continuous solutions of $\Delta f = -h$ that are bounded from below are $f = Gh + r$ where r is bounded from below and $\Delta r = 0$. In particular the only bounded solutions are $f = Gh + \beta$. For recurrent processes in general there are no such solutions. For non-singular processes there is such a solution if and only if $J(h) \leqslant 0$ and in that case the only such solutions are $f = -Ah + \beta$ in the type I case and $f = -Ah - (\alpha J(h)/\sigma^2)\psi + \beta$ in the type II case where $|\alpha| \leqslant 1$. Suppose that $B \neq \emptyset$. Let $C_c(B')$ be the continuous functions having compact support contained in B'. For φ a Borel function that is bounded on B we show that the only bounded solutions of the equation system $\Delta_B f = -h$, $f = \varphi$ on B, $h \in C_c(B')$ are

$$ f(x) = G_B h(x) + H_B \varphi(x) + \alpha P_x(T_B = \infty) $$

for non-singular processes and in the general case every solution coincides a.e. with such a function. In general there are no continuous solutions because the functions $G_B h$, $H_B \varphi$ and $P_x(T_B = \infty)$ need have no continuity properties. We do show however that these functions always possess the following stochastic regularity properties: Let τ_n be stopping times such that $\tau_n \uparrow T_B$ a.s. P_x. Then a.s. P_x, $H_B \varphi(X_{\tau_n}) \to \varphi(X_{T_B})$

on $[T_B < \infty]$, $P_{X_{\tau_n}}(T_B = \infty) \to 1$ on $[T_B = \infty]$, and $G_B h(X_{\tau_n}) \to 0$. A measurable function f is said to be harmonic on the complement B' of a closed set B if for every open set U having compact closure contained in B',

$$f(x) = E_x[f(X_{T_{U'}})].$$

A harmonic function is said to be stochastically regular if for any sequence τ_n of stopping times $\tau_n \uparrow T_B$ a.s. P_x, it is true that $f(X_{\tau_n}) \to f(X_{T_B})$ a.s. P_x on $[T_B < \infty]$ and for some constant α, $f(X_{\tau_n}) \to \alpha$ a.s. P_x on $[T_B = \infty]$. We show that every function f of the form

$$f(x) = H_B\varphi(x) + \alpha P_x(T_B = \infty)$$

for φ bounded on B is a stochastically regular harmonic function and conversely every bounded stochastically regular harmonic function is of this form. Using results from § 25 we show that if B is a compact set such that

$$P_x(X_{T_B} \in B^r | T_B < \infty) = 1 \quad \text{for all} \quad x \in B',$$

then for φ a bounded function that is continuous at each point of B^r, the only bounded harmonic functions f on B' such that $\lim_{x \to r} f(x) = \varphi(r)$ are $f(x) = H_B\varphi(x) + \alpha P_x(T_B = \infty)$.

In Section 25 we show that for arbitrary Borel sets B the functions $G_B h$, $H_B\varphi$ etc., have desirable continuity properties whenever the i.d. process is a strong Feller process, i.e. whenever $P^t f \in C(\mathfrak{G})$ for f a bounded Borel function. Every process such that X_t has a density for each t is such a process. For these processes and for closed sets B we can then find solutions to the equation system $\Delta_B f = -h$, $f = \varphi$ on B that are continuous on $B' \cup (B^r \cap C_\varphi)$ where C_φ denotes the set of continuity points of φ and B^r denotes the regular points of B.

2. Notation.

In this section we introduce the notation and basic concepts that will be used throughout this paper.

$\mathfrak{G}$ will be a fixed locally compact, non-compact Abelian group. The *Borel sets* of $\mathfrak{G}$ are the elements of the minimal

σ-field generated by the open sets. Haar measure on $\mathfrak{G}$ will be denoted by $|\cdot|$ or dx. The phrase almost everywhere (a.e.) will always be with respect to Haar measure, and the phrase essentially will mean except on a set of Haar measure 0.

The complement of a set B will be denoted by B′ or B^c.

For a Borel set B, $T_B = \inf \{t \geqslant 0 : X_t \in B\}$ ($= \infty$ if no such t) and $V_B = \inf \{t > 0 : X_t \in B\}$ ($= \infty$ if no such t). For $T_B < \infty$ we define $W_B = \sup \{t \geqslant 0 : X_t \in B\}$. If $T_B = \infty$ the random time W_B is undefined.

A function f defined on $\mathfrak{G}$ is called universally measurable if for any finite measure γ on $\mathfrak{G}$ there are Borel functions $f_1 \leqslant f_2$ such that $f_1 \leqslant f \leqslant f_2$ and

$$\int_{\mathfrak{G}} (f_2(x) - f_1(x))\, dx = 0.$$

These functions are needed because in general some of the quantities we deal with e.g. $E_x[f(X_{T_B}); T_B < \infty]$ for B a Borel set and f a Borel function are not Borel functions but only universally measurable. We will state and prove our results for Borel functions. In a few instances it will be necessary to apply some of these results to universally measurable functions. In the places where this occurs no difficulty arises and we shall just do so without further explicit mention.

In our work we will need various classes of functions. These are

Φ : All bounded Borel functions having compact support. (The support of f is $\overline{\{x : f(x) \neq 0\}}$.)

$C(\mathfrak{G})$: All bounded continuous function on $\mathfrak{G}$.

$C_0(\mathfrak{G})$: All continuous functions vanishing at ∞.

$C_c(\mathfrak{G})$: All continuous functions having compact support.

$\mathcal{F}$: A certain collection of integrable functions defined in Section 16 whose Fourier transforms have compact support and satisfy some additional technical requirements.

$$\begin{aligned} \Phi^* &= \Phi && \text{if the process is non-singular,} \\ &= C_c && \text{if the process is singular.} \\ \mathcal{F}^* &= \Phi && \text{if the process is non-singular,} \\ &= \mathcal{F} && \text{if the process is singular.} \end{aligned}$$

If χ is any of the above class of functions χ^+ denotes the collection of non-negative functions in χ.

We introduce the notation

$$\begin{aligned} J(f) &= \int_{\mathfrak{G}} f(x)\, dx \\ (f, g) &= \int_{\mathfrak{G}} f(x)g(x)\, dx \\ (\mu, f) &= \int_{\mathfrak{G}} f(x)\mu\,(dx) \\ f_y(x) &= f(x-y). \end{aligned}$$

For $\lambda > 0$ we define operators on bounded Borel functions or non-negative Borel functions as follows:

$$\begin{aligned} G^\lambda f(x) &= \int_0^\infty e^{-\lambda t} P^t f(x)\, dt = E_x \int_0^\infty f(X_t)\, dt \\ A^\lambda f(x) &= J(f)c^\lambda - G^\lambda f(x), \end{aligned}$$

where c^λ is an appropriately choosen positive constant

$$\begin{aligned} H_B^\lambda f(x) &= E_x[e^{-\lambda T_B} f(X_{T_B});\ T_B < \infty] \\ \Pi_B^\lambda f(x) &= E_x[e^{-\lambda V_B} f(X_{V_B});\ V_B < \infty] \\ G_B^\lambda f(x) &= E_x \int_0^{T_B} e^{-\lambda t} f(X_t)\, dt \\ U_B^\lambda f(x) &= E_x \int_0^{V_B} e^{-\lambda t} f(X_t)\, dt. \end{aligned}$$

If any of the above quantities are finite for $\lambda = 0$ we denote that operator by the same symbol without the λ, e.g. $H_B f(x) = H_B^0 f(x)$.

Other operators we will use are:

$$\begin{aligned} Q_B^t f(x) &= E_x[f(X_t);\ T_B > t] \\ {}_B P^t f(x) &= Q_B^t f(x) + E_x[f(X_{T_B});\ T_B \leqslant t] \\ R^t f(x) &= \int_t^\infty P^s f(x)\, ds. \end{aligned}$$

We define

$$\begin{aligned} E_B(t, A) &= \int_{\mathfrak{G}} P_x(T_B \leqslant t,\ X_{T_B} \in A)\, dx \\ E_B(t) &= E_B(t, \overline{B}) \\ e_B^h(t, A) &= E_B(t+h, A) - E_B(t, A) \\ e_B(t) &= e_B^1(t, \overline{B}) \\ E_B^\lambda(A) &= \int_0^\infty e^{-\lambda t} E_B\,(dt, A) = \int_{\mathfrak{G}} H_B^\lambda(x, A)\, dx \\ \mu_B^\lambda(A) &= \lambda E_B^\lambda(A) \\ C^\lambda(B) &= \mu_B^\lambda(\overline{B}) \\ L_B^\lambda(x) &= \lambda c^\lambda \int_0^\infty P_x(T_B > t) e^{-\lambda t}\, dt \end{aligned}$$

where c^λ is the constant that enters into the definition of A^λ.

Some constants associated with sets are

C(B) : The capacity of B. This is defined only for transient processes (see § 8).

k(B) : The Robin's constant of B. This is defined only for recurrent processes (see § 20).

$k^{\pm}$(B) : Constants related to the Robin's constant for type II recurrent processes (see § 20).

μ_B : The equilibrium measure of B. This is defined for transient processes in § 8 and for recurrent processes in § 19.

$\mu_B^{\pm}$: Measures supported on $\overline{B}$ related to the equilibrium measure for type II recurrent processes (see § 19).

L_B and $L_B^{\pm}$: Functions that occur in our study of recurrent processes (see § 18).

Various classes of Borel sets will be used in our work. These are

$\mathcal{B}$: all relatively compact Borel sets.

$\mathcal{B}_1$: all relatively compact sets that are not essentially polar.

$\mathcal{B}_2$: all relatively compact set such that $G_B(x, A)$ is locally integrable for A a compact set.

$\mathcal{B}_3$: all relatively compact sets such that $G_B(x, A)$ is bounded for A a compact set.

$\mathcal{B}_4$: all relatively compact sets having non-empty interior such that $P_x(T_{\overline{B}} = T_{\mathring{B}}) = 1$ a.e. x.

$\mathcal{B}^* = \mathcal{B}$ in the non-singular case and $\mathcal{B}^* = \mathcal{B}_4$ in the singular case.

$\mathcal{B}_1^* = \mathcal{B}_1$ in the non-singular case and $\mathcal{B}_1^* = \mathcal{B}_4$ in genearl.

$\mathcal{A}$: all relatively compact sets whose boundaries have zero Haar measure.

Of all groups $\mathfrak{G}$, two particular compactly generated groups play a distinguished role. These are when $\mathfrak{G}$ is isomorphic to either $R \oplus H$ or $Z \oplus H$, where H is a compact group. In this case we will identify $\mathfrak{G}$ with either $R \oplus H$ or $Z \oplus H$. Let ψ denote the natural projection of $R \oplus H$ onto R or of $Z \oplus H$ onto Z. The i.d. process induced on R or Z is the process $\psi(X_t)$. If $\psi(X_t)$ has finite mean then $E\psi(X_t - X_0) = tm$ for some constant m. Similarly if $\psi(X_t)$ has finite variance then Var $\psi(X_t - X_0) = t\sigma^2$. We set $\mathfrak{G}^+ = \{x : \psi(x) \geqslant 0\}$ and $\mathfrak{G}^- = \{x : \psi(x) < 0\}$.

By $\lim_{x\to\infty} f(x) = f(\infty)$ we mean that given $\varepsilon > 0$ we can find a compact set K such that $|f(x) - f(\infty)| < \varepsilon$ for $x \notin K$. When $\mathfrak{G}$ can be identified with either $R \oplus H$ or $Z \oplus H$ we define $\lim_{x\to\pm\infty} f(x) = f(\pm\infty)$ as $\lim_{x\in\mathfrak{G}^\pm, x\to\infty} f(x)$. We introduce the convention that

$$\mathrm{Lim}_x f(x) = (f(+\infty) + f(-\infty))/2,$$

when $\mathfrak{G}$ is one of our distinguished groups and $\mathrm{Lim}_x = \lim_x$ otherwise.

The process $\tilde{X}_t = -X_t$ is called the dual process. Quantities that refer to this process are denoted by $\tilde{}$. For example the quantity $H_B f$ for the dual process is denoted by $\tilde{H}_B f$. The quantity $E_x\left[\int_0^{T_B} f(X_t)\,dt\right]$ for the dual process is denoted by either $\tilde{E}_x\left[\int_0^{\tilde{T}_B} f(X_t)\,dt\right]$ or $E_x\left[\int_0^{\tilde{T}_B} f(\tilde{X}_t)\,dt\right]$. Quantities that pertain to the dual process are prefixed by co-. For example the quantity $\tilde{C}(B)$, which is the quantity $C(B)$ for the dual process, is called the co-capacity of B.

A point $x \in \mathfrak{G}$ is said to be regular for B if $P_x(V_B = 0) = 1$. The collection of all regular points of B is denoted by B^r. The collection of all co-regular points is denoted by ${}^rB$.

If γ is a bounded measure then the Fourier transform $\hat{\gamma}(\theta)$ of γ is $\hat{\gamma}(\theta) = \int_{\mathfrak{G}} \langle\theta, x\rangle\gamma(dx)$ where θ is a character of $\mathfrak{G}$. The Fourier transform f of a function $f \in L_1(\mathfrak{G})$ is $\hat{f}(\theta) = \int_{\mathfrak{G}} \langle\theta, x\rangle f(x)\,dx$. Haar measure is choosen so that $f(x) = \int_{\hat{\mathfrak{G}}} \langle\theta, x\rangle \hat{f}(\theta)\,d\theta$, whenever f is continuous and $\hat{f}$ is integrable.

3. Preliminaries.

In this section we will gather together some preliminary facts of a technical nature that will be used throughout the sequel.

The transition operator P^t of an i.d. process has the property that $P^t f \in C_0$ if $f \in C_0$. Consequently, by a fundamental result in the construction of Markov processes there is a realization of the process as a standard Markov process. Henceforth X_t will always denote this realization of the process, and in the future we will freely use the properties of standard processes. For a full discussion of the properties of standard processes we refer the reader to [2] Chapter. 1.

The dual process to X_t is the process $\tilde{X}_t = - X_t$. It follows at once that for any two functions f, $g \in \Phi$ or any two non-negative functions that for any $t > 0$,

$$(f, P^t g) = (g, \tilde{P}^t f),$$

and thus for any $\lambda > 0$ ($= 0$ also in the transient case) that $(f, G^\lambda g) = (g, \tilde{G}^\lambda f)$. A slightly deeper duality relation will be given a shatly. These relations are some of the key tools used in our development.

The hitting times T_B and V_B are stopping times. An application of the strong Markov property (valid for any standard process) yields the first passage relations

$$G^\lambda - H_B^\lambda G^\lambda = G_B^\lambda \tag{3.1}$$

and

$$G^\lambda - \Pi_B^\lambda G^\lambda = U_B^\lambda. \tag{3.2}$$

These equations are the Laplace transform versions of the relations

$$P_x(X_t \in A) = \int_0^t \int_{\bar{B}} P_x(T_B \in ds,\ X_s \in dy) P_y(X_{t-s} \in A) + P_x(T_B > t,\ X_t \in A) \tag{3.3}$$

and

$$P_x(X_t \in A) = \int_0^t \int_{\bar{B}} P_x(V_B \in ds,\ X_{V_B} \in dy) P_y(X_{t-s} \in A) + P_x(V_B > t,\ X_t \in A) \tag{3.4}$$

respectively.

A zero-one law for stopping times (see [2], p. 30) asserts that $P_x(V_B = 0) = 1$ or 0. *A point x is called regular* for B if $P_x(V_B = 0) = 1$. Let B^r denote the set of all regular points of B. It is clear that $\dot{B} \subset B^r \subset \bar{B}$. Our next result shows that $B \cap (B^r)^c$ has Haar measure zero.

PROPOSITION. 3.1. — *For any Borel set* B *and any* $t \geqslant 0$, $P_x(V_B \leqslant t) = P_x(T_B \leqslant t), x \notin B$, *and* $P_x(T_B \leqslant t) = P_x(V_B \leqslant t)$ *a.e.* $x \in B$. *In particular,* $P_x(V_B = 0) = P_x(T_B = 0) = 1$ *a.e.* $x \in B$.

Proof. — It is clear that $P_x(V_B \leqslant t) = P_x(T_B \leqslant t)$ for $x \notin B$. To establish the last assertion we proceed as follows.

On the one hand for $h > 0$ and $t > 0$,

$$\int_{\mathfrak{S}} \mathrm{P}^h(x, dy)\mathrm{P}_y(\mathrm{T_B} < t) = \mathrm{P}_x(\mathrm{X}_s \in \mathrm{B} \quad \text{for some} \quad s \in [h, t+h)) \\ \to \mathrm{P}_x(\mathrm{X}_s \in \mathrm{B} \quad \text{for some} \quad s \in (0, t]), \quad h \to 0,$$

and

$\mathrm{P}_x(\mathrm{V_B} < t) \leqslant \mathrm{P}_x(\mathrm{X}_s \in \mathrm{B}$ for some $s \in [0, t]) \leqslant \mathrm{P}_x(\mathrm{V_B} \leqslant t)$ and thus for any $f \in \mathrm{C}_{\mathrm{C}}^+$

$$(3.5) \qquad \int_{\mathfrak{S}} f(x)\mathrm{P}_x(\mathrm{V_B} < t)\, dx \leqslant \lim_{h \downarrow 0} \mathrm{P}_f^h\,(dy)\mathrm{P}_y(\mathrm{T_B} < t) \\ \leqslant \int_{\mathfrak{S}} f(x)\mathrm{P}_x(\mathrm{V_B} \leqslant t)\, dx\cdot$$

On the other hand

$$\int_{\mathfrak{S}} \mathrm{P}_f^h(dy)\mathrm{P}_y(\mathrm{T_B} < t) = \int_{\mathfrak{S}} \mathrm{P}_y(\mathrm{T_B} < t)\tilde{\mathrm{P}}^h f(y)\, dy.$$

Let K be open, $\overline{\mathrm{K}}$ compact, contain the support of f. Then

$$\int_{\mathrm{K^C}} \mathrm{P}_y(\mathrm{T_B} < t)\tilde{\mathrm{P}}^h f(y)\, dy \leqslant \int_{\mathfrak{S}} f(x)\mathrm{P}^h(x, \mathrm{K}^c)\, dx \to 0,\ h \to 0$$

so

$$\lim_{h \downarrow 0} \int_{\mathfrak{S}} \mathrm{P}_y(\mathrm{T_B} < t)\tilde{\mathrm{P}}^h f(y)\, dy = \int_{\mathfrak{S}} \mathrm{P}_y(\mathrm{T_B} < t) f(y)\, dy.$$

Thus from (3.5) and the above computation we see that

$$\int_{\mathfrak{S}} f(x)\mathrm{P}_x(\mathrm{V_B} < t)\, dx \leqslant \int_{\mathfrak{S}} f(x)\mathrm{P}_x(\mathrm{T_B} < t)\, dx \\ \leqslant \int_{\mathfrak{S}} f(x)\mathrm{P}_x(\mathrm{V_B} \leqslant t)\, dx,$$

and thus for any $t \geqslant 0$

$$\int_{\mathfrak{S}} f(x)\mathrm{P}_x(\mathrm{V_B} \leqslant t)\, dx = \int_{\mathfrak{S}} f(x)\mathrm{P}_x(\mathrm{T_B} \leqslant t)\, dx.$$

Since $f \in \mathrm{C}_c^+$ is arbitrary $\mathrm{P}_x(\mathrm{V_B} \leqslant t) = \mathrm{P}_x(\mathrm{T_B} \leqslant t)$ a.e.

PROPOSITION 3.2. — *Let* B *be any Borel set and let* $f \in \Phi^+$. *Then* $\mathrm{P}_f(\mathrm{T_B} \leqslant t) = \mathrm{P}_f(\mathrm{V_B} \leqslant t)$ *is continuous on* $(0, \infty)$.

Proof. — Suppose $\exists t > 0$ such that $\mathrm{P}_f(\mathrm{T_B} = t) = \varepsilon > 0$. Then for any δ, $0 < \delta < t$

$$\varepsilon = \mathrm{P}_f(\mathrm{T_B} = t) \leqslant \int_{\mathfrak{S}} \mathrm{P}_f^{t-\delta}\,(dy)\mathrm{P}_y(\mathrm{T_B} = \delta) \\ = \int_{\mathfrak{S}} \mathrm{P}_y(\mathrm{T_B} = \delta)\tilde{\mathrm{P}}^{t-\delta} f(y)\, dy,$$

and thus $\int_{\mathfrak{S}} P_y(T_B = \delta)\, dy > 0$ for all δ, $0 < \delta < t$. Let K_n be relatively compact with union $\mathfrak{S}$. Since

$$\int_{K_n} P_y(T_B \leqslant t)\, dy \leqslant |K_n| < \infty$$

it can only be that

$$\int_{K_n} P_y(T_B = \delta)\, dy > 0 \quad \text{for countably many } \delta \text{ in } (0, t).$$

Consequently

$$\int_{\mathfrak{S}} dy\, P_y(T_B = \delta) \leqslant \sum_n \int_{K_n} P_y(T_B = \delta)\, dy$$

can only be positive for countably many $\delta \in (0, t)$ a contradiction. Thus $P_f(T_B = t) = 0$ for all $t > 0$, as desired.

We are now in a position to establish the duality relation alluled to above.

PROPOSITION 3.3. — *Let* B *be a Borel set. Then for any two functions* f, $g \in \Phi^+$, *and* $\lambda > 0$

$$(f, G^\lambda_B g) = (g, \tilde{G}^\lambda_B f), \tag{3.6}$$
$$(f, H^\lambda_B G^\lambda g) = (g, \tilde{H}^\lambda_B \tilde{G}^\lambda f), \tag{3.7}$$
$$(f, U^\lambda_B g) = (g, \tilde{U}^\lambda_B f) \tag{3.8}$$

and

$$(f, \Pi^\lambda_B G^\lambda g) = (g, \tilde{\Pi}^\lambda_B \tilde{G}^\lambda f). \tag{3.9}$$

Before proving this proposition we point out that it follows at once from the proposition that it holds for $\lambda = 0$, and for f, g arbitrary non-negative measurable functions, whenever the quantities involved are finite. Also, the following holds.

COROLLARY 3.1. — *Let* f, g *be any two non-negative measurable functions. Then for any* $t > 0$

$$\int_{\mathfrak{S}} \int_{\mathfrak{S}} f(x) P_x(T_B > t, X_t \in dy) g(y)\, dx = \int_{\mathfrak{S}} \int_{\mathfrak{S}} g(y) \tilde{P}_y(T_B > t, X_t \in dx) f(x)\, dy \tag{3.10}$$

and

$$\int_{\mathfrak{S}} \int_{\mathfrak{S}} f(x) P_x(V_B > t, X_t \in dy) g(y)\, dx = \int_{\mathfrak{S}} \int_{\mathfrak{S}} g(y) \tilde{P}_y(V_B > t, X_t \in dx) f(x)\, dy. \tag{3.11}$$

Proof of Corollary. — It suffices to prove (3.10) since $P_x(V_B > t,\ X_t \in A) = P_x(T_B > t,\ X_t \in A)$, $x \notin B \cap (B^r)'$ and $|B \cap (B^r)'| = 0$. Also (3.10) holds for all non-negative f and g if and only if it holds for $f, g \in C_c^+$. But for such f, g, both terms in (3.10) are right continuous in t and thus (3.10) follows from (3.6) by the uniqueness of the Laplace transform.

Proof of Proposition. — The first passage relations show that (3.6) and (3.7) are equivalent as are (3.8) and (3.9). Since $|B \cap (B^r)'| = 0$ it follows that (3.6) and (3.8) are equivalent, so it suffices to establish (3.6). We will first do this for an open set B. Let B be an open set and let $f, g \in \Phi^+$. Then as the paths are right continuous and B is open

$$
\begin{aligned}
(3.12)\quad & \int_{\mathfrak{S}} f(x)\,dx \int_{\mathfrak{S}} P_x(X_s \notin B \text{ for all } s \in (0, t),\ X_t \in dy) g(y) \\
&= \lim_n \int_{\mathfrak{S}} f(x)\,dx \int_{\mathfrak{S}} P_x(X_{tj/n} \notin B \text{ all } j,\ 0 < j < n,\ X_t \in dy) g(y) \\
&= \lim_n \int_{\mathfrak{S}} g(y)\,dy \int_{\mathfrak{S}} P_y(\tilde{X}_{tj/n} \notin B, \text{ all } j,\ 0 < j < n,\ \tilde{X}_t \in dx) f(x) \\
&= \int_{\mathfrak{S}} g(y)\,dy \int_{\mathfrak{S}} P_y(\tilde{X}_s \notin B \quad \text{all} \quad s \in (0, t),\ \tilde{X}_t \in dx) f(x).
\end{aligned}
$$

Also (again because X_t is right continuous and B is open)

$$
\begin{aligned}
[\tilde{X}_s \notin B \quad \text{all} \quad s \in (0, t)] &= [\tilde{X}_s \notin B \quad \text{all} \quad s \in [0, t)], \\
[X_s \notin B \quad \text{all} \quad s \in (0, t)] &= [\tilde{X}_s \notin B \quad \text{all} \quad s \in [0, t)]
\end{aligned}
$$

and

$$
\begin{aligned}
(3.13)\quad & [\tilde{X}_s \notin B \quad \text{all} \quad s \in [0, t]] \\
&= \bigcap_n \left[\tilde{X}_s \notin \quad \text{Ball} \quad s \in \left(0,\ t + \frac{1}{n}\right)\right] \\
(3.14)\quad & [X_s \notin B \quad \text{all} \quad s \in [0, t]] \\
&= \bigcap_n \left[X_s \notin \quad \text{Ball} \quad s \in \left(0,\ t + \frac{1}{n}\right)\right].
\end{aligned}
$$

It now follows from (3.12)-(3.14) that (3.6) holds for B open, and thus (3.7) also holds for B open. We may rewrite (3.7) as

$$
(3.15)\qquad E_f[e^{-\lambda T_B} G^\lambda g(X_{T_B})] = E_g[e^{-\lambda \tilde{T}_B} \tilde{G}^\lambda f(X_{\tilde{T}_B})]
$$

Now let B be any Borel set. By theorem 10.20 of Chapter 1 of [2] there is a decreasing sequence of open sets B_n such that $T_{B_n} \uparrow T_B$ a.e. P_f and $\tilde{T}_{B_n} \uparrow T_B$ a.e. $\tilde{P}_g$. Also

$$G^\lambda g(X_{T_{B_n}}) = \int_{T_{B_n}}^\infty e^{-\lambda t} f(X_t)\, dt \downarrow G^\lambda h(X_{T_B}),\ n \to \infty,$$

and likewise for $\tilde{G}^\lambda f(\tilde{X}_{\tilde{T}_{B_n}})$. Thus, setting $B = B_n$ and passing to the limit we see that (3.7) holds for all B. Since (3.7) and (3.6) are equivalent, (3.6) holds for all Borel sets B. This establishes the proposition.

Another useful relation is the following

Proposition 3.4. — *Let* A, B *be Borel sets and let* $A \subset B$. *Then for any* $f \in \Phi^+$ *and* $\lambda \geqslant 0$

$$H^\lambda_A f(x) = H^\lambda_B(H^\lambda_A f)(x) \tag{3.16}$$
$$G^\lambda_A f = G^\lambda_B f + H^\lambda_B G^\lambda_A f \tag{3.17}$$

Proof. — If $A \subset B$ then $T_A \geqslant T_B$ so $T_A = T_B + T_A . \theta_{T_B}$. Hence

$$\begin{aligned} H^\lambda_A f(x) &= E_x[e^{-\lambda T_A} f(X_{T_A});\ T_A < \infty] \\ &= E_x\{e^{-\lambda T_B} E_{X(T_B)}[e^{-\lambda T_A} f(X_{T_A});\ T_A < \infty];\ T_B < \infty\} \\ &= H^\lambda_B(H^\lambda_A f)(x). \end{aligned}$$

In a similar manner

$$\begin{aligned} G^\lambda_A f(x) &= E_x \int_0^{T_A} e^{-\lambda t} f(X_t)\, dt = E_x \int_0^{T_B} e^{-\lambda t} f(X_t)\, dt \\ &\quad + E_x \int_{T_B}^{T_A} e^{-\lambda t} f(X_t)\, dt = G^\lambda_B f(x) \\ &\quad + E_x \Big\{ e^{-\lambda T_B} E_{X(T_B)}\Big[\int_0^{T_A} e^{-\lambda t} f(X_t)\, dt\Big]\Big\} \\ &= G^\lambda_B f(x) + H^\lambda_B G^\lambda_A f(x). \end{aligned}$$

The following is a useful fact to know.

Proposition 3.5. — *Let* B *be any relatively compact set. Then* $E_x T_{B'} < \infty$.

Proof. — Let K be a compact set such that $B - B \subset K$. Then for any $x \in B$,

$$P_x(X_t \in B) = P_0(X_t \in B - x) \leqslant P_0(X_t \in K),$$

and thus for $x \in B$,

$$P_x(X_t \in B') \geqslant P_0(X_t \in K').$$

Since K is compact there is a $t_0 > 0$ such that

$$P_0(X_{t_0} \in K^C) = \delta > 0$$

and thus $\inf_{x \in B} P_x(X_{t_0} \in B') \geqslant \delta > 0$. Hence for $x \in B$, $P_x(T_{B'} \leqslant t_0) \geqslant P_x(X_{t_0} \in B') \geqslant \delta$ so

$$\sup_{x \in B} P_x(T_{B'} > t_0) \leqslant 1 - \delta.$$

It easily follows that

$$\sup_{x \in B} P_x(T_{B'} > nt_0) \leqslant (1 - \delta)^n$$

and thus for any $x \in B$,

$$E_x T_{B'} \leqslant t_0 \frac{1}{1 - \delta} < \infty.$$

If $x \notin B$ then $P_x(T_B = 0) = 1$ so $E_x T_{B'} = 0$.

The following simple estimates are of frequent use.

PROPOSITION 3.6. — *Let* B *be relatively compact and let* $t > 0$. *Then there is a compact set* K_t *such that*

$$P_x(T_B \leqslant t) \leqslant 2P_x(X_t \in K_t). \tag{3.18}$$

Also there is a compact set K *such that for all* $t > 0$,

$$P_x(T_B \leqslant t) \leqslant 2 \int_0^{t+1} P_x(X_s \in K)\, ds. \tag{3.19}$$

Proof. — Since the paths are bounded we can choose K_t compact such that $P_y(X_s \in K_t \text{ for } s \leqslant t) > \frac{1}{2}$ for $y \in \overline{B}$ and thus

$$\begin{aligned} P_x(X_t \in K_t) &\geqslant \int_0^t \int_{\overline{B}} P_x(T_B \in ds, X_{T_B} \in dy) P_y(X_{t-s} \in K_t) \\ &\geqslant \left(\frac{1}{2}\right) P_x(T_B \leqslant t). \end{aligned}$$

Similarly we can choose K compact such that

$$P_y(X_s \in K \quad \text{for} \quad 0 \leqslant s \leqslant 1) \geqslant \frac{1}{2} \qquad \text{for} \qquad y \in \overline{B}$$

and thus

$$\begin{aligned}\int_0^{t+1} P_x(X_s \in K)\, ds & \\ &\geqslant \int_{\bar{B}_t}\int_t^{t+1} P_x(T_B \leqslant s,\ X_{T_B} \in dy) P_y(X_{t+1-s} \in K)\, ds \\ &\geqslant \left(\frac{1}{2}\right) P_x(T_B \leqslant t).\end{aligned}$$

When the resolvant $G^\lambda(0, dx)$ is absolutely continuous with respect to Haar measure then we expect that the first passage relations should hold for nice versions of the densities of $G^\lambda(x, dy)$ and $G^\lambda_B(x, dy)$. We will spell out these details in the next five propositions. Throughout this discussion $\lambda > 0$ in general and $\lambda \geqslant 0$ in the transient case.

Recall that a non-negative function f is called λ-excessive if $e^{-\lambda t}P^t f \leqslant f$ and $e^{-\lambda t}P^t f \uparrow f$, $t \downarrow 0$.

PROPOSITION 3.7. — *If* $G^\lambda(0, dx) \ll dx$ *then* λ*-excessive functions are lower semi-continuous.*

Proof. — Let $g^\lambda(x)$ be a density of $G^\lambda(0, dx)$. Then if φ is bounded and measurable $G^\lambda\varphi(x)$ is continuous. Indeed,

$$|G^\lambda\varphi(x + x_0) - G^\lambda\varphi(x_0)| \leqslant \|\varphi\|_\infty \int_{\mathfrak{G}} |g^\lambda(y - x) - g^\lambda(y)|\, dy$$

and translations are continuous in the $L_1(\mathfrak{G})$ norm. The assertion now follows from this fact and the fact that given f λ-excessive there is a sequence φ_n of bounded measurable functions such that $G^\lambda\varphi_n \uparrow f$. (See [2] Chapter 2, Proposition 2.6 and Exercise 2.19.)

PROPOSITION 3.8. — *Let* $G^\lambda(0, dx) \ll dx$. *If* f *and* g *are* λ*-excessive and* $f = g$ *a.e. then* $f(x) = g(x)$ *for all* x. *Similarly, if* $f \geqslant g$ a.e. *then* $f(x) \geqslant g(x)$ *for all* x.

Proof. — It follows at once from the resolvent equation that $G^\beta(0, dx) \ll dx$ for all $\beta > 0$. The assertions follow at once from this and the fact that if f is λ-excessive then $\beta G^{\lambda+\beta} f \uparrow f$, $\beta \to \infty$.

PROPOSITION 3.9. — *Assume* $G^\lambda(0, dx) \ll dx$. *Then there is a version* $g^\lambda(x)$ *of the density of* $G^\lambda(0, dx)$ *such that* $g^\lambda(y - x)\, dy = G^\lambda(x, dy)$ *and* $g^\lambda(y - x)$ *is* λ*-excessive in* x, λ*-co-excessive in* y *and* $\tilde{g}^\lambda(x) = g^\lambda(-x)$.

Proof. — By Theorem 1.4 of Chapter 6 of [2] we know that there is a function $u^\lambda(x, y)$ such that

$$G^\lambda(x, dy) = u^\lambda(x, y)\, dy,\ \hat{G}^\lambda(y, dx) = u^\lambda(x, y)\, dx,$$

and $u^\lambda(x, y)$ is λ-excessive in x and λ-co-excessive in y. To establish the proposition we need only show that $u^\lambda(x, y) = u^\lambda(0, y - x)$ for all x and y. To this end note that $u^\lambda(x, y) = u^\lambda(0, y - x)$ a.e. y and $u^\lambda(x, y) = u^\lambda(x - y, 0)$ a.e. x. Thus

$$\begin{aligned}\int_{\mathfrak{G}} u^\beta(x + a, z)u^\lambda(z, y + a)\, dz \\ &= \int_{\mathfrak{G}} u^\beta(0, z - x - a)u^\lambda(z - y - a, 0)\, dz \\ &= \int_{\mathfrak{G}} u^\beta(x, t)u^\lambda(t, y)\, dt.\end{aligned}$$

Since $u^\lambda(., y)$ is λ-excessive it now follows from the above by multiplying through by β and then taking the limit as $\beta \to \infty$ that

$$u^\lambda(x + a, y + a) = u^\lambda(x, y).$$

Thus $g^\lambda(x) = u^\lambda(0, x)$ is the required density.

Note. — In view of Proposition 3.8 $g^\lambda(x)$ is the unique density with the stipulated properties.

PROPOSITION 3.10. — *Assume* $G^\lambda(0, dx) \ll dx$. *Then for any Borel set* B

$$\int_{\bar{B}} \Pi_B^\lambda(x, dz)g^\lambda(y - z) = \int \tilde{\Pi}_B^\lambda(y, dz)\tilde{g}^\lambda(x - z).$$

Proof. — This is Theorem 1.16 of Chapter VI of [2].

PROPOSITION 3.11. — *Assume* $G^\lambda(0, dx) \ll dx$ *and let* B *be any Borel set. Then* $U_B^\lambda(x, dy)$ *has a density* $u_B^\lambda(x, y)$ *such that for all* x *and* y,

$$u_B^\lambda(x, y) = \tilde{u}_B^\lambda(y, x) \tag{3.20}$$

$$g^\lambda(y - x) - \int_{\bar{B}} \Pi_B^\lambda(x, dz)g^\lambda(y - z) = u_B^\lambda(x, y). \tag{3.21}$$

Moreover $u_B^\lambda(x, y) = 0$ *if either* x *is a regular point or* y *is a co-regular point of* B.

Proof. — It is clear that for each fixed x (3.21) holds for a.e. y. We can define $u_B^\lambda(x, .)$ by the left hand side provided we know that it is non-negative for all y. Thus we must show that for all y

$$(3.22) \quad g^\lambda(y-x) \geqslant \int_{\bar{B}} \Pi_B^\lambda(x, dz) g^\lambda(y-z) = \int_{\bar{B}} \Pi_B^\lambda(x, dz) \tilde{g}^\lambda(z-y).$$

We know that (3.22) holds for a.e. y. The function $g^\lambda(y-x)$ is λ-co-excessive in y. Also it is easily checked that for any measure μ the function $\int_{\mathfrak{G}} \tilde{g}^\lambda(z-y)\mu(dz)$ is λ-co-excessive. Thus both sides of (3.22) are λ-co-excessive functions of y. The desired conclusion now follows from Proposition 3.8. Now if x is regular for B, then $\Pi_B^\lambda(x, dz)$ is the unit mass at x so it follows from (3.21) that $u_B^\lambda(x, y) = 0$. By Proposition 3.10 and the fact that $\tilde{\Pi}_B^\lambda(y, dz)$ is the unit mass at y for y a co-regular point of B we see that $u_B^\lambda(x, y) = 0$ for y a co-regular point. Finally (3.20) follows from (3.21) and Proposition 3.10.

When the process X_t is non-singular we know that there is a $t_0 > 0$ and a non-trivial density $p_{t_0}(x)$ such that

$$P^{t_0}(0, dx) = p_{t_0}(x)\, dx + Q^t(0, dx).$$

Since

$$Q^{t+s}(0, \mathfrak{G}) \leqslant Q^t(0, \mathfrak{G}) Q^s(0, \mathfrak{G})$$

it follows that $Q^t(0, \mathfrak{G})$ is decreasing. Since $Q^{t_0}(0, \mathfrak{G}) < 1$ it follows that $Q^{nt_0}(0, \mathfrak{G}) \leqslant [Q^{t_0}(0, \mathfrak{G})]^n \downarrow 0$ and thus

$$(3.23) \qquad \lim_{t\to\infty} \int_{\mathfrak{G}} p_t(x)\, dx = 1.$$

4. Classification of an i.d. process.

In this section we will characterize an i.d. process as being recurrent or transient analogous to the corresponding classification for a random walk.

DÉFINITION 4.1. — *A point $x \in \mathfrak{G}$ is called possible if for each neighborhood* N *of* 0 *there is a* $t > 0$ *such that* $P_0(X_t \in N + x) > 0$. We denote the set of all possible points by Σ.

PROPOSITION 4.1. — *The set* Σ *is a closed sub-semi-group of* $\mathfrak{G}$.

Proof. — Let $x \in \Sigma'$. Then there is a neighborhood N of 0 such that $P_0(X_t \in N + x) = 0$ for all $t > 0$. Let N_1 be a neighborhood of 0 such that $N_1 + N_1 \subset N$ and let $y \in N_1 + x$. Then for any $t > 0$,

$$P_0(X_t \in N_1 + y) \leqslant P_0(X_t \in N + x) = 0,$$

and thus $y \in \Sigma'$. Thus $N_1 + x \in \Sigma'$ so Σ' is open. To see that Σ is a semi-group let N be a neighborhood of 0 and let N_1 be a neighborhood of 0 such that $N_1 - N_1 \subset N$. Then

$$\begin{aligned} P_0(X_{t+s} \in N + x + y) & \\ &\geqslant \int_{N_1+x} P_0(X_t \in dz) P_0(X_s \in N + x + y - z) \\ &\geqslant P_0(X_t \in N_1 + x) P_0(X_s \in N_1 + y). \end{aligned}$$

Thus if x any $y \in \Sigma$ so is $x + y$.

Basic Assumption. Throughout this paper we assume that the group generated by Σ *is* $\mathfrak{G}$. This entails no loss in generality and is essential to the proper formulation of our results.

PROPOSITION 4.2. — *If for some relatively compact open neighborhood* N *of* 0, $G(0, N) < \infty$, *then* $G(x, K) < \infty$ *for all* x *and all compact sets* K. *On the other hand if* $G(0, N) = \infty$ *for all open neighborhoods of* 0, *then* $G(0, N + x) = \infty$ *for all* $x \in \Sigma$.

Proof. — Suppose that $G(0, N) < \infty$ for an open neighborhood N of 0. Let N_1 be an open neighborhood of 0 such that $\overline{N}_1 - \overline{N}_1 \subset N$, where $\overline{N}_1$ is the closure of N_1. Then for any $x \in \mathfrak{G}$,

$$\begin{aligned} G(x, N_1) = \int_{\overline{N}_1} H_{N_1}(x, dz) G(z, N_1) &\leqslant \sup_{z \in \overline{N}_1} G(z, N_1) \\ &\leqslant G(0, \overline{N}_1 - \overline{N}_1) \leqslant G(0, N) < \infty. \end{aligned}$$

Given any compact set K we can cover K by finitely many of the open sets $N_1 - x$. Hence $G(x, K) < \infty$ for all $x \in \mathfrak{G}$ and all compact sets K.

Suppose now that $G(0, N) = \infty$ for all open neighborhoods of 0. Let $x \in \Sigma$. Then for any neighborhood N_1 of 0, $P_0(T_{N_1+x} < \infty) > 0$. Let N_1 be an open neighborhood of 0 such that $\overline{N}_1 - \overline{N}_1 \subset N$. Then

$$\begin{aligned} G(0, N + x) &\geqslant \int_{\overline{N_1+x}} H_{N_1+x}(0, dy) G(0, N + x - y) \\ &\geqslant P_0(T_{N_1+x} < \infty) \inf_{y \in \overline{N_1+x}} G(0, N + x - y) \\ &\geqslant P_0(T_{N_1+x} < \infty) G(0, N_1). \end{aligned}$$

Thus $G(0, N + x) = \infty$.

DÉFINITION 4.2. — *An i.d. process is called transient if* $G(0, N) < \infty$ *for some relatively compact open neighborhood of* 0. *Otherwise the process is called recurrent.*

It follows from Proposition 4.2 that this is a disjunct classification. For general transient processes Σ need not be a group. However for recurrent processes Σ is always a group, and under our basic assumption, $\Sigma = \mathfrak{G}$.

PROPOSITION 4.3. — *For a recurrent process* $\Sigma = \mathfrak{G}$ *and* $G(x, K) = \infty$ *a.e.* x *whenever* $|K| > 0$. *Moreover, a process is recurrent if and only if for every open neighborhood* N *of* 0 *and every* $x \in \mathfrak{G}$, $P_x(X_s \in N$ *for some* $s \geqslant t) = 1$ *for all* $t \geqslant 0$.

Proof. — Let $\mathfrak{R}$ denote the collection of all points $x \in \mathfrak{G}$, such that $P_0(X_s \in N + x$ for some $s \geqslant t) = 1$, for all neighborhoods N of 0 and all $t > 0$. We claim that if $x \in \Sigma$ and $y \in \mathfrak{R}$ then $y - x \in \mathfrak{R}$. To see this suppose it is false. There then exists a $t_0 > 0$ and a neighborhood N of 0 such that $P_0(X_s \notin N + (y - x))$ for all $s \geqslant t_0) > 0$. Let N_1 be an open neighborhood of 0 such that $\overline{N}_1 - \overline{N}_1 \subset N$. Now if $X_t \in N_1 + x$ and $X_s - X_t \notin N + (y - x)$ then $X_s \notin N_1 + y$. Hence if we choose t such that

$$P_0(X_t \in N_1 + x) > 0$$

we see that

$$\begin{aligned} &P_0(X_s \notin N_1 + y \text{ for all } s \geqslant t_0 + t) \\ &\quad \geqslant P_0(X_t \in N_1 + x) P(X_s - X_t \notin N + (y - x) \text{ for all } s \geqslant t + t_0) \\ &\quad = P_0(X_t \in N_1 + x) P_0(X_s \notin N + (y - x) \text{ for all } s \geqslant t_0) > 0. \end{aligned}$$

This contradicts the fact that $y \in \mathfrak{R}$. From this fact it follows at once that either $\mathfrak{R} = \emptyset$ or if $\mathfrak{R} \neq \emptyset$ then $\mathfrak{R}$ is a group and $\mathfrak{R} = \Sigma$. Indeed, if $\mathfrak{R} \neq \varphi$, then we see that if $y \in \mathfrak{R}$ so is $y - y = 0$. Hence also $-y = 0 - y \in \mathfrak{R}$, and thus for any $x \in \Sigma$, $x - 0 = x \in \mathfrak{R}$. Thus we have shown that either $\Sigma = \mathfrak{R}$ or $\mathfrak{R} = \emptyset$.

Suppose that X_t is recurrent. Let N be a neighborhood of 0 and choose the sub neighborhood $N_1 \subset N$ such that $\overline{N}_1 + \overline{N}_1 \subset N$. Since for $x \in N_1$

$$P_x(X_s \in N \quad \text{for all} \quad s \leqslant h) \geqslant P_0(X_s \in N_1 \quad \text{for all} \quad s \leqslant h)$$

and $P(\lim_{s \downarrow 0} X_s = X_0) = 1$ we can choose $h > 0$ so that $P_x(X_s \in N$ for all $s \leqslant h) = \delta > 0$ for all $x \in \overline{N}_1$. But then for $(n-1)h \leqslant t \leqslant nh$,

$$P_0(X_{nh} \in N) \geqslant \int_{N_1} P_0(X_t \in dy) P_y(X_{nh-t} \in N) \geqslant P_0(X_t \in N_1)\delta.$$

Hence

$$\sum_{n=1}^{\infty} P_0(X_{nh} \in N) \geqslant \frac{\delta}{h} G(0, N_1).$$

Thus the random walk X_{nh}, $n \geqslant 0$, is recurrent, and by a well-known result on random walks $P_0(X_{nh} \in N$ for some $n \geqslant n_0) = 1$ for all n_0. But then $P_0(X_t \in N$ for some $t \geqslant t_0) = 1$ for all t_0 and consequently $0 \in \mathfrak{R}$, so $\mathfrak{R} \neq \emptyset$, and therefore $\mathfrak{R} = \Sigma$.

Now suppose that $\mathfrak{R} = \Sigma$ and let N and N_1 be as before. Define stopping times $T_1 < T_2 < \cdots$ as follows:

$$T_1 = \inf \{t > 0 : X_t \in N_1\}$$

and

$$T_{n+1} = \inf \{t > T_n + h : X_t \in N_1\}.$$

Since $P_y(X_s \in N_1$ for some $s \geqslant t) = 1$ for all $y \in \mathfrak{G}$ and all $t \geqslant 0$ it easily follows that

$$P_0(X_s \in N_1 \qquad \text{for some} \qquad s \geqslant T_n + h) = 1$$

for all n. Now

$$E_0\left[\int_{T_n}^{T_n+h} 1_N(X_s)\, ds \,|\, X(T_n) = y\right] = \int_0^h P_y(X_s \in N)\, ds \geqslant h\delta;$$

and thus

$$E_0\left[\int_{T_n}^{T_n+h} 1_N(X_s)\, ds\right] \geqslant h\delta P_0(T_n < \infty) = h\delta.$$

Hence

$$G(0, N) \geqslant \sum_{n=1}^{\infty} \left[E_0 \int_{T_n}^{T_n+h} 1_N(X_s)\, ds \right] \geqslant \sum_{n=1}^{\infty} h\delta = \infty$$

and thus the process X_t is recurrent.

Finally suppose K is compact and $|K| > 0$. Then for any compact set C, $|C| > 0$,

$$(4.1) \quad \int_C G(x, K)\, dx = \int_{\mathfrak{G}} G(0, dy) \int_C 1_K(y + x)\, dx.$$

Since $\int_C 1_K(y + x)\, dx$ is a continuous function it follows from Proposition 4.2 and the fact that $\Sigma = \mathfrak{G}$ in the recurrent case that the right hand side of (4.1) is infinite whenever the process is recurrent. Thus $\int_C G(x, K)\, dx = \infty$ for every compact set C, $|C| > 0$ and thus $G(x, K) = \infty$ a.e. (If not, there is a compact set C and an $N < \infty$ such that $G(x, K) \leqslant N$ for all $x \in C$, and so

$$\int_C G(x, K)\, dx \leqslant N|C| < \infty.)$$

Proposition 4.4. — *The process is transient if and only if for every compact set* B,

$$\lim_{t\to\infty} P_x(X_s \in B \quad \textit{for some} \quad s \geqslant t) = 0,\ x \in \mathfrak{G}.$$

Proof. — If (4.2) holds then by Proposition 4.3 the process is transient. On the other hand if the process is transient then from (3.19) we see that there is a compact set K such that

$$P_x(T_B < \infty) \leqslant 2G(x, K)$$

and so

$$P_x(X_s \in B \quad \text{for some} \quad s \geqslant t) = \int_{\mathfrak{G}} P^t(x, dy) P_y(T_B < \infty) \leqslant 2P^tG(x, K).$$

Since $P^tG(x, K) \downarrow 0$ as $t \to \infty$ the result follows.

5. Periodicities and the ratio limit theorem.

In this section it will be convenient to work with the probability measures μ^t defined by $\mu^t(dx) = P^t(0, dx)$. Then

$\mu^{s+t} = \mu^s * \mu^t$ for $s \geqslant 0$ and $t \geqslant 0$ and μ^t is continuous in t, in the sense of weak convergence. Let S_t be the support of μ^t. Then $\mathfrak{G}$ is generated by

$$\bigcup_{t>0} S_t.$$

PROPOSITION 5.1. — *The groups* $\mathfrak{G}_t$, $t > 0$, *generated by* $S_t - S_t$ *are all equal to some fixed group* $\mathfrak{G}_1$.

Proof. We note first that $S_{s+t} = S_s + S_t$ for all $s, t \geqslant 0$. Thus

$$S_{s+t} - S_{s+t} = S_s - S_s + S_t - S_t.$$

Consequently

$$\mathfrak{G}_s \subseteq \mathfrak{G}_{s+t} \subseteq \mathfrak{G}_s + \mathfrak{G}_t$$

and also $\mathfrak{G}_t \subseteq \mathfrak{G}_{s+t}$. It follows that $\mathfrak{G}_{s+t} = \mathfrak{G}_{\max(s,\, t)}$ and, by a simple induction argument, that $\mathfrak{G}_t$ is independent of t.

From now on $\mathfrak{G}_1$ will denote the group generated by $S_t - S_t$.

PROPOSITION 5.2. — *In the non-singular case* $\mathfrak{G}_1 = \mathfrak{G}$.

Proof. — Let the process be non-singular. Then, for some $t > 0$, μ^t has a density component that is positive on some non-empty open set. Thus some S_t has a non-empty interior and hence some $S_t - S_t$ has a non-empty interior. From this it follows easily that $\mathfrak{G}_1$ is an open subgroup. Now μ^t converges weakly to the probability measure concentrated at the origin as $t \to 0$. Thus for t sufficiently small S_t contains a point of $\mathfrak{G}_1$ and hence $S_t \subseteq \mathfrak{G}_1$. Since $S_{s+t} = S_s + S_t$, s, $t > 0$, it follows that $S_t \subseteq \mathfrak{G}_1$ for all $t \geqslant 0$. Therefore $\mathfrak{G}_1 = \mathfrak{G}$, as desired.

Let Λ denote the annihilator of $\mathfrak{S}_1$.

PROPOSITION 5.3. — *If* $\mathfrak{G}/\mathfrak{G}_1$ *is compact, then* Λ *is countable and there are only a countable number of times* t *such that* S_t *does not generate* $\mathfrak{G}$.

Proof. — Let $\mathfrak{G}/\mathfrak{G}_1$ be compact. Since Λ is isomorphic to $\mathfrak{G}/\mathfrak{G}_1$, it follows that Λ is discrete. Now $\mathfrak{G}$ is second countable and hence so is $\hat{\mathfrak{G}}$. This implies that Λ is countable.

For $t > 0$, S_t fails to generate $\mathfrak{G}$ if and only if there is a $\theta \in \hat{G}$ such that $\theta \neq 0$ and $\hat{\mu}^t(\theta) = 1$. (Here $\hat{\mu}^t$ denotes the characteristic function of μ^t.) This can happen only if $\theta \in \Lambda$. For each such θ there are only a countable number of times t such that $\hat{\mu}^t(\theta) = 1$. Since Λ is countable, the proof is complete.

Set $H = \mathfrak{G}/\mathfrak{G}_1$. Let M be the natural map from $\mathfrak{G}$ to H. Then $M(S_t)$ is a single point. Thus we can define a function $T : [0, \infty) \to H$ by setting $T(t) = M(S_t)$.

Proposition 5.4. — *The function* T *is continuous.*

Proof. — Let U be an open set of H and let $t_0 \in [0, \infty)$ be such that $T(t_0) \in U$. We need only find an $\varepsilon > 0$ such that $T(t) \in U$ for $t \geqslant 0$ and $|t - t_0| \leqslant \varepsilon$.

Let $P = M^{-1}(U)$. Then P is open and $P + \mathfrak{G}_1 = P$. By assumption μ^{t_0} is supported by P. Thus we can find a compact set $C \subseteq P$ such that $\mu^{t_0}(C) > 0$. By continuity of μ^t we can find an $\varepsilon > 0$ such that $\mu^t(P) > 0$ for $t \geqslant 0$ and $|t - t_0| \leqslant \varepsilon$. It follows that μ^t is supported by P for such values of t or, equivalently, that $T(t) \in U$ for $t \geqslant 0$ and $|t - t_0| \leqslant \varepsilon$, as desired.

We can extend T to $(-\infty, \infty)$ by setting $T(-t) = -T(t)$ for $t > 0$. Then T is a continuous homomorphism from R to H.

Proposition 5.5. — $T(R)$ *is dense in* H.

Proof. — Let U be a non-empty open subset of H. We need only prove that there is a t such that $T(t) \in U$.

Set $P = M^{-1}(U)$. Then P is open and $P + \mathfrak{G}_1 = P$. There exist $r \geqslant 0$, $s \geqslant 0$, $x \in S_r$, and $y \in S_s$ such that $x - y \in P$. Set $t = r - s$. Then

$$T(t) = T(r - s) = T(r) - T(s) = M(x) - M(y) = M(x - y) \in U,$$

as desired.

Proposition 5.6. — *Either* H *is compact or* H *is isomorphic to* R.

Proof. — By the previous proposition H is solenoidal and the result follows (see Hewitt and Ross [4, pp. 84-5]).

PROPOSITION 5.7. — *Suppose* H *is not compact. Then* S_t, $t \geqslant 0$, *lies in a closed semigroup of* $\mathfrak{S}$. *Also, for any compact set* C, $\mu^t(C) = 0$ *for* t *sufficiently large.*

Proof. — Since H is isomorphic to R, $T(t)$, $t \geqslant 0$, ranges over a proper closed semigroup in H and hence S_t, $t \geqslant 0$, lies in a proper closed semigroup in $\mathfrak{G}$. We also have (since $H \cong R$) that $T(t) \to \infty$ as $|t| \to \infty$. If C is compact then M(C) is compact and hence $T(t) \cap M(C)$ is empty for $|t|$ sufficiently large. Thus for t sufficiently large $S_t \cap C$ is empty and hence $\mu^t(C) = 0$.

PROPOSITION 5.8. — *Suppose* H *is compact. Then* $T(t)$, $t \geqslant 0$, *ranges over a dense subset of* H.

Proof. — Let S be the closure of the range of $T(t)$, $t \geqslant 0$. Then S is a closed and hence compact sub-semigroup of H. By Hewitt and Ross [4, p. 99] S must be a subgroup of H. Since $T(-t) = -T(t)$ for $t > 0$, S contains the range of $T(t)$, $-\infty < t < \infty$, and hence by Proposition 5.5 S is all of H.

PROPOSITION 5.9. — *For* $t \geqslant 0$ *define the operators* $U(t) : H \to H$ *by* $U(t)h = h + T(t)$. *Then* $U(t)$, $t \geqslant 0$, *is an ergodic semigroup.*

Proof. — Clearly $U(t)$, $t \geqslant 0$, defines a continuous semigroup of invertible measure preserving operators.

Let $I : H \to R$ be a bounded measurable function such that, for all $t \geqslant 0$, $I(h + T(t)) = I(h)$ a.e. h. We want to prove that I is constant a.e.

Let $\{c_k(h),\ h \in H\}$ be a complete orthonormal basis of continuous characters in $\mathcal{L}_2(H)$. Then there are constants a_k such that in $\mathcal{L}_2(H)$ for $t \geqslant 0$.

$$\begin{aligned}\sum_k a_k c_k(h) = I(h) &= I(h + T(t)) \\ &= \sum_k a_k c_k(h + T(t)) \\ &= \sum_k a_k c_k(T(t)) c_k(h).\end{aligned}$$

Consequently $a_k(c_k(T(t)) - 1) = 0, t \geqslant 0$. Thus either $a_k = 0$ or $c_k(T(t)) = 1$, $t \geqslant 0$. In the later case the fact that the range of $T(t)$, $t \geqslant 0$, is dense in H implies that $c_k(h) = 1$, $h \in H$. In other words, I is constant a.e., as desired.

PROPOSITION 5.10. — *Let* H *be compact, let* dh *be normalized Haar measure on* H *and let* f *be a continuous real-valued function on* H. *Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(h + T(s))\, ds = \int_H f(h)\, dh$$

uniformly for $h \in H$. *Moreover*

$$\lim_{t,\tau\to\infty} \frac{1}{\tau} \int_t^{t+\tau} f(T(s))\, ds = \int_H f(h)\, dh.$$

Proof. — The first conclusion of the proposition follows from the previous proposition, the pointwise ergodic theorem and the fact that f, being a continuous function on a compact group, is uniformly continuous. The second conclusion follows from the first since $T(s + t) = T(s) + T(t)$ for $s,\ t \geqslant 0$.

PROPOSITION 5.11. — *Let* H, dh, *and* f *be as in the previous proposition. Let* g *and* h *be continuous bounded non-negative functions on* $[0, \infty)$ *such that* $g(t)$ *is positive for* t *sufficiently large,*

$$\lim_{t\to\infty} \frac{g(s + t)}{g(t)} = 1$$

uniformly for s *in compacts, and*

$$\lim_{t\to\infty} \left(\frac{h(t)}{g(t)} - f(T(t))\right) = 0.$$

Under these conditions

$$\lim_{t\to\infty} \frac{\int_t^\infty h(s)\, ds}{\int_t^\infty g(s)\, ds} = \int_H f(h)\, dh$$

or

$$\lim_{t\to\infty} \frac{\int_0^t h(s)\, ds}{\int_0^t g(s)\, ds} = \int_H f(h)\, dh$$

according as the integral of g *over* $[0, \infty)$ *converges or diverges.*

Proof. — The proof of this proposition is a straightforward application of the second part of the previous proposition. Set $D_t = S_t + \mathfrak{G}_1$, $t \geqslant 0$.

THEOREM 5.1. — *Let* A *be a compact subset of* $\mathfrak{G}_1$ *and* B *an open subset of* $\mathfrak{G}_1$ *such that* $|A|_{\mathfrak{G}_1} > 0$ *and* $0 < |B|_{\mathfrak{G}_1} < \infty$. *Then for any* $0 < \tau < \infty$, $\varepsilon > 0$, *and compact subset* C *of* $\mathfrak{G}$, *there is a* $\delta > 0$ *such that for* t *sufficiently large*

$$\mu^{s+t}(x + y + A)/|A|_{\mathfrak{G}_1} \leqslant (1 + \varepsilon)\mu^t(x + B)/|B|_{\mathfrak{G}_1} + e^{-\delta t}$$

for $x \in D_t$, $y \in D_s \cap C$, *and* $-\tau \leqslant s \leqslant \tau$.

We begin the proof of this result with

LEMMA 5.1. — *The conclusion of Theorem* 5.1 *holds if* s, t *are restricted to integer multiples of any fixed* $\alpha > 0$.

Proof. — This lemma reduces immediately to Theorem 1 of Stone [10].

Let $\alpha > 0$ be fixed. For $t \geqslant 0$ set $t^+ = \min\,[n\alpha | n\alpha \geqslant t]$ and $t^- = \max\,[n\alpha | n\alpha \leqslant t]$.

LEMMA 5.2. — *Let* A *be a compact subset of* $\mathfrak{G}_1$ *such that* $|A|_{\mathfrak{G}_1} > 0$ *and let* $\varepsilon > 0$. *Then for sufficiently small* $\alpha > 0$ *there is a compact subset* A_1 *of* $\mathfrak{G}_1$ *such that* $|A_1|_{\mathfrak{G}_1} > 0$ *and a compact subset* C *of* $\mathfrak{G}$ *such that* $D_{t^+-t} \cap C \neq \emptyset$, $t \geqslant 0$, *and for* t *sufficiently large*

$$\mu^t(x + A)/|A|_{\mathfrak{G}_1} \leqslant (1 + \varepsilon)\mu^{t^+}(x + y + A_1)/|A_1|_{\mathfrak{G}_1}$$

for $x \in D_t$ *and* $y \in D_{t^+-t} \cap C$.

Proof. — There is a compact subset A_1 of $\mathfrak{G}_1$ such that $A \subseteq \mathring{A}_1$ and

$$|A_1|_{\mathfrak{G}_1} = (1 + \varepsilon)^{\frac{1}{2}}\,|A|_{\mathfrak{G}_1}.$$

The conclusion of the lemma now follows easily from the fact that μ^t converges weakly as $t \to 0$ to a probability measure concentrated at the origin.

LEMMA 5.3. — *Let* B *be an open subset of* $\mathfrak{G}_1$ *such that* $0 < |B|_{\mathfrak{G}_1} < \infty$ *and let* $\varepsilon > 0$. *Then for sufficiently small* $\alpha > 0$ *there is an open subset* B_1 *of* $\mathfrak{G}_1$ *such that*

$0 < |B_1|_{\mathfrak{G}_1} < \infty$ *and a compact subset* C *of* $\mathfrak{G}$ *such that* $D_{t-t^-} \cap C \neq \emptyset$, $t \geqslant 0$, *and for* t *sufficiently large*

$$\mu^{t^-}(x + y + B_1)/|B_1|_{\mathfrak{G}_1} \leqslant (1 + \varepsilon)\mu^t(x + B)/|B|_{\mathfrak{G}_1}$$

for $x \in D_t$ *and* $-y \in D_{t-t^-} \cap C_1$.

Proof. — There is a relatively compact open subset B_1 of $\mathfrak{G}_1$ such that $\overline{B}_1 \subseteq B$ and

$$|B|_{\mathfrak{G}_1} \leqslant (1 + \varepsilon)^{\frac{1}{2}}|B_1|_{\mathfrak{G}1}.$$

The conclusion of the lemma again follows easily from the fact that μ^t converges weakly as $t \to 0$ to a probability measure concentrated at the origin.

Proof of Theorem 5.1. — The theorem follows easily from Lemmas 5.1-5.3.

We will use Theorem 5.1 only when the process satisfies

Condition 1. — For some compact set C

$$\limsup_{t\to\infty} (\mu^t(C))^{1/t} = 1.$$

By Proposition 5.7 we see that if Condition 1 holds, then $\mathfrak{G}/\mathfrak{G}_1$ is compact. It follows from Proposition 1 of Stone [13] that for sufficiently large compact sets C

$$D_t \cap C \neq \emptyset, \qquad t \geqslant 0.$$

In the next several results $x_t \in D_t$, $t \geqslant 0$ and the x_t's all lie in some fixed compact set.

PROPOSITION 5.12. — *Suppose Condition* 1 *holds and let* B *be a non-empty open subset of* $\mathfrak{G}_1$. *Then*

$$\lim_{t\to\infty} (\mu^t(x_t + B))^{1/t} = 1.$$

Proof. — Let B_1 be a non-empty relatively compact open subset of $\mathfrak{G}_1$ such that $\overline{B}_1 \subseteq B$. Let C_1 be a compact set containing all x_t, $t \geqslant 0$. Let C be a compact set containing $(C_1 - C_1 - B_1) \cup C_1$. By Theorem 5.1 for any $\delta > 0$ there is a $t_0 > 0$ such that

$$\mu^{t_0}(B_1 + y) \geqslant e^{-\delta t_0}, \qquad y \in D_{t_0} \cap C.$$

Consequently

$$\mu^{kt_0}(x_{kt_0} + B_1) \geqslant \int_{x_{(k-1)t_0}+B_1} \mu^{(k-1)t_0}(dz)\mu^{t_0}(x_{kt_0} - z + B_1)$$
$$\geqslant e^{-\delta t_0}\mu^{(k-1)t_0}(x_{(k-1)t_0} + B_1).$$

Thus by induction

$$\mu^{kt_0}(x_{kt_0} + B_1) \geqslant e^{-k\delta t_0}, \qquad k = 1, 2, \ldots.$$

It now follows from Theorem 5.1 that

$$\liminf_{t\to\infty} (\mu^t(x_t + B))^{1/t} \geqslant e^{-\delta}.$$

Since δ can be made arbitrarily small the proof of the proposition is complete.

From Theorem 5.1 and Proposition 5.12 we obtain immediately

Proposition 5.13. — *Suppose Condition* 1 *holds and let* A *and* B *be respectively compact and non-empty open subsets of* $\mathfrak{G}_1$. *Then*

$$\limsup_{t\to\infty} \frac{\mu^{s+t}(x_{s+t} + A)}{\mu^t(x_t + B)} \leqslant \frac{|A|}{|B|}$$

uniformly for s *in compacts.*

From this proposition we obtain immediately

Theorem 5.2. — *Suppose Condition* 1 *holds and let* A *and* B *be relatively compact sets in* $\mathfrak{G}_1$ *such that* $|\partial A|_{\mathfrak{G}_1} = |\partial B|_{\mathfrak{G}_1} = 0$ *and* $|B|_{\mathfrak{G}_1} > 0$. *Then*

$$\lim_{t\to\infty} \frac{\mu^{s+t}(x_{s+t} + A)}{\mu^t(x_t + B)} = \frac{|A|_{\mathfrak{G}_1}}{|B|_{\mathfrak{G}_1}}$$

uniformly for s *in compacts.*

Let f_0 denote a continuous non-negative function on $\mathfrak{G}_1$ having compact support and such that

$$\int_{\mathfrak{G}_1} f_0(x)\, dx = 1$$

(where dx here represents Haar measure on $\mathfrak{G}_1$). Set

$$g(t) = \int \mu^t(dy) f_0(y - x_t), \qquad t \geqslant 0.$$

If Condition 1 holds then by Theorem 5.2,

$$\lim_{t\to\infty} \frac{g(s+t)}{g(t)} = 1$$

uniformly for s in compacts. From Theorem 5.2 we also have

PROPOSITION 5.14. — *Suppose Condition 1 holds and let f be a continuous function on $\mathfrak{G}_1$ having compact support. Then*

$$\lim_{t\to\infty} \frac{1}{g(t)} \int_{x_t+\mathfrak{G}_1} \mu^t (dy) f(y - x_t) = \int_{\mathfrak{G}} f(x)\, dx.$$

From this proposition we have

PROPOSITION 5.15. — *Suppose Condition 1 holds. Let F be a collection of continuous functions on $\mathfrak{G}_1$ such that the functions in F are uniformly bounded, equicontinuous, and supported by a common compact set. If $f_t \in \mathrm{F}$ for $t \geqslant 0$, then*

$$\lim_{t\to\infty} \left(\frac{1}{g(t)} \int_{x_t+_1\mathfrak{G}} \mu^t (dy) f_t(y - x_t) - \int_{\mathfrak{G}_1} f_t(x)\, dx \right) = 0.$$

Let φ be a continuous function on $\mathfrak{G}$ having compact support. Then as x ranges over a compact subset of $\mathfrak{G}$ the family F of functions $\varphi(x+y)$, $y \in \mathfrak{G}_1$, satisfies the conditions of Proposition 5.15. Thus from that proposition we obtain immediately

PROPOSITION 5.16. — *Let Condition 1 hold and let φ be a continuous function in $\mathfrak{G}$ having compact support. Then*

$$\lim_{t\to\infty} \left(\frac{1}{g(t)} \int \mu^{(t)} (dy) \varphi(y) - \int_{\mathfrak{G}_1} \varphi(y + x_t)\, dy \right) = 0.$$

We now wish to apply Proposition 5.11 to the above result. Let φ be a continuous function on $\mathfrak{G}$ having compact support. We can define a function f on H by setting

$$f(h) = \int_{\mathfrak{G}_1} \varphi(y + x)\, dy \qquad \text{if} \qquad h = \mathrm{M}(x).$$

Then f is well defined and continuous on H. Furthermore the functional I defined by

$$\mathrm{I}\varphi = \int_{\mathrm{H}} f(h)\, dh$$

is a non-trivial translation invariant non-negative linear functional on the continuous functions of $\mathfrak{G}$ having compact support. Thus for some positive number c we have

$$\int_{\mathrm{H}} f(h)\, dh = c \int_{\mathfrak{G}} \varphi(x)\, dx.$$

Finally we observe that

$$\int_{\mathfrak{G}_1} \varphi(y + x_t)\, dy = f(\mathrm{T}(t)), \qquad t \geqslant 0.$$

Therefore by Propositions 5.11 and 5.16 we have

PROPOSITION 5.17. — *Suppose Condition* 1 *holds and let* φ_1 *and* φ_2 *be continuous functions having compact support and such that* $\mathrm{J}(\varphi_2) \neq 0$. *Then*

$$\lim_{t\to\infty} \frac{\int_t^\infty (\mu^s, \varphi_1)\, ds}{\int_t^\infty (\mu^s, \varphi_2)\, ds} = \frac{\mathrm{J}(\varphi_1)}{\mathrm{J}(\varphi_2)}$$

or

$$\lim_{t\to\infty} \frac{\int_0^t (\mu^s, \varphi_1)\, ds}{\int_0^t (\mu^s, \varphi_2)\, ds} = \frac{\mathrm{J}(\varphi_1)}{\mathrm{J}(\varphi_2)}$$

according as the process is transient or recurrent.

In the non-singular case $\mathfrak{G}_1 = \mathfrak{G}$ and the discrete time results of Stone [10] are easily extended to continuous time. In particular we have

PROPOSITION 5.18. — *Suppose Condition* 1 *holds and the process is non-singular. Let* $\mathrm{A} \in \mathfrak{B}$ *and* $\mathrm{B} \in \mathfrak{B}$ *with* $|\mathrm{B}| > 0$. *Then*

$$\lim_{t\to\infty} \frac{\mu^{s+t}(x + \mathrm{A})}{\mu^t(y + \mathrm{B})} = \frac{|\mathrm{A}|}{|\mathrm{B}|}$$

and the convergence is uniform for x *and* y *in compact subsets of* $\mathfrak{G}$ *and* s *in compact subsets of* $(-\infty, \infty)$.

Proof. — Let $\mathrm{A} \in \mathfrak{B}$. Then for $s \geqslant 0$

$$\int_{\mathfrak{G}} \mu^s(\mathrm{A} - x)\, dx = |\mathrm{A}|.$$

For any $0 \leqslant s_0 < \infty$ and $\varepsilon > 0$ there is a compact set C such that

$$\int_{C} \mu^{s}(A - x)\, dx \geqslant |A| - \varepsilon, \qquad 0 \leqslant s \leqslant s_0.$$

Using these results we easily reduce Proposition 5.18 to the corresponding discrete time result, Corollary 5 of [10].

Let φ be a continuous function in $\mathfrak{G}$ having compact support. Then as x ranges over a compact the collection $\{\varphi_x\}$ is uniformly bounded, equicontinuous, and has common compact support. Thus from Proposition 5.17 and 5.18 we obtain

THEOREM 5.3. — *Suppose Condition 1 holds and let $f \in \varphi^*$ and $g \in \Phi^*$ with $J(g) \neq 0$. If the process is transient, then*

$$\lim_{t\to\infty} \frac{\int_t^\infty P^s f(x)\, ds}{\int_t^\infty P^s g(y)\, ds} = \frac{J(f)}{J(g)}$$

uniformly for x and y in compacts. If the process is recurrent, then

$$\lim_{t\to\infty} \frac{\int_0^t P^s f(x)\, ds}{\int_0^t P^s g(y)\, ds} = \frac{J(f)}{J(g)}$$

uniformly for x and y in compacts.

Closely related to ratio limit theorems are local limit theorems. We will assume that $\mathfrak{G}$ is a closed subgroup of Euclidean space R^d. For simplicity we will also assume that

$$\mathfrak{G} = Z^{d_1} \oplus R^{d-d_1}$$

and Haar measure on $\mathfrak{G}$ is chosen as the product of counting measure on Z^{d_1} and Lebesgue measure on R^{d-d_1}.

THEOREM 5.4. — *Let $\mathfrak{G}$ be a closed subgroup of R^d normalized as indicated above. Suppose there is a continuous strictly positive function B_t, $t \geqslant 0$, such that $B_t^{-1}X_t$ is asymptotically distributed as a stable distribution having density p. Let $f \in \Phi^*$. In the transient case*

$$\lim_{t\to\infty} \int_t^\infty P^s f(x)\, ds \Big/ \int_t^\infty B_s^{-d}\, ds = p(0)J(f)$$

uniformly for x *in compacts. In the recurrent case*

$$\lim_{t\to\infty} \int_0^t \mathrm{P}^s f(x)\, ds \Big/ \int_0^t \mathrm{B}^{-d}\, ds = p(0)\mathrm{J}(f)$$

uniformly for x *in compacts.*

Proof. — By arguing as in the usual local limit theorems (Stone [10], [11], [12]) one can show that for an appropriate positive constant c_1 as $t \to \infty$

$$\mathrm{P}^t f(x) = c_1 p(0) \mathrm{B}_t^{-d} \int_{\mathfrak{G}_1} f(x + x_t + y)\, dy + o(\mathrm{B}_t^{-d})$$

uniformly for x in compacts. It is necessarily true that

$$\lim_{t\to\infty} \frac{\mathrm{B}_{s+t}}{\mathrm{B}_t} = 1$$

uniformly for s in compacts. It now follows from the ergodic theorem that as $t, \tau \to \infty$ for an appropriate positive constant c_2

$$\int_t^{t+\tau} \mathrm{P}^s f(x)\, ds = c_2 p(0)\mathrm{J}(f) \int_t^{t+\tau} \mathrm{B}_s^{-d}\, ds + o\left(\int_t^{t+\tau} \mathrm{B}_s^{-d}\, ds\right)$$

uniformly for x in compacts. By the same methods one can show that as $t, \tau \to \infty$

$$\int_t^{t+\tau} \mathrm{P}^s f_y(0)\, ds = c_2 \int_t^{t+\tau} \mathrm{B}_s^{-d} p(y/\mathrm{B}_s)\, ds + o\left(\int_t^{t+\tau} \mathrm{B}_s^{-d}\, ds\right)$$

uniformly for $y \in \mathfrak{G}$. The only value of c_2 which is compatible with this last formula, the assumptions on $\mathfrak{G}_1$, and the fact that $\mathrm{X}_t/\mathrm{B}_t$ has as asymptotic distribution with density p is $c_2 = 1$. Thus

$$\int_t^{t+\tau} \mathrm{P}^s f(x)\, ds = p(0)\mathrm{J}(f) \int_t^{t+\tau} \mathrm{B}_s^{-d}\, ds + o\left(\int_t^{t+\tau} \mathrm{B}_s^{-d}\, ds\right)$$

uniformly for x in compacts as $t, \tau \to \infty$. In the transient case

$$\int_0^\infty \mathrm{B}_t^{-d}\, dt < \infty$$

and in the recurrent case

$$\int_0^\infty \mathrm{B}_t^{-d}\, dt = \infty,$$

from which the conclusion of the theorem follows easily.

The remaining two results of this section will be used in Section 10 in reducing results in continuous time to the corresponding results in discrete time.

PROPOSITION 5.19. — *For fixed* $\tau > 0$, *let* $\mathfrak{G}_2$ *be the group generated by* S_τ. *If* $\mathfrak{G}_2$ *is compactly generated, then so is* $\mathfrak{G}$.

Proof. — Let C_1 be a compact subset of $\mathfrak{G}_2$ that generates $\mathfrak{G}_2$. Let C_2 be a compact subset of $\mathfrak{G}$ such that $P_0(X_t \in C_2) > 0$ for $0 \leqslant t \leqslant \tau$. Let $\mathfrak{G}_3$ be the subgroup of $\mathfrak{G}$ generated by $C_1 + C_2$. Then $\mathfrak{G}_3$ contains C_1 and hence $\mathfrak{G}_3$ contains $\mathfrak{G}_2$. Thus $\mathfrak{G}_3$ contains $\mathfrak{G}_1$, where $\mathfrak{G}_1$ is defined as usual. Since $\mathfrak{G}_3 \cap S_t$ is non-empty for $0 \leqslant t \leqslant \tau$, it follows that $\mathfrak{G}_3$ contains S_t, $0 \leqslant t \leqslant \tau$. Thus $\mathfrak{G}_3$ contains S_t, $0 \leqslant t < \infty$, and hence $\mathfrak{G}_3 = \mathfrak{G}$ as desired.

Our final result is rather special and will be needed only in discussing type II transient processes.

Suppose $\mathfrak{G} = R \oplus H$ or $\mathfrak{G} = Z \oplus H$, where H is a compact group. If $\mathfrak{G}/\mathfrak{G}_1$ is not compact, then $\mathfrak{G} = R \oplus H$, $\mathfrak{G}_1 = H$, and the induced process on R moves deterministically. Under these conditions we have the following

THEOREM 5.5. — *Let* $\mathfrak{G} = R \oplus H$, *where* H *is a compact group and suppose that* $\mathfrak{G}_1 = H$. *Then there is some non-zero constant* m *such that* μ^t *is supported by* $mt + H$ *for* $t \geqslant 0$. *Let* D *and* E *be Borel subsets of* $\mathfrak{G}$ *with* $|D| < \infty$. *Then*

$$\lim_{t \to \infty} \int_{D+y} \mu^t(x + mt + E)\, dx, \qquad y \in H,$$

exists uniformly in y *and the limit is independent of* y.

Proof. — Let φ^t be the probability measure induced on H by μ^t. Since $\mathfrak{G}_1 = H$ none of the measures φ^t are supported by the translate of a proper closed subgroup of H. It follows from the Ito-Kawata Theorem that if f is a continuous function on H, then

$$\lim_{t \to \infty} \int_H \varphi^t\,(dw) f(w - y) = \int_H f(w)\, dw$$

uniformly in y, where dw is normalized Haar measure on H.

For any subset A of $\mathfrak{G}$ let $A_r = \{z \in H \mid r + z \in A\}$.

Then for $r \in R$ and $z \in H$

$$\mu^t(r + z + E) = \varphi^t(z + E_{mt-r}).$$

Thus

$$\begin{aligned}\int_{D+y} \mu^t(x + mt + E)\, dx. & \\ &= \int_{-\infty}^{\infty} dr \int_{y+D_r} \mu^t(r + z + mt + E)\, dz \\ &= \int_{-\infty}^{\infty} dr \int_{y+D_r} \varphi^t(z + E_{-r})\, dz \\ &= \int_{-\infty}^{\infty} dr \int_H \varphi^t\,(dw) \int_H 1_{E_{-r}}\,(w - y - z) 1_{D_r}(z)\, dz.\end{aligned}$$

For each r

$$\int_H 1_{E_{-r}}(w - z) 1_{D_r}(z)\, dz, \qquad w \in H,$$

defines a continuous function. Therefore

$$\begin{aligned}\lim_{t\to\infty} \int_H \varphi^t\,(dw) \int_H 1_{E_{-r}}(w - y - z) 1_{D_r}(z)\, dz & \\ &= \int_H dw \int_H 1_{E_{-r}}(w - z) 1_{D_r}(z)\, dz \\ &= |E_{-r}|_H |D_r|_H\end{aligned}$$

uniformly in y. Since

$$\int_{-\infty}^{\infty} |D_r|_H\, dr = |D|_{\mathfrak{G}} < \infty$$

it follows that

$$\lim_{t\to\infty} \int_{D+y} \mu^t(x + mt + E)\, dx = \int_{-\infty}^{\infty} |D_r||E_{-r}|\, dr$$

uniformly for $y \in H$, as desired.

6. λ-Capacities.

Let μ be a Radon measure on $\mathfrak{G}$, i.e. μ is a regular measure on $\mathfrak{G}$ such that $\mu(K) < \infty$ for all compact sets K. The measure μG^λ is called the λ-potential of μ. For transient processes we can also take $\lambda = 0$. The measure μG is called the potential of μ. It is of vital importance to know that the λ-potential of a measure determines the measure under quite general conditions.

THEOREM 6.1. — *Let* $\lambda > 0$, *and in the transient case* $\lambda \geqslant 0$. *Suppose* μ *is a Radon measure such that* μG^λ *is also a Radon measure. Then* μG^λ *determines* μ. *In particular, if* μ *is a finite measure then* μG^λ *determines* μ.

Proof. — Suppose $\mu(\mathfrak{G}) < \infty$. Then for any compact set K,

$$\mu G^\lambda(K) \leqslant \mu(\mathfrak{G}) \sup_{x \in K} G(\lambda\, x, K) < \infty,$$

and thus μG^λ is a Radon measure. Let μ be any measure satisfying the conditions of the theorem. Let K be any compact set. We can then find $f \in C_c^+$ such that

$$\inf_{x \in K} G^\lambda f(x) \geqslant \delta > 0,$$

and thus from (3.2)

$$\mu \Pi_K^\lambda(\mathfrak{G}) = \mu \Pi_K^\lambda(K) \leqslant \delta^{-1} \mu G^\lambda f < \infty.$$

Hence the measures $\mu \Pi_K^\lambda$ are finite for all compact sets. The assertion of the theorem now follows from Proposition 7.6 of [5] if $\lambda > 0$. An examination of the proof of this proposition shows that it is also valid for $\lambda = 0$ provided that for any excessive function f there is an increasing sequence of bounded non-negative functions φ_n such that $G\varphi_n \uparrow f$. That that is so in our case follows from Exercise 2.19 of Chapter 2 of [2] and the fact that $\sup_{x \in K} G(x, K) < \infty$ for all compact sets K.

The following useful result is due to Hunt [5]

PROPOSITION 6.1. — *If* μ *and* ν *are two Radon measures such that* $\mu G^\lambda \leqslant \nu G^\lambda$ *then* $\mu \Pi_B^\lambda G^\lambda \leqslant \nu \Pi_B^\lambda G^\lambda$ *for any Borel set* B. *If the process is transient then this is also true for* $\lambda = 0$.

Proof. — Since for any $f \in \Phi^+$, $\Pi_B^\lambda G^\lambda f$ is λ-excessive, we can find bounded $f_n \geqslant 0$ such that $G^\lambda f_n \uparrow \Pi_B^\lambda G^\lambda f$. The result follows from this.

Let B be a Borel set and let $\lambda > 0$. Define the measure E_B^λ by

$$E_B^\lambda(A) = \int_{\mathfrak{G}} H_B^\lambda(x, A)\, dx,$$

and set $\mu_B^\lambda = \lambda E_B^\lambda$.

THEOREM 6.2. — *Let* $\lambda > 0$. *The measure* μ_B^λ *is the unique measure supported on* $\bar{B}$ *whose* λ *potential has the density (relative to Haar measure)* $E_x(e^{-\lambda \tilde{T}_B})$.

Proof. — The measure E_B^λ and hence μ_B^λ is a Radon measure. Indeed, let K be a compact set and let $f \in C_c^+$ be such that $G^\lambda f(x) \geqslant \delta > 0$, $x \in K$. Then from (3.1)

$$H_B^\lambda(x, K) \leqslant \delta^{-1} H_B^\lambda G^\lambda f(x) \leqslant G^\lambda f(x)$$

and thus $E_B^\lambda(K) \leqslant |J(f)|/\lambda\delta < \infty$. Also from (3.1) we see that for any $f \in \Phi^+$

$$J(f)\lambda^{-1} = E_B^\lambda G^\lambda f + \int_{\mathfrak{G}} G_B^\lambda f(x)\, dx.$$

Now from (3.6) we have

$$\int_{\mathfrak{G}} G_B^\lambda f(x)\, dx = (1, G_B^\lambda f) = (f, \tilde{G}_B^\lambda 1)$$
$$= \int_{\mathfrak{G}} f(x)\, dx \int_0^\infty P_x(\tilde{T}_B > t) e^{-\lambda t}\, dt = \int_{\mathfrak{G}} f(x)\, dx\, [1 - E_x(e^{-\lambda \tilde{T}_B})]\lambda^{-1}.$$

Thus

$$\mu_B^\lambda G^\lambda f = \int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_B}) f(x)\, dx. \tag{6.1}$$

The uniqueness of the measure μ_B^λ follows at once from Theorem 6.1.

DEFINITION 6.1. — *The measure* μ_B^λ *is called the* λ*-capacitory measure of* B; *its total mass* $\mu_B^\lambda(\bar{B}) = C^\lambda(B)$, *is called the* λ*-capacity of* B. *The corresponding quantities for the dual process are called the co-*λ*-capacitory measure and co-*λ*-capacity respectively and denoted by* $\tilde{\mu}_B$ *and* $\tilde{C}^\lambda(B)$ *respectively.*

PROPOSITION 6.2. — *For any Borel set* B, $C^\lambda(B) = \tilde{C}^\lambda(B)$.

Proof. — By definition, $C^\lambda(B) = \mu_B^\lambda(\mathfrak{G}) = \lambda \int_{\mathfrak{G}} E_x(e^{-\lambda T_B})\, dx$. Let $f_n \in \Phi$ and $f_n \uparrow 1$. Then from Theorem 6.2

$$C^\lambda(B) = \mu_B^\lambda(\mathfrak{G}) = \lim_n \lambda E_B^\lambda G^\lambda f_n = \lim_n \lambda \int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_B}) f_n(x)\, dx$$
$$= \lambda \int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_B})\, dx = \tilde{\mu}_B^\lambda(\mathfrak{G}) = \tilde{C}^\lambda(B).$$

PROPOSITION 6.3. — *If for some* $\lambda > 0$, $C^\lambda(B) = 0$ *then* $C^\lambda(B) = 0$ *for all* $\lambda > 0$ *and* $P_x(T_B < \infty) = P_x(\tilde{T}_B < \infty) = 0$, *a.e.* x. *Conversely, if* $P_x(T_B < \infty) = 0$, *a.e.* x, *then* $C^\lambda(B) = 0$ *for all* $\lambda > 0$.

Proof. — From (6.1) we see that $C^\lambda(B) = 0$ if and only if $\int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_B})\, dx = 0$. Now, $\int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_B})\, dx = 0$ if and only if $E_x(e^{-\lambda \tilde{T}_B}) = 0$ a.e. x, and $E_x(e^{-\lambda \tilde{T}_B}) = 0$ if and only if $P_x(\tilde{T}_B = \infty) = 1$. The assertions of the theorem now follow from these facts.

THEOREM 6.3. — *Assume* $\Sigma = \mathfrak{G}$. *Then for* $\lambda > 0$, $C^\lambda(B) > 0$ *if and only if* $P_x(T_B < \infty) > 0$ *a.e.* x *and* $P_x(\tilde{T}_B < \infty) > 0$ *a.e.* x. *In the non-singular case the a.e.* x *can be strengthened to all* x.

To prove this theorem we will need the following.

LEMMA 6.1. — *Assume* $\Sigma = \mathfrak{G}$. *Then if* A *is a non-empty open set,* $G^\lambda(x, A) > 0$ *for all* x. *If* $|K| > 0$ *then* $G^\lambda(x, K) > 0$ *a.e.* x *and in the non-singular case for all* x.

Proof. — The first assertion of the theorem follows at once from the fact that $\Sigma = \mathfrak{G}$ and the fact that the paths are right continuous. Now let $|K| > 0$ and let $g \in \Phi^+$ be such that $g(x) > 0$ on a set of positive measure. Then

$$\int_{\mathfrak{G}} g(x) G^\lambda(x, K)\, dx = \int_{\mathfrak{G}} G^\lambda(0, dy) \int_{\mathfrak{G}} 1_K(x+y) g(x)\, dx.$$

The function $\int_{\mathfrak{G}} 1_K(x + y) g(x)\, dx$ is continuous (because C_c is dense in L_1) and not identically 0. Thus by first part of the theorem

$$\int_{\mathfrak{G}} G^\lambda(0, dy) \int_{\mathfrak{G}} 1_K(x + y) g(x)\, dx > 0.$$

Hence for any such g, $(g, G^\lambda 1_K) > 0$ and thus $G^\lambda 1_K(x) > 0$ a.e. x. Finally if the process is non-singular, then for some $t > 0$,

$$P^t(0, dy) = p(t, y)\, dy + \gamma_t(dy)$$

where $p(t, y) > 0$ for all y on some set of positive measure.

But

$$G^\lambda(x, K) \geqslant \int_{\mathfrak{G}} e^{-\lambda t} P^t(x, dy) G^\lambda(y, K)$$
$$\geqslant \int_{\mathfrak{G}} e^{-\lambda t} p(t, y - x) G^\lambda(y, K)\, dy$$

and by what was just proved $G^\lambda(y, K) > 0$ a.e. y. Hence $\int_{\mathfrak{G}} p(t, y - x) G^\lambda(y, K)\, dy > 0$. This establishes the lemma.

Proof of Theorem. — Suppose that $C^\lambda(B) > 0$. Then by Proposition 6.3 $P_x(\tilde{T}_B < \infty) > 0$ on some set of positive measure. Consequently, there is a compact set K having positive measure such that $\inf_{x \in K} P_x(\tilde{T}_B < \infty) \geqslant \delta > 0$. But then

$$tP_x(\tilde{T}_B < \infty) \geqslant \int_K \int_0^t P_x(\tilde{X}_s \in dy) P_y(\tilde{T}_B < \infty)\, ds$$
$$\geqslant \delta \int_0^t P_x(\tilde{X}_s \in K)\, ds.$$

By Lemma 6.1, we know that $\tilde{G}^\lambda(x, K) > 0$ a.e. x (and for all x in the non-singular case). For each x such that $\tilde{G}^\lambda(x, K) > 0$ there is a $t > 0$ such that $\int_0^t P_x(\tilde{X}_s \in K)\, ds > 0$. Hence $P_x(\tilde{T}_B < \infty) > 0$ a.e. x (and for all x in the non-singular case).

Conversely, if $P_x(\tilde{T}_B < \infty) > 0$ a.e. x then by Proposition 6.3 $C^\lambda(B) > 0$. This establishes the theorem.

We will now show that $C^\lambda(\cdot)$ is a Choquet capacity.

Proposition 6.4. — *Let* $\lambda > 0$. *Then* $C^\lambda(\cdot)$ *has the following properties.*

(*a*) *If* $A \subset B$, *then* $C^\lambda(A) \leqslant C^\lambda(B)$.

(*b*) $C^\lambda(A \cup B) + C^\lambda(A \cap B) \leqslant C^\lambda(A) + C^\lambda(B)$.

(*c*) $C^\lambda(B) = \sup_{K \subset B} C^\lambda(K)$, K *compact.*

(*d*) $C^\lambda(B) = \inf_{U \supset B} C^\lambda(U)$, U *open.*

(*e*) $C^\lambda(B + y) = C^\lambda(B)$, *all* y.

(*f*) $C^\lambda(B) = C^\lambda(-B)$.

Proof. — Suppose $A \subset B$. Then $T_A \geqslant T_B$, so

$$E_x(e^{-\lambda T_A}) \leqslant E_x(e^{-\lambda T_B}).$$

Assertion (a) follows at once from this last inequality. Assertion (b) follows from the inequality

$$
\begin{aligned}
(6.2)\quad P_x(T_{A\cap B} \leqslant t) \leqslant P_x(T_A \leqslant t,\ T_B \leqslant t) \\
= P_x(T_A \leqslant t) + P_x(T_B \leqslant t) - P_x(T_{A\cup B} \leqslant t).
\end{aligned}
$$

Let K be a compact set and let A_n be relatively compact open sets such that $A_n \supset \overline{A}_{n+1} \supset A_{n+1}$ and $\bigcap_n A_n = \bigcap_n \overline{A}_n = K$. The T_{A_n} are an increasing sequence of stopping times. Let $T = \lim_n T_{A_n}$. Then clearly $T \leqslant T_K$. By quasi-left continuity, $X(T_{A_n}) \to X(T)$ a.s. P_x for every x. Now $X(T) \in \bigcap_n A_n = K$, so $T \geqslant T_K$. Thus $T = T_K$ and therefore

$$P_x(X(T_{A_n}) \to X(T_K)) = 1.$$

Hence

$$E_x(e^{-\lambda T_{A_n}}) \downarrow E_x(e^{-\lambda T_K})$$

and therefore $C^\lambda(A_n) \downarrow C^\lambda(K)$. This shows that (d) holds whenever B is compact. Similarly, it is quite easy to verify that (c) holds whenever B is an open set. Indeed, if B is open there is a sequence of compact sets $K_1 \subset K_2 \subset \ldots$, such that $\bigcup_n K_n = B$, so $T_{K_n} \downarrow T_B$ and thus

$$E_x(e^{-\lambda T_{K_n}}) \uparrow E_x(e^{-\lambda T_B}).$$

Hence $C^\lambda(K_n) \uparrow C^\lambda(B)$. Properties (a), (b) and (d) for compact sets show that $C^\lambda(\cdot)$ is a Choquet capacity on the compacts and thus by Choquet's capacity theorem there is a unique extension of $C^\lambda(\cdot)$ to the Borel sets. Denote this extension by $C^\lambda_*(\cdot)$. For any Borel set B we then have

$$(6.3)\qquad C^\lambda_*(B) = \sup_{K\subset B} C^\lambda(K), \quad K \text{ compact}$$

$$(6.4)\qquad C^\lambda_*(B) = \inf_{U\supset B} C^\lambda(U), \quad U \text{ open}.$$

But by (a) if $K \subset B$, $C^\lambda(K) \leqslant C^\lambda(B)$ and thus by (6.3) $C^\lambda_*(B) \leqslant C^\lambda(B)$. Also if $U \supset B$, then $C^\lambda(U) \geqslant C^\lambda(B)$ and so by (6.4) $C^\lambda_*(B) \geqslant C^\lambda(B)$. Hence $C^\lambda(B) = C^\lambda_*(B)$, so $C^\lambda(\cdot)$ is itself its extension from the compact sets to the Borel sets and therefore (c) and (d) hold for any Borel set B. To see that (e) holds note that

$$E_{x+y}(e^{-\lambda T_{B+y}}) = E_x(e^{-\lambda T_B})$$

and thus integrating on x we see that (e) holds. Finally (f) holds because $E_x(e^{-\lambda T_B}) = E_{-x}(e^{-\lambda \tilde{T}_{-B}})$, and thus

$$C^\lambda(B) = \tilde{C}^\lambda(-B) = C^\lambda(-B).$$

This completes the proof.

The following proposition is a simple consequence of Proposition 6.3.

Proposition 6.5. — *Assume* B_n, $n \geqslant 1$ *all have* λ*-capacity* 0. *Then* $B = \bigcup_n B_n$ *also has* λ*-capacity* 0.

Proof. — $C^\lambda(B_n) = 0$ if and only if $P_x(T_{B_n} < \infty) = 0$ a.e. Since this is true for all n, $P_x(T_B < \infty) = 0$ a.e., and thus $C^\lambda(B) = 0$.

Proposition 6.6. — *Let* $\lambda > 0$ *and let* B *be any Borel set. Let* K_n *be relatively compact sets such that* $K_n \uparrow$, $K_n \subset B$ *and* $P_x(\tilde{T}_{K_n} \downarrow \tilde{T}_B) = 1$ *a.e.* x. *Then* $\mu^\lambda_{K_n} \to \mu^\lambda_B$ *vaguely and* $C^\lambda(K_n) \uparrow C^\lambda(B)$.

Proof. — From the assumption it follows that

$$E_x(e^{-\lambda \tilde{T}_{K_n}}) \uparrow E_x(e^{-\lambda \tilde{T}_B}) \quad \text{a.e.}$$

and thus by montone convergence and Proposition 6.2

$$\lim_{n\to\infty} C^\lambda(K_n) = \lim_n \lambda \int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_{K_n}})\, dx = \lambda \int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_B})\, dx = C^\lambda(B).$$

Also for any $f \in C_c^+$,

$$(\mu^\lambda_{K_n}, G^\lambda f) \uparrow (\mu^\lambda_B, G^\lambda f). \tag{6.5}$$

Let K be any compact set. Then we can find $f \in C_c^+$ such that $G^\lambda f(x) \geqslant \delta > 0$, $x \in K$, and thus

$$\delta \mu^\lambda_{K_n}(K) \leqslant (\mu^\lambda_B, G^\lambda f) < \infty.$$

Thus there is a subsequence $\mu^\lambda_{K_{n_j}}$ that vaguely converges to a measure μ. Fatou's lemma shows that $(\mu, G^\lambda f) \leqslant (\mu^\lambda_B, G^\lambda f)$

and thus $\mu G^\lambda(K) < \infty$ for any compact set K. Now by Proposition 6.1. for any $f \in C_c^+$,

$$\mu^\lambda_{K_n}\Pi^\lambda_{K'}G^\lambda f \leqslant \mu^\lambda_B\Pi^\lambda_{K'}\, G^\lambda f,$$

and since $\mu^\lambda_B\Pi^\lambda_{K'}G^\lambda f \downarrow 0$ as $K \uparrow \mathfrak{S}$, we see that given $\varepsilon > 0$ we can find a compact K such that for all $n \geqslant 1$,

$$\mu^\lambda_{K_n}\Pi^\lambda_{K'}G^\lambda f \leqslant \varepsilon.$$

But then

$$(6.6)\quad \int_{K'} \mu^\lambda_{K_n}(dx)G^\lambda f(x) = \int_{K'} \mu^\lambda_{K_n}(dx)\ \Pi^\lambda_{K'}G^\lambda f(x) \leqslant (\mu^\lambda_{K_n},\ \Pi^\lambda_{K'}G^\lambda f) \leqslant \varepsilon.$$

Also,

$$(6.7)\quad \lim_{K\uparrow\mathfrak{S}}\lim_{n_j\to\infty}\int_K \mu^\lambda_{K_{n_j}}(dx)G^\lambda f(x) = \lim_{K\uparrow\mathfrak{S}}\int_K \mu(dx)G^\lambda f(x) = (\mu,\ Gf).$$

It now follows from (6.6) and (6.7) that

$$\lim_{n_j\to\infty}(\mu^\lambda_{K_{n_j}},\ G^\lambda f) = (\mu,\ G^\lambda f).$$

Thus from (6.5), $(\mu,\ G^\lambda f) = (\mu^\lambda_B,\ G^\lambda f)$ and it now follows from Theorem 6.1 that $\mu = \mu^\lambda_B$. If we had another vaguely converging sequence we would again obtain that the limit measure was μ^λ_B so $\mu^\lambda_{K_n} \to \mu^\lambda_B$ vaguely.

PROPOSITION 6.7. — *Let* $\lambda > 0$ *and let* B *be a Borel set and let* U_n *be open,* $U_n \downarrow$ *and such that* $P_x(\tilde{T}_{U_n} \uparrow \tilde{T}_B) = 1$ *a.e. Then* $\mu_{U_n} \to \mu_B$ *vaguely and* $C^\lambda(U_n) \downarrow C^\lambda(B)$.

Proof. — The proof is similar to the previous proposition and will be omitted.

7. Applications of λ-capacities.

In this section we will illustrate the use of the λ-capacity theory of the last section in finding criteria for when various sets are hit or not. Mainly, we will focus our attention on one-point sets but we shall also indicate how analogous results can be given for other sets. These results are to be considered only as examples of what can be done. No attempt has been made to be exhaustive.

By Proposition 6.4., for any $x \in \mathfrak{G}$, $C^\lambda(\{x\}) = C^\lambda(\{0\}) = C^\lambda$ so either every one-point set has positive capacity or every such set has 0 capacity. Our first result gives a necessary and sufficient condition for $C^\lambda > 0$.

THEOREM 7.1. — *In order that* $C^\lambda > 0$ *it is both necessary and sufficient that* $G^\lambda(0, dx)$ *have a bounded density. In that case there is a version* $g^\lambda(x)$ *of the density such that* $g^\lambda(y - x)$ *is* λ*-excessive in* x, λ*-co-excessive in* y *and* $\tilde{g}^\lambda(x) = g^\lambda(-x)$. *For this version of the density*

$$(7.1) \qquad E_x[e^{-\lambda V_{\{0\}}}; V_0 < \infty] = C^\lambda g^\lambda(-x), \quad x \in \mathfrak{G}.$$

Proof. — Assume $C^\lambda > 0$. By Theorem 6.2 applied to $B = \{0\}$ we then see that

$$(7.2) \qquad E_x[e^{-\lambda V_{\{0\}}}; V_0 < \infty]\, dx = C^\lambda \tilde{G}^\lambda(0, dx).$$

Thus $\tilde{G}^\lambda(0, dx)$ and hence $G^\lambda(0, dx)$ has a bounded density. By Proposition 3.9 we may assume that the density g^λ is choosen to have the properties stated in the theorem. From (7.2) it follows that (7.1) holds for a.e. x. Since both sides are λ-excessive it follows from Proposition 3.8 that (7.1) holds for all x.

Suppose now that $G^\lambda(0, dx)$ has a bounded density g^λ. Again we may assume g^λ has the excessive function properties stated in the theorem. By Proposition 3.11, for all x and y,

$$g^\lambda(y - x) - \int_{\overline{B}} \Pi_B^\lambda(x, dz) g^\lambda(y - z) = u_B^\lambda(x, y).$$

Let B_n be open, $\overline{B}_n$ compact,

$$B_1 \supset \overline{B}_2 \supset B_2 \supset \cdots, \bigcap_n B_n = \bigcap_n \overline{B}_n = \{0\}.$$

Then as $u_{B_n}^\lambda(x, 0) = 0$ we see that for some K, $0 < K < \infty$

$$\begin{aligned}(7.3) \quad g^\lambda(-x) &= E_x[\exp(-\lambda V_{B_n}) g^\lambda(-X_{V_{B_n}}); V_B < \infty] \\ &\leqslant K E_x[\exp(-\lambda V_{B_n}); V_{B_n} < \infty].\end{aligned}$$

Quasi-left continuity shows that for $x \neq 0$, $V_{B_n} \uparrow V_0$ and $X_{V_{B_n}} \to 0$ a.s. P_x on $[V_{\{0\}} < \infty]$. Thus for $x \neq 0$

$$g^\lambda(-x) \leqslant K E_x[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty].$$

Since $g^\lambda(-x) > 0$ on a set of positive measure we see that $E_x[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty] > 0$ on a set of positive measure and thus by Proposition 6.3 $C^\lambda > 0$. This completes the proof.

COROLLARY 7.1. — *Suppose* $C^\lambda > 0$. *Then*

$$P_x(V_{\{0\}} < \infty) > 0$$

if and only if $g^\lambda(-x) > 0$. *The set* $\Sigma_0 = \{x: g^\lambda(-x) > 0\}$ *is a sub-semigroup of* $\mathfrak{G}$ *contained in* $-\Sigma$. *If* $\Sigma = \mathfrak{G}$ *then* $\Sigma_0 = \mathfrak{G}$ *and for all* $x \in \mathfrak{G}$

$$(7.4)\quad E_x[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty] = E_0[e^{-\lambda V_{\{0\}}}; V_0 < \infty]\frac{g^\lambda(-x)}{g^\lambda(0)}.$$

Proof. — The first statement follows at once from equation (7.1). Also by (3.21) applied to $B = \{b\}$ we see that

$$g^\lambda(x) \geqslant E_0[e^{-\lambda V_{\{b\}}}; V_{\{b\}} < \infty]g^\lambda(x-b).$$

Using (7.1) again it follows that

$$(7.5)\qquad g^\lambda(x+y) \geqslant C^\lambda g^\lambda(x)g^\lambda(y).$$

Hence $x + y \in \Sigma_0$ whenever x and $y \in \Sigma_0$. Now if $g^\lambda(x) > 0$ then for any neighborhood N of 0

$$G^\lambda(0, N+x) \geqslant E_0(e^{-\lambda V_x})G^\lambda(x, N+x) = E_0(e^{-\lambda V_x})G^\lambda(0, N) > 0$$

so $x \in \Sigma$. Finally, if $\Sigma = \mathfrak{G}$ then by Theorem 6.3

$$E_x[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty] > 0$$

a.e. x and thus $g^\lambda(-x) > 0$ a.e. x. Given any $x \in \mathfrak{G}$ there are then points a, b such that $x = a + b$ and $g^\lambda(a) > 0$, $g^\lambda(b) > 0$. It follows from (7.5) that $g^\lambda(x) > 0$ for all $x \in \mathfrak{G}$. Thus $\Sigma_0 = \mathfrak{G}$. Equation (7.4) now follows from this fact and equation (7.1). This establishes the corollary.

If $C^\lambda > 0$ it is natural to inquire if x is regular for $\{x\}$. Now x is regular for $\{x\}$ if and only if

$$E_x[e^{-\lambda V_{\{x\}}}; V_{\{x\}} < \infty] = 1.$$

Since $E_x[e^{-\lambda V_{\{x\}}}; V_{\{x\}} < \infty] = E_0[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty]$ either every point is regular or no point is regular.

COROLLARY 7.2. — *If* $C^\lambda > 0$ *and* $E_0[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty] = 1$ *then the density* g^λ *in Theorem* 7.1 *is continuous on* $\mathfrak{G}$ *and* $g^\lambda(0) > 0$. *Conversely if* $G^\lambda(0, dx)$ *has a bounded continuous density then* $C^\lambda > 0$ *and* $E_0[e^{-\lambda V_{\{0\}}}, V_{\{0\}} < \infty] = 1$.

Proof. — Suppose $C^\lambda > 0$ and $E_0[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty] = 1$. From (7.1) we then see that

$$1 = E_0[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty] = C^\lambda g^\lambda(0)$$

and thus $g^\lambda(0) > 0$. Again from (7.1) we then see that

$$E_x[e^{-\lambda V_{\{0\}}}; V_0 < \infty] = \frac{g^\lambda(-x)}{g^\lambda(0)}.$$

As λ-excessive functions are lower semi-continuous in this case (Proposition 3.7) we see that

$$\varliminf_{x \to x_0} E_x[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty] \geq E_0[e^{-\lambda V_{\{0\}}}; V_0 < \infty] = 1.$$

Thus $E_x[e^{-\lambda V_{\{0\}}}; V_{\{0\}} < \infty]$ and consequently $g^\lambda(-x)$ is continuous at 0. Now from (7.5) and the fact that here $C^\lambda = [g^\lambda(0)]^{-1}$ we have

$$g^\lambda(x+y) \geq g^\lambda(x) g^\lambda(y) [g^\lambda(0)]^{-1}. \tag{7.6}$$

Setting $x = a$ in (7.6) we see that

$$\varliminf_{y \to 0} g^\lambda(a+y) \geq g^\lambda(a).$$

Now set $x + y = a$ in (7.6) to obtain

$$g^\lambda(a) \geq g^\lambda(a-y) g^\lambda(y) [g^\lambda(0)]^{-1}.$$

Thus

$$g^\lambda(a) \geq \varlimsup_{y \to 0} g^\lambda(a-y).$$

Hence g^λ is continuous at a.

Suppose now that $G^\lambda(0, dx)$ has a bounded continuous density $u^\lambda(x)$. We will now show that $u^\lambda(x) = g^\lambda(x)$ where g^λ is the density given in Theorem 7.1. Since

$$g^\lambda(-x) = u^\lambda(-x)$$

a.e. x and $g^\lambda(-x)$ is λ-excessive it suffices by Proposition 3.8 to show that $u^\lambda(-x)$ is λ-excessive. Now as $g^\lambda(-x)$

is λ-excessive

$$e^{-\lambda t} \int_{\mathfrak{G}} \mathrm{P}^t(x, dy) g^\lambda(-y) \leqslant g^\lambda(-x),$$

and thus for a.e x,

$$e^{-\lambda t} \int_{\mathfrak{G}} \mathrm{P}^t(x, dy) u^\lambda(-y) \leqslant u^\lambda(-x).$$

As both sides are continuous in x this inequality must hold for all x. Also as u^λ is bounded and continuous

$$\int_{\mathfrak{G}} \mathrm{P}^t(x, dy) u^\lambda(-y) \to u^\lambda(-x)$$

as $t \downarrow 0$. Thus $u^\lambda(-x)$ is λ-excessive. Now let B_n be open relatively compact neighborhoods of 0 such that $\mathrm{B}_n \downarrow \{0\}$. Then from (7.3) we see that for $x \neq 0$,

$$g^\lambda(-x) = \mathrm{E}_x[e^{-\lambda \mathrm{V}_{\{0\}}}; \mathrm{V}_{(0)} < \infty] g^\lambda(0). \tag{7.7}$$

Hence $g^\lambda(0) > 0$ since otherwise $g^\lambda(-x) \equiv 0$. On the other hand from (7.1) for $x = 0$ we see that

$$\mathrm{E}_0[e^{-\lambda \mathrm{V}_{\{0\}}}; \mathrm{V}_{\{0\}} < \infty] = \mathrm{C}^\lambda g^\lambda(0)$$

and thus (again by (7.1))

$$\mathrm{E}_x[e^{-\lambda \mathrm{V}_{\{0\}}}; \mathrm{V}_{\{0\}} < \infty] = \frac{\mathrm{E}_0[e^{-\lambda \mathrm{V}_{\{0\}}}; \mathrm{V}_{\{0\}} < \infty] g^\lambda(-x)}{g^\lambda(0)}. \tag{7.8}$$

Comparing (7.7) with (7.8) we see that

$$\mathrm{E}_0[e^{-\lambda \mathrm{V}_{\{0\}}}; \mathrm{V}_{\{0\}} < \infty] = 1.$$

This completes the proof.

Remark. — In the above proof we used the continuity of the density to show that it was the density g^λ. Only the continuity at 0 of the density g^λ was needed to establish the regularity at 0. Thus alternately we could assume that $\mathrm{G}^\lambda(0, dx)$ has a bounded density g^λ such that $g^\lambda(-x)$ is λ-excessive and continuous at 0.

The remainder of this section will be devoted to finding a simple sufficient condition to guarantee that a set have zero capacity. We will confine our attention to processes on R^d.

PROPOSITION 7.1. — *Let* $X(t)$, $t \geq 0$, *be an infinitely divisible process on* R^d. *Then for any positive random variable* T

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \int_0^\infty P(|X(t)| \leq \varepsilon, T > t)\, dt = \infty. \tag{7.9}$$

Proof. — It suffices to prove this result under the added assumption that $X(t)$ does not have any jumps of magnitude larger than 1 (for the time S to the first such jump is a positive random variable and we can consider the time $\min(S, T)$). Under this assumption the logarithm of the characteristic function of $X(t)$ can be written as

$$t\left(i\alpha . \theta + \int_{R^d} (e^{i\theta . x} - 1 - i\theta . x)\nu\,(dx)\right).$$

For $0 < \delta < \infty$ let $X_\delta(t)$, $t \geq 0$, denote an infinitely divisible process whose characteristic function has logarithm

$$t\left(i\alpha . \theta + \int_{|x| < \delta} (e^{i\theta . x} - 1 - i\theta . x)\nu\,(dx)\right).$$

Then $X_\delta(t)$ has mean αt and

$$E|X_\delta(t) - \alpha t|^2 = t \int_{|x| \leq \delta} |x|^2 \nu(dx) = \sigma_\delta^2 t,$$

where $\sigma_\delta \to 0$ as $\delta \to 0$. By Tchebychef's inequality for $\varepsilon > 0$

$$P\left(|X_\delta(t) - \alpha t| \geq \frac{\varepsilon}{2}\right) \leq \frac{4\sigma_\delta^2 t}{\varepsilon^2}, \quad t \geq 0.$$

If $0 \leq t \leq \varepsilon/2|\alpha|$, then $|\alpha t| \leq \varepsilon/2$. Consequently

$$P(|X_\delta(t)| \geq \varepsilon) \leq \frac{4\sigma_\delta^2 t}{\varepsilon^2}, \qquad 0 \leq t \leq \varepsilon/2|\alpha|.$$

If $\sigma_\delta = 0$ for some $\delta > 0$, then $X(t)$, $t \geq 0$, is a pure jump process and

$$\int_0^\infty P(X(t) = 0, T > t)\, dt > 0,$$

from which (7.9) follows immediately. Thus in proving Proposition 7.1 we can assume that $\sigma_\delta > 0$ for all $\delta > 0$.

Let $\delta > 0$ be fixed. There is an $\varepsilon_0 > 0$ such that

$$\frac{\varepsilon^2}{8\sigma_\delta^2} \leq \frac{\varepsilon}{2|\alpha|}, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

Then for $0 < \varepsilon \leqslant \varepsilon_0$

$$P(|X_\delta(t)| \geqslant \varepsilon) \leqslant \frac{1}{2}, \qquad 0 \leqslant t \leqslant \frac{\varepsilon^2}{8\sigma_\delta^2}.$$

Consequently there is an $\varepsilon_1 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_1$

$$P(|X(t)| \leqslant \varepsilon \quad \text{and} \quad T > t) \geqslant \frac{1}{4}, \qquad 0 \leqslant t \leqslant \frac{\varepsilon^2}{8\sigma_\delta^2}.$$

This shows that for $0 < \varepsilon \leqslant \varepsilon_1$

$$\int_0^\infty P(|X(t)| \leqslant \varepsilon, T > t)\, dt \geqslant \frac{\varepsilon^2}{32\sigma_\delta^2}.$$

Since $\sigma_\delta^2 \to 0$ as $\delta \to 0$ we see that (7.9) holds as desired.

In the next result $\mathfrak{G} = R^{d_1} \oplus Z^{d_2}$ and S_r, $r > 0$, denotes the points in $\mathfrak{G}$ of the form $(x_1, \ldots, x_{d_1+d_2})$, where $x_{d_1+1}, \ldots, x_{d_2+2}$ are integers and

$$\sum_{i=1}^{d_1+d_2} x_i^2 < r.$$

THEOREM 7.2. — *Let* $\mathfrak{G} = R^{d_1} \oplus Z^{d_2}$ *and let* B *be a Borel set in* $\mathfrak{G}$. *If*

$$\limsup_{r\to 0} \frac{|B + S_r|}{r^2} < \infty, \tag{7.10}$$

then $C^\lambda(B) = 0$.

Proof. — We can assume that $C^\lambda(B) < \infty$. Then for any Borel set A

$$\int_{\bar B} \mu_B^\lambda (dz) G^\lambda(z, A) = \int_A E_x e^{-\lambda \tilde T_B}\, dx \leqslant |A|.$$

Setting $A = \bar B + S_r$, we see that

$$|\bar B + S_r| \geqslant \int_{\bar B} \mu_B^\lambda (dz) G^\lambda(z, \bar B + S_r)$$
$$\geqslant \mu_B^\lambda(\bar B) G^\lambda(0, S_r) = C^\lambda(B) G^\lambda(0, S_r).$$

By Proposition 7.1

$$\lim_{r\to 0} \frac{G^\lambda(0, S_r)}{r^2} = \infty.$$

If (7.10) holds, then

$$\limsup_{r\to 0} \frac{|\bar B + S_r|}{r^2} < \infty$$

and hence $C^\lambda(B) = 0$ as desired.

COROLLARY 7.3. — *Let* $\mathfrak{G} = R^2$. *Then* $C^\lambda(\{x\}) = 0$ *for all* $x \in \mathfrak{G}$.

COROLLARY 7.4. — *Let* $\mathfrak{G} = R^3$ *and let* B *be the range of a continuous curve having bounded variation. Then* $C^\lambda(B) = 0$.

Proof. — We can write

$$B = \{\varphi(t), 0 \leqslant t \leqslant 1\},$$

where φ is a continuous function of total variation $M < \infty$. Choose $r > 0$ and set $x_0 = \varphi(0)$. Let x_1 be the first (in the sense of least value of t) point, if any, along the curve whose distance from x_0 is r. If x_{n-1} exists let x_n be the first point, if any, along the curve beyond x_{n-1} whose distance from x_{n-1} is r. Let $x_0, \ldots, x_N$ be all points obtained by this procedure. Then $Nr \leqslant M$ and

$$B + S_r \subseteq \bigcup_{j=0}^{N} (x_j + S_{2r}).$$

Consequently

$$|B + S_r| \leqslant (N+1)\frac{4}{3}\pi(2r)^3 \leqslant r^2\left(\frac{32\pi M}{3} + \frac{32\pi r}{3}\right)$$

and it follows from Theorem 7.2 that $C^\lambda(B) = 0$.

8. Transient and recurrent sets.

In this section we will first show that a Borel set B is either such that $P_x(V_B < \infty) = 1$ a.e. x or $\lim_{t\to\infty} P_x(X_s \in B$ for some $s \geqslant t) = 0$ a.e. x. For a recurrent process, $P_x(V_B < \infty) = 1$ a.e. for any set having $C^\lambda(B) > 0$. In the transient case a Borel set may be of either type. A Borel set such that $P_x(X_s \in B$ for some $s \geqslant t) \downarrow 0$, a.e. as $t \to \infty$ is called a transient set. Most of this section is devoted to showing that associated with each such set is a unique Radon measure $\tilde{\mu}_B$, called the co-capacitory measure of B, such that $P_x(T_B < \infty)\, dx = \tilde{\mu}_B \tilde{G}(dx)$ and in investigating associated capacity theory.

PROPOSITION 8.1. — *Let* B *be a Borel set. Then either* $P_x(V_B < \infty) = 1$ *a.e. (and for all* x *in the non-singular case) or* $\lim_{t\to\infty} P_x(X_s \in B$ *for some* $s \geq t) = 0$ *a.e.* (and for all x in the non-singular case).

Proof. — Let $\varphi_B(x) = P_x(V_B < \infty)$. Then as φ_B is an excessive function, $P^t\varphi_B(x) \downarrow r(x)$, $t \to \infty$. Let $h(x) = \varphi_B(x) - r(x)$. Then $\varphi_B = r + h$, $P^t h \downarrow 0$, and by dominated convergence

$$P^t r(x) = \int_{\mathfrak{G}} P^t(x, dy)\left[\lim_{s\to\infty} P^s\varphi_B(y)\right] = \lim_{s\to\infty} P^{t+s}\varphi_B(x) = r(x),$$

so $r(x)$ is P^t invariant for each t. But then $\lambda G^\lambda r(x) = r(x)$. Now $\lambda G^\lambda(0, dx)$ is a probability measure on $\mathfrak{G}$ and r is bounded. Thus by the Choquet-Deny theorem there is a constant α such that $r(x) = \alpha$ a.e. on the group generated by the support of $\lambda G^\lambda(0, dx)$. It is quite easy to see however that the support of this measure is just Σ. Thus

$$(8.1) \qquad \varphi_B(x) = \alpha + h(x) \qquad \text{a.e.} \qquad x \in \mathfrak{G}.$$

In the non-singular case it follows from (3.23) that $r(x) \equiv \alpha$ so (8.1) holds for all $x \in \mathfrak{G}$. The conclusion of the theorem now follows at once from (8.1) and the following

LEMMA 8.1. — *If* $\alpha > 0$ *then* $P_x(V_B < \infty) = 1$ *a.e. (and for all* x *in the non-singular case).*

Proof. — Let $g \in \Phi^+$ be such that $J(g) = 1$. Then the measure $P_g(V_B > t, X_t \in dy)$ is absolutely continuous and thus

$$P_g(t < V_B < \infty) = \int_{\mathfrak{G}} P_g(V_B > t, X_t \in dy) P_y(V_B < \infty) \geq \alpha P_g(V_B > t).$$

Thus for any $t > 0$,

$$P_g(V_B < \infty) \geq P_g(V_B \leq t) + \alpha P_g(V_B > t)$$

and so $\alpha P_g(V_B = \infty) \leq 0$. Since $\alpha > 0$, $P_g(V_B = \infty) = 0$. Since g was arbitrary, $P_x(V_B = \infty) = 0$ a.e. x, and so $P_x(V_B < \infty) = 1$, a.e. x. If the process is also non-singular, then for some $t_0 > 0$,

$$P^{t_0}(0, dy) = p_{t_0}(y)\, dy + \mu_{t_0}(dy)$$

where $p_t(y) > 0$ on a set of positive measure. Hence for $t \geqslant t_0$

$$P_x(V_B < \infty) \geqslant \int_{\mathfrak{G}} d_t(y)\, dy.$$

Since (see 3.23) $\lim\limits_{t\to\infty} \int_{\mathfrak{G}} p_t(y)\, dy = 1$ we see that

$$P_x(V_B < \infty) = 1$$

for all x. This establishes the lemma.

COROLLARY 8.1. — *If the process is recurrent then* $P_x(V_B < \infty) = 1$ *a.e.* (*all* x *in the non-singular case*) *whenever* $C^\lambda(B) > 0$.

Proof. — Suppose false. Then in (8.1), $\alpha = 0$ so $P^t\varphi_B(x) \downarrow 0$ a.e. Then for any $g \in \Phi^+$

$$(8.2) \qquad (g_2 G(\varphi_B - P^1\varphi_B)) = \int_0^1 (g_3 P^t\varphi_B)\, dt.$$

Since by Proposition 4.3 $G(x, K) = \infty$ a.e. if $|K| > 0$ it must be that $\varphi_B - P^1\varphi_B = 0$ a.e. for otherwise the left hand side of (8.2) would be infinite which cannot be because the right hand side is bounded by $J(g)$. But then the left hand side of (8.2) is 0. Let $g_n \in \Phi^+$, $J(g_n) > 0$ be such that $g_n \uparrow 1$. Then by monotone convergence,

$$0 = \lim_n \int_0^1 (g_n, P^t\varphi_B)\, dt = \int_0^1 (1, P^t\varphi_B)\, dt = \int_{\mathfrak{G}} \varphi_B(x)\, dx.$$

Thus $\varphi_B(x) = 0$ a.e. That is impossible since we are assuming $C^\lambda(B) > 0$. This establishes the corollary.

DÉFINITION 8.1. — *A Borel set* B *is called recurrent if* $P_x(T_B < \infty) = 1$ *a.e. It is called co-recurrent if* $P_x(\tilde{T}_B < \infty) = 1$ *a.e. It is called transient* (*respectively co-transient*) *if it is not recurrent* (*respectively co-recurrent*).

From Corollary 8.1 we see that if the process is recurrent then any set B such that $C^\lambda(B) > 0$ is recurrent. Since $-X_t$ is also a recurrent process and $\tilde{C}^\lambda(B) = C^\lambda(B)$ every such set is also co-recurrent.

Throughout the remainder of this section we will assume that X_t is a transient process. Our aim is to show that there is a capacitory measure that is attached to every co-transient set and to develop the relevant capacity theory. The major results are summarized in Theorem 8.1.

PROPOSITION 8.2. — *Let* B *be a co-transient set. Then there is a unique Radon measure* μ_B *whose potential* $\mu_B G$ *has* $P_x(\tilde{T}_B < \infty)$ *as its density function.*

Proof. — Let $\tilde{\varphi}_B(x) = P_x(\tilde{V}_B < \infty)$ and let $\psi_h = [\tilde{\varphi}_B - \tilde{P}^h \tilde{\varphi}_B] \frac{1}{h}$. Then $\tilde{G}\psi_h \leqslant \tilde{\varphi}_B$ and $\tilde{G}\psi_h \downarrow \tilde{\varphi}_B$ a.e. Thus for any $f \in C_C^+$,

$$(\mu_h, Gf) = (f, \tilde{G}\psi_h) \uparrow (f, \tilde{\varphi}_B), \tag{8.3}$$

where $\mu_h(dx) = \psi_h(x)\,dx$. Given any compact set K we can find $f \in C_C^+$ such that $Gg(x) \geqslant \delta > 0$, $x \in K$ and thus

$$\mu_h(K) \leqslant \delta^{-1}(f, \varphi_B) < \infty. \tag{8.4}$$

Let K_r be a family of compacts $K_1 \subset \dot{K}_2 \subset K_2 \subset \cdots$, $\bigcup_n K_n = \mathfrak{G}$. Then for $f \in C_C^+$,

$$\begin{aligned} \int_{K_r'} \mu_h(dx) Gf(x) &= \int_{\mathfrak{G}} f(x)\,dx \int_{K_r'} \tilde{G}(x, dy)\psi_h(y) \\ &= \int_{\mathfrak{G}} f(x)\,dx \int_{K_r'} \tilde{H}_{K_r'}\tilde{G}(x, dy)\psi_h(y) \leqslant \int_{\mathfrak{G}} f(x)\,dx\, \tilde{H}_{K_r'}\tilde{\varphi}_B(x). \end{aligned} \tag{8.5}$$

Now for any fixed $t > 0$,

$$\tilde{H}_{K_r'}\tilde{\varphi}_B(x) = E_x[\tilde{\varphi}_B(\tilde{X}_{T'_{K_r}})] \leqslant P_x(\tilde{T}_{K_r'} \leqslant t) + \tilde{P}^t\tilde{\varphi}_B(x).$$

But $\tilde{P}^t\tilde{\varphi}_B \downarrow 0$ a.e. and $\tilde{T}_{K_r'} \uparrow \infty$ a.s. P_x as $r \to \infty$. Thus for any $f \in C_c^+$,

$$\lim_{r \to \infty} \int_{\mathfrak{G}} \tilde{H}_{K_r'}\tilde{\varphi}_B(x) f(x)\,dx = 0. \tag{8.6}$$

Let $\varepsilon > 0$ be given. Then from (8.5) and (8.6) we see that there is an r_0 such that for *all* h, $0 < h \leqslant 1$, and $r \geqslant r_0$,

$$\int_{K_r'} \mu_h\,(dx) Gf(x) \leqslant \varepsilon. \tag{8.7}$$

From (8.4) we see that there is a subsequence $h_n \downarrow 0$ and a Radon measure μ_B such that $\mu_{h_n} \to \mu_B$ vaguely. Now

$$\begin{aligned} \left|(f, \tilde{\varphi}_B) - \int_{K_r} \mu_{h_n}\,(dx) Gf(x)\right| &\leqslant |(f, \tilde{\varphi}_B) - (\mu_{h_n}, Gf)| \\ &\quad + \int_{K_r'} \mu_{h_n}\,(dx) Gf(x). \end{aligned} \tag{8.8}$$

Since $\tilde{G}f$ is a bounded continuous function it follows from (8.3) and (8.7) that for all $r \geqslant r_0$,

$$\left|(f, \tilde{\varphi}_B) - \int_{K_r} \mu_B (dx) Gf(x)\right| \leqslant \varepsilon,$$

and thus letting $r \to \infty$

$$|(f, \tilde{\varphi}_B) - (\mu_B, Gf)| \leqslant \varepsilon.$$

Since ε was arbitrary we see that $(f, \tilde{\varphi}_B) = (\mu_B, Gf)$. It is clear that $\mu_B G$ is a Radon measure and thus by Theorem 6.1 the measure μ_B is unique. This establishes the proposition.

COROLLARY 8.2. — *Let* B *be a co-transient set and let* $\mu_h(dx) = \frac{1}{h}[\tilde{\varphi}_B(x) - \tilde{P}^h\tilde{\varphi}_B(x)]\,dx$. *Then for all* $h > 0$, $\mu_h(\mathfrak{G}) = \mu_B(\mathfrak{G})$ *and the measures* μ_h *converge vaguely to* μ_B as $h \downarrow 0$.

Proof. — During the course of the proof of Proposition 8.2. it was shown that $\mu_h \to \mu_B$ vaguely. To complete the proof we must show that the μ_h have the common total mass $\mu_B(\mathfrak{G})$. To this end let $f_n \in C_C^+$ be such that $f_n \uparrow 1$. Then

$$(\mu_h, f_n = \left(\frac{[\tilde{\varphi}_B - \tilde{P}^h\tilde{\varphi}_B]}{h}, f_n\right).$$

By Proposition 8.1 $(\tilde{\varphi}_B, f_n) = (\mu_B, Gf_n)$ and so

$$(\tilde{P}^h\tilde{\varphi}_B, f_n) = (\tilde{\varphi}_B, P^h f_n) = (\mu_B, GP^h f_n).$$

Thus

$$(\mu_h, f_n) = \frac{1}{h}\int_0^h (\mu_B, P^s f_n)\,ds.$$

Letting $n \to \infty$ we see by monotone convergence that

$$\mu_h(\mathfrak{G}) = \mu_B(\mathfrak{G})$$

as desired.

DÉFINITION 8.2. — *Let* B *be a co-transient set. The measure* μ_B *in Proposition* 8.2 *is called the capacitory measure or equilibrium measure of* B. *Its total mass* C(B) *is called the capacity of* B. *Similarly if* B *is transient the corresponding measure*

$\tilde{\mu}_B$ is called the co-capacitory measure or co-equilibrium measure of B and its total mass $\tilde{C}(B)$ is called the co-capacity of B.

Proposition 8.3. — *If B is relatively compact then μ_B is finite and supported on $\overline{B}$. The measures $\mu_B^\lambda \to \mu_B$ weakly and $C(B) = \tilde{C}(B)$.*

Proof. — Let B be relatively compact. Then B is both transient and co-transient. Consequently, for any $f \in C_c^+$,

$$\infty > \int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty) f(x)\, dx \geqslant \int_{\mathfrak{G}} E_x(e^{-\lambda \tilde{T}_B}) f(x)\, dx = \mu_B^\lambda G^\lambda f(x).$$

Choose $f \in C_c^+$ such that $G^1 f(x) \geqslant \delta$, $x \in \overline{B}$. Then for any $x \in \overline{B}$ and $\lambda \leqslant 1$, $G^\lambda f(x) \geqslant G^1 f(x) \geqslant \delta$, so

$$\mu_B^\lambda(\overline{B}) \leqslant \delta^{-1} \int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty) f(x)\, dx.$$

Hence there is a sequence $\lambda_n \downarrow 0$ and a finite measure μ supported on $\overline{B}$ such that $\mu_B^{\lambda_n} \to \mu$ weakly. Since $G^\lambda f \uparrow Gf$, and Gf is a bounded continuous function, $G^\lambda f \to Gf$ uniformly on $\overline{B}$ and thus

$$\int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty) f(x)\, dx = \lim_{\lambda \downarrow 0} (\mu_B^\lambda, G^\lambda f) = \mu, Gf).$$

Thus by the uniqueness of μ_B, $\mu = \mu_B$. It mow follows that $\mu_B^\lambda \to \mu_B$ weakly. Since B is relatively compact,

$$\lim_{\lambda \downarrow 0} C^\lambda(B) = \lim_{\lambda \downarrow 0} (\mu_B^\lambda,\ 1) = (\mu_B,\ 1) = C(B),$$

and as $C^\lambda(B) = \tilde{C}^\lambda(B)$ we see that $C(B) = \tilde{C}(B)$.

Remark. — In point of fact stronger results are true. For any co-transient set B and for any relatively compact set A, $\mu_B^\lambda(A) \to \mu_B(A)$. This will be established in § 11.

Proposition 8.4. — *The set function $C(\cdot)$ is a Choquet capacity on the relatively compact sets that has the additional properties that $C(B + x) = C(B)$ for all x and $C(-B) = C(B)$.*

Proof. — We must show that properties (a) — (f) of Proposition 6.4 hold for $C(\cdot)$. That (a) and (b) hold for $C(B)$ follows at once from the fact that $C^\lambda(B) \to C(B)$, $\lambda \downarrow 0$. Let

K be compact. We can then find relatively compact open sets $A_1 \supset \bar{A}_2 \supset A_2 \supset \ldots$ such that $\bigcap_n A_n = K$. The times $\tilde{T}_{A_n} \uparrow \tilde{T}_K$ a.s. P_x and thus $P_x(\tilde{T}_{A_n} < \infty) \downarrow P_x(\tilde{T}_K < \infty)$. Consequently, for any $f \in C_G^+$,

$$\int_{\mathfrak{G}} P_x(\tilde{T}_{A_n} < \infty) f(x)\, dx \downarrow \int P_x(\tilde{T}_K < \infty) f(x)\, dx,$$

and thus by Proposition 8.2,

$$\lim_{n\to\infty} (\mu_{A_n}, Gf) = \mu_K, Gf). \tag{8.10}$$

Now $\mu_{A_1}(\bar{A}_1) < \infty$, and $\mu_{A_n}(\bar{A}_1)$ and $\mu_K(\bar{A}_1)$ are dominated by $\mu_{A_1}(\bar{A}_1)$. Thus there is a subsequence $\mu_{A_{n_j}}$ of the μ_{A_n} that converge weakly to a measure μ supported on $\bar{A}_1$. If follows from (8.10) that $\mu = \mu_K$, and thus $\mu_{A_n} \to \mu_K$ weakly. Hence $C(A_n) \downarrow C(K)$. If U is a relatively compact open set, then there are compact sets K_n, $K_1 \subset K_2 \subset \cdots$ such that $\bigcup_n K_n = U$. Arguing as above we find that $\mu_{K_n} \to \mu_U$ weakly and thus $C(K_n) \uparrow C(U)$. Thus $C(\bullet)$ is a Choquet capacity on the compact sets. Let C_* be its extension to the Borel sets. What we have just proved about relatively compact open sets shows that if U is such a set then $C_*(U) = C(U)$. Arguing now as in the case of λ-capacities we see that (*c*) and (*d*) hold for all relatively compact sets. Properties (*e*) and (*f*) follow from the fact that they are true for $C^\lambda(B)$ and $C^\lambda(B) \to C(B)$.

The next result is a corollary of the proof of the previous proposition.

PROPOSITION 8.5. — *Let* K *be compact and let* U_n *be open relatively compact set* $\supset K$ *such that* $U_n \downarrow K$. *Then* $\mu_{U_n} \to \mu_K$ *completely. Let* U *be an open relatively compact set and let* K_n *be compact* $K_n \subset U$ *and* $K_n \uparrow U$. *Then* $\mu_{K_n} \to \mu_U$ *completely.*

We will now show that for any co-transient set B, approximations from below are always possible. The approximation from above may fail since there need not be any co-transient open set $\supset B$ if B is not relatively compact.

PROPOSITION 8.6. — *Let* B *be a co-transient set and suppose* $A \subset B$. *Then* A *is co-transient and* $C(A) \leqslant C(B)$.

Proof. — Clearly A is co-transient and

$$P_x(\tilde{T}_A < \infty) \leqslant P_x(\tilde{T}_B < \infty)$$

and thus for any $f \geqslant 0$,

$$(\mu_A, Gf) \leqslant (\mu_B, Gf).$$

Since 1 is excessive we can find $f_n \geqslant 0$ and bounded such that $Gf_n \uparrow 1$. Thus $C(A) = \mu_A(\mathfrak{G}) \leqslant \mu_B(\mathfrak{G}) = C(B)$.

PROPOSITION 8.7. — *Let* B *be a co-transient set. Then* $\sup\{C(K) : K \subset B, K$ *compact*$\} = C(B)$. *Moreover if* $K_n \subset B$, K_n *relatively compact, and* $K_1 \subset K_2 \subset \cdots$, *are such that* $P_x(\tilde{T}_{K_n} \downarrow \tilde{T}_B) = 1$ *a.e. then* $\mu_{K_n} \to \mu_B$ *vaguely and* $C(K_n) \uparrow C(B)$, *so if* $C(B) < \infty$ *then the convergence is complete.*

Proof. — Let K_n satisfy the hypothesis of the theorem. Then

$$(\mu_{K_n}, Gf) \uparrow (\mu_B, Gf). \tag{8.11}$$

By essentially the same argument as used in the proof of Proposition 6.6 we can show that $\mu_{K_n} \to \mu_B$ vaguely so we will omit these details. Now from Proposition 8.6 we know that $C(K_n) \leqslant C(B)$ so $\lim_n C(K_n) \leqslant C(B)$. On the other hand, if $f \in C_c^+$, $0 \leqslant f \leqslant 1$, we know that

$$(\mu_{K_n}, f) \leqslant C(K_n)$$

and thus $(\mu_B, f) \leqslant \lim_n C(K_n)$. Letting $f \uparrow 1$ we see that $C(B) \leqslant \lim_n C(K_n)$. Hence $\lim_n C(K_n) = C(B)$. If $C(B) < \infty$ vague convergence becomes weak convergence and since we have just shown there is no escape of mass the weak converge is complete. Finally, by 10.16 of Chapter I [2] we can find compacts $K_n \subset B$ such that $P_x(\tilde{T}_{K_n} \uparrow \tilde{T}_B) = 1$ a.e. x and thus $C(K_n) \uparrow C(B)$. Hence

$$\sup\{C(K) : K \subset B, K \text{ compact}\} = C(B).$$

This establishes the proposition.

COROLLARY 8.3. — *If* B *is transient and co-transient then* $\tilde{C}(B) = C(B)$.

Proof. — Immediate from the previous proposition and the fact that for relatively compact sets K, $C(K) = \tilde{C}(K)$.

PROPOSITION 8.8. — *Let* B_n *be co-transient sets such that* $C(B_n) = 0$, $n = 1, 2, \ldots$ *Then* $\bigcup_n B_n$ *is co-transient and* $C\left(\bigcup_n B_n\right) = 0$.

Proof. — Since $C(B_n) = 0$ it follows from Proposition 8.2 that $P_x(\tilde{T}_{B_n} < \infty) = 0$ a.e., and thus $P_x\left(\tilde{T}_{\bigcup_n B_n} < \infty\right) = 0$ a.e. Hence $\bigcup_n B_n$ is co-transient and $C\left(\bigcup_n B_n\right) = 0$.

The following is one of the most fundamental facts about transient i.d. processes.

THEOREM 8.1. — *Let* B *be a co-transient set. Then there is a unique Radon measure* μ_B *supported on* $\bar{B}$ *such that* $\mu_B G\,(dx) = \tilde{\varphi}_B(x)\,dx$. *The total mass* C(B) *of* μ_B *is finite whenever* $\bar{B}$ *is compact. Whenever* B *is both transient and co-transient* $C(B) = \tilde{C}(B)$.

Proof. — It follows from Proposition 8.2 that there is a unique Radon measure μ_B whose potential is $\tilde{\varphi}_B(x)\,dx$. From Proposition 8.3 we see that if B is relatively compact then $\mu_B(\mathfrak{G}) < \infty$ and μ_B is supported on $\bar{B}$. The fact that in general μ_B is supported on $\bar{B}$ follows from this fact and Proposition 8.7. The final assertion in the theorem is just Corollary 8.3. This establishes the theorem.

9. On sets in $\mathcal{B}_4$.

Recall that $\mathcal{B}_4$ consists of those sets $B \in \mathcal{B}$ having a non-empty interior and such that $P_x(T_{\mathring{B}} = T_{\bar{B}}) = 1$ for almost all $x \in \mathfrak{G}$. In this section we will develop some properties of sets in $\mathcal{B}_4$ and also find a sufficient condition for sets to be in $\mathcal{B}_4$ in the special case that $\mathfrak{G}$ be isomorphic to a closed subgroup in Euclidean space. We will establish at the end of this section that, in general, if B is a relatively compact set there is a compact set $B_4 \in \mathcal{B}_4$ containing B.

PROPOSITION 9.1. — *Let* B *be a Borel set in* $\mathfrak{G}$ *and let* $y_n \in G$ *with* $y_n \to 0$ *as* $n \to \infty$. *Then*

$$P_x(\lim_{n\to\infty} T_{B-y_n} = T_B) = 1$$

holds at all x *such that* $P_x(T_{\mathring{B}} = T_{\overline{B}}) = 1$.

Proof. — Let B_n be the closed set consisting of all points whose distance from $\overline{B}$ is no larger than the maximum distance from the origin to y_k, $k \geqslant n$. Then B_n are closed sets and $B_n \downarrow B$ as $n \to \infty$. By quasi-left-continuity

$$P_x(T_{B_n} \uparrow T_B \quad \text{as} \quad n \to \infty) = 1, \qquad x \in \mathfrak{G}.$$

Consequently

$$P_x(\liminf_{n\to\infty} T_{B-y_n} \geqslant T_{\overline{B}}) = 1, \qquad x \in \mathfrak{G}.$$

It is also clear that

$$P_x(\limsup_{n\to\infty} T_{B-y_n} \leqslant T_{\mathring{B}}) = 1, \qquad x \in \mathfrak{G}.$$

In other words

$$P_x(T_{\overline{B}} \leqslant \liminf_{n\to\infty} T_{B-y_n} \leqslant \limsup_{n\to\infty} T_{B-y_n} \leqslant T_{\mathring{B}}) = 1.$$

Thus if $P_x(T_{\mathring{B}} = T_{\overline{B}}) = 1$, then

$$P_x(\lim_{n\to\infty} T_{B-y_n} = T_B) = 1,$$

as desired.

PROPOSITION 9.2. — *Let* B *be a Borel set having a non-empty interior and let* $f \in C_c$. *Then* $G_B f$ *and* $H_B f$ *are continuous at every* x *such that* $P_x(T_{\mathring{B}} = T_{\overline{B}}) = 1$.

To prove that $G_B f$ is continuous at x, we need only prove that if $y_n \to 0$ as $n \to \infty$, then $G_B f(x + y_n) \to G_B f(x)$. Now

$$G_B f(x + y_n) = E_x \int_0^{T_{B-y_n}} f(X_t + y_n)\, dt.$$

Since B has a non-empty interior and f has compact support the desired result follows by dominated convergence and Proposition 9.1.

To prove that $H_B f$ is continuous at x, we need only prove that if $y_n \to 0$ as $n \to \infty$, then $H_B f(x + y_n) \to H_B f(x)$. Now

$$H_B f(x + y_n) = E_x\{f(X_{T_{B-y_n}} + y); T_{B-y_n} < \infty\}.$$

To prove the desired result we need only show that

$$(9.1) \quad P_x(T_B = \infty \text{ or } X_{T_{B-y_n}} \to X_{T_B} \text{ as } n \to \infty) = 1.$$

To see that this is the case note that, except for a set of P_x probability zero, $X_{T_{B_n}} \to X_{T_B}$ as $n \to \infty$ on the set $(T_B < \infty)$, where B_n is defined as in the proof of Proposition 9.1. Since $T_{B_n} \leqslant T_B$, if $X_{T_{B_n}} \to X_{T_B}$, then either X_t is continuous at $t = T_B$ or $T_{B_n} = T_B$ for n sufficiently large. Since $T_{B-y_n} \geqslant T_{B_n}$ and $T_{B-y_n} \to T_B$ as $n \to \infty$ (except on a P_x null set), it follows that (9.1) holds, as desired.

From Proposition 9.2 we obtain immediately

THEOREM 9.1. — *Let* $B \in \mathcal{B}_4$ *and let* $f \in C_c$. *Then* $G_B f$ *and* $H_B f$ *are continuous for almost all* $x \in \mathfrak{G}$.

For $t \geqslant 0$ set

$$T_B^t = \inf[s \geqslant t \mid X_s \in B].$$

PROPOSITION 9.3. — *Let* $B \in \mathcal{B}_4$, $t \geqslant 0$, *and* $y_n \in G$ *with* $y_n \to 0$ *as* $n \to \infty$. *Then*

$$P_x\left(\lim_{n\to\infty} T_{B+y_n}^t = T_B^t\right) = 1 \qquad \text{a.e.} \qquad x \in \mathfrak{G}.$$

Proof. — Let f be any probability density function on $\mathfrak{G}$ and set $\mu = fP^t$. Then μ is absolutely continuous, so by Proposition 9.1

$$\begin{aligned} 1 &= P_\mu\left(\lim_{n\to\infty} T_{B+y_n} = T_B\right) \\ &= \int_{\mathfrak{G}} f(x)\, dx P_x\left(\lim_{n\to\infty} T_{B+y_n}^t = T_B^t\right). \end{aligned}$$

Since f is an arbitrary probability density on $\mathfrak{G}$, the conclusion of Proposition 9.1 is valid.

COROLLARY 9.1. — *Suppose the process is transient. Let* $B \in \mathcal{B}_4$, $t \geqslant 0$, *and* $y_n \in \mathfrak{G}$ *with* $y_n \to 0$ *as* $n \to \infty$. *Then* *a.e.* $x \in \mathfrak{G}$ $P_x(T_B^t = \infty$ *iff* $T_{B+y_n}^t = \infty$ *for* n *sufficiently large*$) = 1$.

Proof. — This result follows from the fact that $P_x(X_t \to \infty$ as $t \to \infty) = 1$ and there is a compact set B_1 such that $B + y_n \subseteq B_1$ for all $n \geqslant 1$.

Recall that if $T_B < \infty$, then

$$W_B = \sup[t | X_t \in B].$$

COROLLARY 9.2. — *Suppose the process is transient. Let* $B \in \mathcal{B}_4$ *and let* $y_n \in \mathfrak{G}$ *with* $y_n \to 0$ *as* $n \to \infty$. *Then*

$$P_x\Big(T_B = \infty \quad or \quad \lim_{n \to \infty} W_{B+y_n} = W_B\Big) = 1, \quad a.e. \quad x \in \mathfrak{G}.$$

Proof. — Let $0 \leqslant s < \infty$. Then

$$(W_B \leqslant s) = (T_B < \infty) \cap (T_B^t = \infty \text{ for } t > s \text{ and } t \text{ rational}).$$

Applying this to $B + y_n$ and using Corollary 9.1 we see that a.e. $x \in \mathfrak{G}$, $P_x(T_B = \infty$ or $W_B \leqslant s$ if $W_{B+y_n} \leqslant s$ for n sufficiently large$) = 1$. The result now follows from this fact.

PROPOSITION 9.4. — *Let* $B \in \mathcal{B}_4$. *Then*

$$P_x(T_B < \infty, X_{W_B-} \neq X_{W_B}, \text{ and } X_{W_B} \in \overline{B}) = 0 \quad a.e. \quad x \in \mathfrak{G}.$$

Proof. — This result is quite obvious. For from almost all points in $\overline{B}$ the process moves with probability one, immediately into $\mathring{B}$. Also if x has an absolutely continuous initial distribution, then for any $\varepsilon > 0$ and positive integer j the location of the process after the j^{th} jump of magnitude at least ε has an absolutely continuous distribution.

PROPOSITION 9.5. — *Suppose the process is transient. Let* $B \in \mathcal{B}_4$, *let* $f \in C$, *and let* $y_n \in \mathfrak{G}$ *with* $y_n \to 0$ *as* $n \to \infty$. *Then a.e.* $x \in \mathfrak{G}$

$$P_x(T_B = \infty \quad or \quad \lim_{n \to \infty} f(X_{W_{B+y_n}-}) = f(x_{W_B-})) = 1.$$

Proof. — We can suppose that $T_B < \infty$, $T_{B+y_n} < \infty$ for n sufficiently large, $W_{B+y_n} \to W_B$ as $n \to \infty$, and if X_t

has a jump at $t = W_B$, then $X_{W_B} \notin \overline{B}$. If $X_{W_B} \notin \overline{B}$, then $W_{B+y_n} \leqslant W_B$ for n sufficiently large. Since X_t has left-hand limits, if follows that

$$f(X_{W_{B+y_n}}) \to f(X_{W_B-}) \qquad \text{as} \qquad n \to \infty$$

as desired.

From the above results we obtain immediately

THEOREM 9.2. — *Suppose the process is transient. Let* $B \in \mathcal{B}_4$, *let* $f \in C$, *and let* $0 < t \leqslant \infty$. *Then*

$$E_x(f(X_{W_B-}); W_B \leqslant t \qquad \textit{and} \qquad T_B < \infty)$$

is continuous at almost all $x \in \mathfrak{G}$.

Suppose now that $\mathfrak{G} = R^{d_1} \oplus Z^{d_2}$. A subset Δ of $\mathfrak{G}$ will be called a sector if there are real numbers $u_1, \ldots, u_{d_1}, a$, and c with $u_1^2 + \cdots + u_{d_1}^2 > 0$, $a > 0$, and $-1 < c < 1$ such that

$$\Delta = \{x \in \mathfrak{G} | x_{d_1+1} = \cdots = x_{d_1+d_2} = 0, 0 < x_1^2 + \cdots + x_{d_1}^2 < a^2 \text{ and } x_1u_1 + \cdots + x_{d_1}u_{d_1} > c(x_1^2 + \cdots + x_{d_1}^2)^{\frac{1}{2}}\}.$$

(Here $x = (x_1, \ldots, x_{d_1+d_2})$.) By definition sectors are open in $\mathfrak{G}$.

Let B be a Borel subset of $\mathfrak{G}$. If Δ is a sector, then

$$D = \{x \in \partial B | x + \Delta \subseteq B\}$$

is a closed subset of $\mathfrak{G}$. Clearly $D + \Delta \subseteq \dot{B}$. We will show next that $|D| = 0$. Let L be a line segment in $\mathfrak{G}$ having one end point at the origin and all other points of L lying in Δ. Then for $x \in \mathfrak{G}$ the line segment $x + L$ contains at most two points of D, as is geometrically evident from the fact that $(D + \Delta) \cap D$ is empty. Consequently

$$|D \cap (x + \Delta)| = 0 \qquad \text{for} \qquad x \in \mathfrak{G}.$$

Thus

$$0 = \int_{\mathfrak{G}} |D \cap (x + \Delta)| \, dx = \int_{\mathfrak{G}} dx \int_{\mathfrak{G}} 1_D(y) 1_\Delta(y - x) \, dy = |D||\Delta|.$$

Since $|\Delta| \neq 0$, it follows that $|D| = 0$ as desired.

THEOREM 9.3. — *Let* $\mathfrak{G} = \mathrm{R}^{d_1} \oplus \mathrm{Z}^{d_2}$. *Let* B *be a Borel set in* $\mathfrak{G}$ *such that for some countable number of sectors* Δ_i

$$\bigcup_i \{x \in \partial\mathrm{B} | x + \Delta_i \in \mathrm{B}\} = \partial\mathrm{B}.$$

Then $|\partial\mathrm{B}| = 0$ *and* $\mathrm{P}_x(\mathrm{T}_{\bar{\mathrm{B}}} = \mathrm{T}_{\mathring{\mathrm{B}}}) = 1$ *a.e.* $x \in \mathfrak{G}$.

Remark 1. — It is easily seen that sets having sufficiently smooth boundaries are of the type covered by the theorem. For example, if $\mathfrak{G} = \mathrm{R}^d$, then polyhedrons and balls are of the required type. It is also clear that the theorem could be extended by allowing the boundary to behave wildly on a subset of the boundary having zero λ-capacity.

Remark 2. — The proof of the theorem does not use the fact that $\mathfrak{G}$ is generated by the union of the supports S_t of X_t, $t \geqslant 0$. This will be relevant to the proof of the following theorem.

Proof of Theorem 9.3. — The fact that $|\partial\mathrm{B}| = 0$ is obvious from the discussion preceding the theorem. Suppose that the other conclusion of the theorem is false. Then we can find a sector Δ, an $\varepsilon > 0$, and an $\mathrm{M} < \infty$ such that if

$$\mathrm{D} = \{x \in \partial\mathrm{B} | x + \Delta \in \mathrm{B}\}$$

and

$$\mathrm{S} = \{x \in \mathrm{G} | \mathrm{P}_x(\mathrm{T}_{\bar{\mathrm{B}}} + \varepsilon \leqslant \mathrm{T}_{\mathring{\mathrm{B}}}, \mathrm{T}_{\bar{\mathrm{B}}} \leqslant \mathrm{M}, \mathrm{X}_{\mathrm{T}\bar{\mathrm{B}}} \in \mathrm{D}) \geqslant \varepsilon\},$$

then $|\mathrm{S}| > 0$. Since

$$\int_{\mathfrak{G}} |\mathrm{S} \cap (x + \Delta)| \, dx = |\mathrm{S}||\Delta| > 0,$$

we can find an $x_0 \in \mathfrak{G}$ such that $|\mathrm{S} \cap (x_0 + \Delta)| > 0$. Consequently we can find a line segment $\mathrm{L} \subseteq \mathfrak{G}$ such that one end point is at x_0, all other points of L are in $x_0 + \Delta$, and L contains infinitely many points of S.

Observe that

$$\mathrm{S} = \{x | \mathrm{P}_0(\mathrm{T}_{\bar{\mathrm{B}}-x} + \varepsilon \leqslant \mathrm{T}_{\mathring{\mathrm{B}}-x}, \mathrm{T}_{\bar{\mathrm{B}}-x} \leqslant \mathrm{M}, \mathrm{X}_{\mathrm{T}_{\bar{\mathrm{B}}-x}} \in \mathrm{D} - x) \geqslant \varepsilon\}.$$

Let Ω be the probability space for the process starting out at the origin. Set

$$\Omega_x = \{\omega | \mathrm{T}_{\bar{\mathrm{B}}-x} + \varepsilon \leqslant \mathrm{T}_{\mathring{\mathrm{B}}-x}, \mathrm{T}_{\bar{\mathrm{B}}-x} \leqslant \mathrm{M}, \mathrm{X}_{\mathrm{T}_{\bar{\mathrm{B}}-x}} \in \mathrm{D} - x\}.$$

Consider $x, y \in \mathfrak{G}$ such that $y - x \in \Delta$ and $\omega \in \Omega_x \cap \Omega_y$. Then

$$X_{T_{\bar{B}-x}} + y \in D - x + y \subseteq D + \Delta \subseteq \mathring{B}.$$

Thus $T_{\mathring{B}-y} \leqslant T_{\bar{B}-x} \leqslant M$ and hence

$$T_{\mathring{B}-y} \leqslant T_{\mathring{B}-y} - \varepsilon \leqslant M - \varepsilon.$$

Consider next $x_1, \ldots, x_n$ all in $\mathfrak{G}$ such that $x_{j+1} - x_j \in \Delta$ for $1 \leqslant j < n$ and $\omega \in \cap_1^n \Omega_{x_j}$. Then

$$0 \leqslant T_{\bar{B}-x_n} \leqslant M - (n-1)\varepsilon$$

and hence $n \leqslant 1 + \varepsilon^{-1}M$.

It follows from the definition of L that for any positive integer ν we can find points $y_1, \ldots, y_\nu \in L \cap S$ such that $y_j - y_i \in \Delta$ for $1 \leqslant i < j \leqslant \nu$. Set $\Omega_j = \Omega_{y_j}$. Then $P(\Omega_j) \geqslant \varepsilon$ for $1 \leqslant j \leqslant \nu$ since $y_j \in S$. By the above paragraph

$$\sum_{j=1}^{\nu} 1_{\Omega_j}(\omega) \leqslant 1 + \varepsilon^{-1}M, \qquad \omega \in \Omega.$$

We thus have the inequality

$$\varepsilon\nu \leqslant \sum_{j=1}^{\nu} P(\Omega_j) = E\left(\sum_{j=1}^{\nu} 1_{\Omega_j}(\omega)\right) \leqslant 1 + \varepsilon^{-1}M,$$

which implies that $\nu \leqslant \varepsilon^{-1} + \varepsilon^{-2}M$. This contradicts the fact that ν can be made arbitrarily large. Therefore the conclusion of the theorem must be true.

It is not clear how to find useful sufficient conditions for a set B in a general locally compact Abelian group $\mathfrak{G}$ to be in B_4. If $\mathfrak{G}$ is compactly generated, then $\mathfrak{G} \cong R^{d_1} \oplus Z^{d_2} \oplus H$, where H is compact. It is clear from Theorem 9.3 that if B_1 is a relatively compact subset of $\mathfrak{G}$ and $\mathfrak{G}$ is compactly generated, then there is a compact subset B of $\mathfrak{G}$ such that $B_1 \subseteq B$ and $B \in \mathcal{B}_4$.

Let $\mathfrak{G}$ be any locally compact Abelian group. Let B_1 be a relatively compact subset of $\mathfrak{G}$. Then there is an open compactly generated subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$ such that $B_1 \subseteq \mathfrak{G}_0$. Consider the process Y_t, $t \geqslant 0$, on $\mathfrak{G}_0$ obtained from X_t, $t \geqslant 0$, by ignoring those times when $X_t \notin \mathfrak{G}_0$. This process satisfies all the assumptions for an infinitely divisible process on $\mathfrak{G}_0$ except that $\mathfrak{G}_0$ is not necessarily generated by the union of the supports of Y_t, $t \geqslant 0$. In Remark 2 following the statement of Theorem 9.3, however, it was pointed out that this is not necessary for Theorem 9.3 to be valid. Thus, there is a compact subset B of $\mathfrak{G}_0$ containing B_1 and in $\mathcal{B}_4$ relative to the process Y_t, $t \geqslant 0$. It is clear that B is in $\mathcal{B}_4$ relative to the process X_t, $t \geqslant 0$, as well. Thus we have the following result (which will be needed later on).

THEOREM 9.4. — *Let* B_1 *be a relatively compact subset of* $\mathfrak{G}$. *Then there is a compact subset* B *of* $\mathfrak{G}$ *such that* $B_1 \subseteq B$ *and* $B \in \mathcal{B}_4$.

10. The Renewal Theorem and Type II Transient Processes.

We say the transient process if type I if

$$\lim_{x \to \infty} Gf(x) = 0, \qquad f \in C_c.$$

Otherwise the process is said to be type II. Suppose the random walk obtained by looking at the process at integer times is type I transient. It is then easily shown that the continuous time process is type I transient. Suppose the random walk is type II transient. Then the group $\mathfrak{G}_2$ generated by the support S_1 of X_1 is isomorphic to either $R \oplus H$ or $Z \oplus H$, where H is compact (by results in [7]). In particular $\mathfrak{G}_2$ is compactly generated. Thus by Proposition 5.19 $\mathfrak{G}$ is compactly generated. It now follows easily from the structure theorem that $\mathfrak{G}$ is isomorphic to $R \oplus H$ or $Z \oplus H$ for some compact group H. If $\mathfrak{G} = R \oplus H$ or $Z \oplus H$, the induced process on R or Z has finite non-zero mean. This also follows easily from the corresponding discrete time results proved in [7].

Suppose now that $\mathfrak{G} = R \oplus H$ or $\mathfrak{G} = Z \oplus H$, where H is compact. Let Haar measure dx on $\mathfrak{G}$ be such that $dx = dr\,dy$, where dy is normalized Haar measure on H and dr is Lebesgue measure on R or counting measure on Z. Define $\psi : \mathfrak{G} \to R$ or $\mathfrak{G} \to Z$ by $\psi(r + y) = r$ if $y \in H$. We set

$$\mathfrak{G}^+ = \{x \in \mathfrak{G} | \psi(x) \geqslant 0\}$$

and

$$\mathfrak{G}^- = \{x \in \mathfrak{G} | \psi(x) < 0\}.$$

By « $x \to +\infty$ » or « $x \to -\infty$ » we mean $x \to \infty$ and $x \in \mathfrak{G}^+$ or $x \in \mathfrak{G}^-$ respectively. The induced process on R or Z has finite mean if and only if for suitably defined m

$$\text{(10.1)} \qquad P^t\psi(0) = mt, \qquad 0 \leqslant t < \infty.$$

THEOREM 10.1. — *Let the process be type* II *transient. Then* $\mathfrak{G} \cong R \oplus H$ *or* $\mathfrak{G} \cong Z \oplus H$, *where* H *is compact. Suppose* $\mathfrak{G} = R \oplus H$ *or* $Z \oplus H$, H *being compact and* ψ *and Haar measure being chosen as indicated above. Then* (10.1) *holds for some finite non-zero* m. *Let* $\pm m > 0$, *then for* $f \in \Phi^*$

$$\lim_{x \to \mp\infty} Gf(x) = |m| J(f) \qquad \textit{and} \qquad \lim_{x \to \pm\infty} Gf(x) = 0.$$

Proof. — By what has been said we can assume that $\mathfrak{G} = R \oplus H$ or $Z \oplus H$, where H is compact and ψ and Haar measure are chosen as indicated above. We can also assume that (10.1) holds with, say, $0 < m < \infty$.

Suppose first that $\mathfrak{G}/\mathfrak{G}_1$ is compact. Then, by Proposition 5.3, S_t generates $\mathfrak{G}$ for some $t > 0$. Without loss of generality we can assume that S_1 generates $\mathfrak{G}$. Set

$$G' = \sum_{n=0}^{\infty} P^n.$$

Then

$$G = \int_0^1 P^t G' \, dt.$$

Choose $f \in \Phi^*$. Then

$$Gf(x) = Gf_{-x}(0) = \int_0^1 P^t G f_{-x}(0) \, dt.$$

Now $G'f_{-x}(y) = G'f(x+y)$ is bounded in x and y and for each y by Theorems 4.1 and 4.2 of [7]

$$\lim_{x \to +\infty} G'f_{-x}(y) = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} Gf_{-x}(y) = mJ(f).$$

Thus

$$\begin{aligned} \lim_{x \to -\infty} Gf(x) &= \lim_{x \to -\infty} \int_0^1 P^t(0, dy) G' f_{-x}(y) \, dt \\ &= mJ(f) \int_0^1 P^t(0, \mathfrak{G}) \, dt \\ &= mJ(f) \end{aligned}$$

and similarly

$$\lim_{x \to \infty} \mathrm{G}f(x) = 0$$

as desired.

Suppose next that $\mathfrak{G}/\mathfrak{G}_1$ is not compact. Then $\mathfrak{G} = \mathrm{R} \oplus \mathrm{H}$, $\mathfrak{G}_1 = \mathrm{H}$, and the process is singular. The induced process on R moves deterministically with velocity m. Let $f \in \mathrm{C}_c$. Then clearly

$$\lim_{x \to +\infty} \mathrm{G}f(x) = 0$$

so we need only prove that

$$\lim_{x \to -\infty} \mathrm{G}f(x) = m\mathrm{J}(f).$$

By translating f if necessary, we can assume that f is supported by $\mathfrak{G}^+$. Then for $t \geqslant 0$

$$\mathrm{G}f(-mt + y) = \mathrm{P}^t\mathrm{G}f(-mt + y), \qquad y \in \mathrm{H}.$$

Let Q^t denote the transition operator for the induced process on H. We can let Q^t act on functions on $\mathfrak{G}$ by considering H as embedded in $\mathfrak{G}$ in the obvious way. Then

$$\mathrm{P}^t\mathrm{G}f(-mt + y) = \mathrm{Q}^t\mathrm{G}f(y), \qquad y \in \mathrm{H}.$$

By the Ito-Kawata Theorem

$$\lim_{t \to \infty} \mathrm{Q}^t\mathrm{G}f(y) = \int_{\mathrm{H}} \mathrm{G}f(z)\, dz$$

uniformly in y. An elementary computation shows that

$$\int_{\mathrm{H}} \mathrm{G}f(z)\, dz = m\mathrm{J}(f).$$

Putting these facts together, we see that

$$\lim_{x \to -\infty} \mathrm{G}f(x) = m\mathrm{J}(f)$$

as desired.

For a type II transient process there is a non-trivial theory of the asymptotic behaviour of $\mathrm{G_B}f\,(x)$ and $\mathrm{H_B}f(x)$ as $x \to \infty$ for any Borel set B. We now proceed to develop this theory. Throughout the remainder of this section the process X_t will be assumed to be a type II transient process having $m > 0$.

PROPOSITION 10.1. — *Let* B *be any co-transient set. Then* $C(B) \leqslant m$ *and* $\lim_{y \to +\infty} (\tilde{\Phi}_B, f_y) = (C(B)/m)\, J(f)$ *for any function* $f \in \Phi^*$.

Proof. — By Theorem 8.1 for any $f \in (\Phi^*)^+$,

$$(\mu_B, Gf_y) = (\tilde{\Phi}_B, f_y) \leqslant J(f).$$

Hence by Fatou's lemma and Theorem 10.1,

$$C(B)\frac{J(f)}{m} \leqslant J(f)$$

so $C(B) \leqslant m$. Since $Gf_y(x) \leqslant M$ for some finite constant M it now follows by dominated convergence and Theorem 10.1 that $\lim_{y \to +\infty} (\tilde{\Phi}_B, f_y) = (C(B)/m)J(f)$.

PROPOSITION 10.2. — *Let* B *be a co-transient and recurrent set. Then* $C(B) = m$.

Proof. — Note that $B \cap \mathfrak{G}^+ \subset B$, $B \cap \mathfrak{G}^+$ is co-transient and thus by Proposition 8.5 $C(B) \geqslant C(B \cap \mathfrak{G}^+)$. To establish the proposition then it suffices to prove if for a recurrent subset of $\mathfrak{G}^+$. Let B be such a set and let K_n be relatively compact sets such that $K_1 \subset K_2 \subset \ldots$, $\cup_n K_n = B$. Then $\Phi_{K_n} \uparrow \Phi_B$ and so if B is recurrent we see that for any $f \in C_C^+$, $(\Phi_{K_n}, f) \uparrow J(f)$. Thus by Theorem 8.1

$$(10.2) \qquad (\tilde{\mu}_{K_n}, \tilde{G}f) \uparrow J(f)$$

and since

$$\tilde{\mu}_{K_n}(\mathfrak{G}) = \tilde{C}(K_n) \leqslant \tilde{C}(B) \leqslant m$$

we can find a subsequence $\tilde{\mu}_{K_{n'}}$, a finite measure γ on $\mathfrak{G}$, and a constant α such that for any bounded continuous function ψ having a limit $\psi(+\infty)$ at $+\infty$,

$$(10.3) \qquad \lim_{n' \to \infty} (\tilde{\mu}_{K_{n'}}, \psi) = (\gamma, \psi) + \alpha\psi(+\infty).$$

Applied to $\psi = \tilde{G}f$ it follows from (10.2) and (10.3) that for any $f \in C_C^+$

$$(\gamma, \tilde{G}f) + \frac{\alpha J(f)}{m} = J(f).$$

Thus

$$\gamma\tilde{G}\,(dx) + \frac{\alpha}{m}\,dx = dx. \tag{10.4}$$

Letting the measures in (10.4) act on f and $\tilde{P}^t f$ and using the fact that $\tilde{G}f - \tilde{G}\tilde{P}^t f = \int_0^t \tilde{P}^s f\,ds$ we see that $\gamma = 0$ and $\alpha = m$. The standard weak compactness argument now shows that

$$\lim_{n\to\infty} (\tilde{\mu}_{K_n}, \psi) = m\psi(+\infty).$$

Thus the measures $\tilde{\mu}_{K_n}$ converge to the mass m at $+\infty$, so $\tilde{C}(K_n) = \tilde{\mu}_{K_n}(\mathfrak{G}) \to m$. By Proposition 8.3 $\tilde{C}(K_n) = C(K_n)$, so by Propositions 10.1 and 8.7

$$m = \lim_n \tilde{C}(K_n) = \lim_n C(K_n) \leqslant C(B) \leqslant m.$$

Thus $C(B) = m$, as desired.

Corollary 10.1. — *If* B *is recurrent and co-transient then for any* $f \in \Phi^*$,

$$\lim_{y\to+\infty} (\tilde{\Phi}_B, f_y) = J(f).$$

Proof. — This follows at once from Proposition 10.1 and 10.2.

Theorem 10.2. — *Let* B *be any Borel set. Then for any* $\varphi \in \Phi^*$,

$$\lim_{y\to+\infty} G_B\varphi_y(x) = \frac{J(\varphi)}{m}\,P_x(T_B = \infty). \tag{10.5}$$

Let $\varphi \in C_c$. *If* B *is a co-transient set and* f *is any continuous bounded function*

$$\lim_{y\to-\infty} \int_{\mathfrak{G}} \varphi_y(x) H_B f(x)\,dx = \frac{J(\varphi)}{m}\,(\mu_B, f). \tag{10.6}$$

If B *is a co-recurrent set then for any bounded Borel function* f *having a limit* $f(-\infty)$ *at* $-\infty$,

$$\lim_{y\to-\infty} \int_{\mathfrak{G}} \varphi_y(x) H_B f(x)\,dx = f(-\infty) J(\varphi) \tag{10.7}$$

and $\lim_{y\to-\infty} (\varphi_y, \Phi_B) = J(\varphi)$.

Proof. — The first passage relation shows that

$$(10.8) \qquad G_B\varphi_y = G\varphi_y - H_BG\varphi_y.$$

Since $G\varphi_y(x) = G\varphi(x-y)$, and $\sup_x G\varphi(x) = M < \infty$ (10.5) follows from (10.8) and Theorem 10.1. Now let $\varphi \in C_c$ and let $f \in C_c$. Then as $(\varphi_y, \tilde{G}_B f) = (f, G_B\varphi_y)$ we see from (10.5) that

$$\lim_{y \to +\infty} (\varphi_y, \tilde{G}_B f) = \frac{J(\varphi)}{m} \int_{\mathfrak{G}} P_x(T_B = \infty) f(x)\, dx.$$

Applied to the dual process $-X_t$ we see that

$$(10.9) \quad \lim_{y \to -\infty} (\varphi_y, G_B f) = \frac{J(f)}{m} \int_{\mathfrak{G}} P_x(\tilde{T}_B = \infty) f(x)\, dx.$$

Thus using the first passage relation again we see that

$$(10.10) \qquad \lim_{y \to -\infty} (\varphi_y H_B, Gf) = \frac{J(\varphi)}{m} (\tilde{\Phi}_B, f).$$

The total mass $(\varphi_y H_B, 1) = (\varphi_y, \Phi_B)$ of the measures $\varphi_y H_B$ is $\leqslant J(\varphi)$ and thus there is a subsequence $y_n \to -\infty$, a finite measure γ supported on $\overline{B}$, and constants $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$ such that for any bounded continuous function ψ having limits at $+\infty$ and $-\infty$,

$$\lim_{n \to \infty} (\varphi_{y_n} H_B, \psi) = (\gamma, \psi) + \alpha_1 \psi(-\infty) + \alpha_2 \psi(+\infty).$$

Applied to $\psi = Gf$ we see that

$$(10.11) \qquad \frac{J(\varphi)}{m} (\tilde{\Phi}_B, f) = (\gamma, Gf) + \alpha_1 \frac{J(f)}{m}.$$

The total mass of the measures $\varphi_y H_B$ is (φ_y, Φ_B) and thus if B is recurrent they have the common mass $J(\varphi)$. If B is transient, it follows from Proposition 10.1 applied to the reverse process that

$$(10.12) \qquad \lim_{y \to -\infty} (\varphi_y, \Phi_B) = \frac{\tilde{C}(B)}{m} J(\varphi).$$

Suppose now that B is co-transient. Then by Theorem 8.1

and (10.11) we see that

$$\frac{J(\varphi)}{m}\mu_B G\,(dx) = \gamma G\,(dx) + \frac{\alpha_1}{m}\,dx. \tag{10.13}$$

Letting this act on f and $P^t f$ for $f \in C_c$ and using the facts that $Gf - GP^t f = \int_0^t P^s f$, and that $P^s f \to f$, $s \to 0$ uniformly, we can conclude from (10.13) that $\alpha_1 = 0$ and that $\gamma = (J(\varphi)/m)\mu_B$. If we had another weakly converging subsequence of the $\varphi_y H_B$ the same argument would again show that it converged to $(J(\varphi)/m)\mu_B$. Thus the measures $\varphi_y H_B$ converge weakly to $(J(\varphi)/m)\mu_B$. The total mass of this measure is $(C(B)/m)J(\varphi)$. To show that the convergence is complete we must now show that this is the same as the limiting total mass of the $\varphi_y H_B$. If B is also recurrent then Proposition 10.2 shows that $C(B) = m$ so in this case $(J(\varphi)/m)\mu_B$ has total mass $J(\varphi)$ which is just the common total mass of the $\varphi_y H_B$. On the other hand if B is transient then Corollary 8.3 shows that $C(B) = \tilde{C}(B)$ so that the total mass of $(J(\varphi)/m)\mu_B$ is the same as the limiting total mass of the $\varphi_y H_B$. Hence in all cases when B is co-transient the $\varphi_y H_B$ converge completely as $y \to -\infty$ to the measure $(J(\varphi)/m)\mu_B$.

Suppose now that B is co-recurrent. Then (10.11) shows that

$$\frac{J(\varphi)}{m}\,dx = \gamma G\,(dx) + \frac{\alpha_1}{m}\,dx.$$

It follows from this that $\gamma = 0$ and $\alpha_1 = J(\varphi)$. Now assume that B is also recurrent. Then $(\varphi_y H_B, 1) = (\varphi_y, \Phi_B) = J(\varphi)$ so in this case $\alpha_2 = 0$ and the measures $\varphi_{y_n} H_B$ converge to the mass $J(\varphi)$ at $-\infty$. On the other hand if B is transient, then $(\varphi_y H_B, 1) = (\varphi_y, \Phi_B) = (\tilde{\mu}_B, \tilde{G}\varphi_y) \to (J(\varphi)/m)\tilde{C}(B)$, as $y \to -\infty$. By Proposition 10.2, as B is also recurrent, $\tilde{C}(B) = m$ so $(\varphi_y H_B, 1) \to J(\varphi)$. Thus here too $\alpha_2 = 0$, so $\varphi_{y_n} H_B$ converges to the mass $J(\varphi)$ at $-\infty$. This establishes 10.7 and thereby completes the proof of Theorem 10.2.

PROPOSITION 10.3. — *Let* B *be any Borel set. Then for any* $f \in \Phi$,

$$\lim_{y \to -\infty} G_B f_y(x) = 0. \tag{10.14}$$

Let $\varphi \in \Phi$. *If* B *is a transient set*

$$\lim_{y \to +\infty} (\varphi_y, \mathrm{H_B} f) = 0. \tag{10.15}$$

If B *is a recurrent set then for any bounded continuous* f *having a limit* $f(+\infty)$ *at* $+\infty$

$$\lim_{y \to +\infty} (\varphi_y, \mathrm{H_B} f) = \mathrm{J}(\varphi) f(+\infty). \tag{10.16}$$

Proof. — Equation (10.14) follows at once from the renewal theorem and the fact that $|\mathrm{G_B} f_y| \leqslant \mathrm{G}|f_y|$. Equation (10.15) follows from the fact

$$|(\varphi_y, \mathrm{H_B} f)| \leqslant \|f\|_\infty (|\varphi_y|, \Phi_\mathrm{B}) = \|f\|_\infty (\tilde{\mu}_\mathrm{B}, \tilde{\mathrm{G}}|\varphi_y|),$$

Proposition 10.2, and the renewal theorem.

Now assume B is recurrent and let $\varphi \in \Phi$. Then $(\varphi_y, \mathrm{H_B} 1) = (\varphi_y, \Phi_\mathrm{B}) = \mathrm{J}(\varphi)$. Let $\psi \in \mathrm{C}_c$. Then

$$(\varphi_y \mathrm{H_B}, \mathrm{G}\psi) = (\varphi_y, \mathrm{G}\psi) - (\varphi_y, \mathrm{G_B}\psi).$$

By (10.14) and duality we see that the right hand side converges to zero as $y \to +\infty$. Thus

$$\lim_{y \to +\infty} (\varphi_y \mathrm{H_B}, \mathrm{G}\psi) = 0. \tag{10.17}$$

There is a subsequence $y_n \to +\infty$ and a finite measure γ on $\mathfrak{G}$ and constants α_1, α_2 such that for any bounded continuous f having limits at $+\infty$ and $-\infty$

$$\lim_n (\varphi_{y_n} \mathrm{H_B}, f) = (\gamma, f) + \alpha_1 f(-\infty) + \alpha_2 f(+\infty).$$

Applied to $\mathrm{G}\psi$ it follows from (10.17) and the above that

$$(\gamma, \mathrm{G}\psi) + \alpha_1 \frac{\mathrm{J}(\psi)}{m} = 0.$$

From this, it follows that $\gamma = 0$ and $\alpha_1 = 0$. Thus as $(\varphi_y, \Phi_\mathrm{B}) = \mathrm{J}(\varphi)$, it must be that $\alpha_2 = \mathrm{J}(\varphi)$. Thus the measures $\varphi_y \mathrm{H_B}$ converge to the mass $\mathrm{J}(\varphi)$ at $+\infty$. This completes the proof.

The previous results may be stated more succinctly by using the two point compactification $\mathfrak{G}^*$ of $\mathfrak{G}$. Take B^* to be B if B is relatively compact and take $\mathrm{B}^* = \mathrm{B} \cup \{+\infty\}$ if $\mathrm{B} \cap \mathfrak{G}^-$ is relatively compact but $\mathrm{B} \cap \mathfrak{G}^+$ is not. Similarly

$B^* = B \cup \{-\infty\}$ if $B \cap \mathfrak{G}^+$ is relatively compact but $B \cap \mathfrak{G}^-$ is not. If neither $B \cap \mathfrak{G}^+$ nor $B \cap \mathfrak{G}^-$ are relatively compact then take $B^* = B \cup \{-\infty, +\infty\}$. Now define $\mu_{B^*} = \mu_B$ if B is co-transient and define μ_{B^*} to be the mass m at $-\infty$ if B is co-recurrent. The measure $\tilde{\mu}_{B^*}$ is defined similarly. The capacity $C(B^*)$ of B^* is taken to be the total mass of μ_{B^*}. The function $G(x, A)$ is extended to $\mathfrak{G}^*$ in the obvious way. Then μ_{B^*} is the unique measure supported on B^* such that for any $f \in L_1(\mathfrak{G})$

$$(\Phi_{B^*}, f) = (\mu_{B^*}, Gf).$$

Also for any $\varphi \in C_c$ and any $f \in C(\mathfrak{G}^*)$

$$\lim_{y \to -\infty} (\varphi_y, H_B f) = \frac{J(\varphi)}{m} (\mu_{B^*}, f).$$

In all cases $C(B^*) = \tilde{C}(B^*)$ and $C(B^*) = m$ in all cases except perhaps when B is both transient and co-transient.

Examples show that the smoothed limits in Theorem 10.2 are the best versions that can be given for general Borel sets. We now examine conditions under which the smoothing can be dropped.

THEOREM 10.3. — *Let* $B \in \mathcal{B}_4$. *Then for any* $f \in C_c$

$$\lim_{x \to -\infty} G_B f(x) = \frac{1}{m} \int_{\mathfrak{G}} P_x(\tilde{T}_B = \infty) f(x)\, dx \tag{10.18}$$

and

$$\lim_{x \to -\infty} H_B f(x) = (\mu_B, f) \frac{1}{m}. \tag{10.19}$$

Proof. — Let K be compact and $K \in \dot{B}$. Then there is a symmetric open neighborhood N of 0 such that $K - N \subset \dot{B}$. (See [11].) Thus for any $y \in N$, $K - y \subset \dot{B}$ so $T_{K-y} \geqslant T_{\dot{B}}$. Hence for $y \in N$ and $f \in C_c^+$

$$\begin{aligned} G_K f(x+y) &= E_{x+y} \int_0^{T_K} f(X_t)\, dt \\ &= E_x \int_0^{T_{K-y}} f(X_t + y)\, dt \geqslant E_x \int_0^{T_{\dot{B}}} f_{-y}(X_t)\, dt. \end{aligned} \tag{10.20}$$

Similarly if U is open, $\overline{U}$ compact, and $\overline{B} \subset U$, then for some symmetric open neighborhood N of 0 $\overline{B} - N \subset U$ so for $y \in \mathrm{N}$ and $f \in \mathrm{C}_{\mathrm{C}}^{+}$

$$(10.21)\quad \mathrm{G}_{\mathrm{U}} f(x+y) \leqslant \mathrm{E}_x \int_0^{\mathrm{T}_{\bar{\mathrm{B}}}} f_{-y}(\mathrm{X}_t)\ dt = \mathrm{G}_{\bar{\mathrm{B}}} f_{-y}(x).$$

Now for $f \in \mathrm{C}_{\mathrm{C}}^{+}$, $\mathrm{G}f(x)$ is uniformly continuous. Let $\varepsilon > 0$ be given. Then there is an open symmetric neighborhood $\mathrm{S} \subset \mathrm{N}$ such that $\|\mathrm{G}f_y - \mathrm{G}f\| < \varepsilon$, $y \in \mathrm{S}$. Now by the first passage relation, for any set A,

$$(10.22)\quad \|\mathrm{G}_{\mathrm{A}} f_{-y} - \mathrm{G}_{\mathrm{A}} f\| \leqslant \mathrm{P}_x(\mathrm{T}_{\mathrm{A}} < \infty)\|\mathrm{G}f_{-y} - \mathrm{G}f\| + \|\mathrm{G}f_{-y} - \mathrm{G}f\| \leqslant 2\varepsilon.$$

Let $\varphi \in \mathrm{C}_{\mathrm{C}}^{+}$ be such that the support of φ is contained in S. Then from (10.20) and (10.22) and 10.9 we see that

$$(10.23)\quad \frac{1}{m}\int \mathrm{P}_y(\tilde{\mathrm{T}}_{\mathrm{K}} = \infty) f(y)\ dy \geqslant \overline{\lim_{y \to -\infty}}\ \mathrm{G}_{\dot{\mathrm{B}}} f(y).$$

Similarly from (10.21) and (10.22) and 10.9 it follows that

$$(10.24)\quad \frac{1}{m}\int_{\mathfrak{G}} \mathrm{P}_y(\tilde{\mathrm{T}}_{\mathrm{U}} = \infty) f(y)\ dy \leqslant \underline{\lim}_{x \to -\infty}\ \mathrm{G}_{\bar{\mathrm{B}}} f(x).$$

Choosing compacts $\mathrm{K}_n \downarrow \dot{\mathrm{B}}$ and open sets $\mathrm{U}_n \downarrow \bar{\mathrm{B}}$ and using the fact that $\mathrm{P}_x(\mathrm{T}_{\bar{\mathrm{B}}} = \mathrm{T}_{\dot{\mathrm{B}}}) = 1$ a.e. we see that (10.18) holds. Using the first passage relation and (10.19) it follows that

$$\lim_{x \to -\infty} \mathrm{H}_{\mathrm{B}} \mathrm{G} f(x) = \frac{1}{m}\int_{\mathfrak{G}} \mathrm{P}_x(\tilde{\mathrm{T}}_{\mathrm{B}} < \infty) f(x)\ dx.$$

There is a measure γ_{B} supported on $\bar{\mathrm{B}}$ and a sequence $x_n \to -\infty$ such that $\mathrm{H}_{\mathrm{B}}(x_n, dy) \to \gamma_{\mathrm{B}}$ weakly. Thus

$$(\gamma_{\mathrm{B}}, \mathrm{G}f) = \frac{1}{m}\int_{\mathfrak{G}} \mathrm{P}_x(\tilde{\mathrm{T}}_{\mathrm{B}} < \infty) f(x)\ dx.$$

Hence by Theorem 8.1 $\gamma_{\mathrm{B}} = \mu_{\mathrm{B}}/m$ and the usual weak compactness argument now shows that (10.19) holds. This completes the proof.

By appealing to results in discrete time we can obtain significant extensions of Theorems 10.2 and 10.3.

THEOREM 10.4. — *Let* B *be any co-transient set, let* $\varphi \in \mathrm{C}_c$, *and let* f *be a bounded Borel function. Then equation* (10.6) *holds.*

THEOREM 10.5. — *Suppose the process is non-singular. Let* $B \in \mathcal{B}$ *and* $f \in \Phi$. *Then* (10.18) *and* (10.19) *hold.*

Remark. — It is also not hard to show that Theorems 10.3 and 10.5 can be extended to include sets B which are unbounded to the right (i.e. $B \cap (x + \mathfrak{G}^+)$ is non-empty for all x). We omit the details concerning this extension.

Proof of Theorem 10.4. — In order to obtain this extension of Theorem 10.2 we need only show that if $B \in \mathcal{B}$ and $f \in \Phi^+$, then

$$\lim_{y \to \infty} \int_{\mathfrak{G}} \varphi_y(x) H_B f(x) \, dx$$

exists.

Suppose first that $\mathfrak{G}/\mathfrak{G}_1$ is compact. Then S_t generates $\mathfrak{G}$ for some $t > 0$. Without loss of generality we can assume that $t = 1$. For any $\varepsilon > 0$ there is a compact set K such that for $f \in \Phi^+$

$$\| H_K^1 H_B f(x) - H_B f(x) \| \leqslant \varepsilon \|f\|, \quad x \in \mathfrak{G}.$$

Here H_K^1 denotes the analogy to H_K for the process viewed at integer times. It is an easily shown result in discrete time that

$$\lim_{y \to -\infty} \int_{\mathfrak{G}} \varphi_y(x) H_K^1 H_B f(x) \, dx = C_f J(\varphi)$$

exists for $f \in \Phi^+$, C_f being a constant depending on f and B. Consequently

$$\limsup_{y \to -\infty} \left| \int_{\mathfrak{G}} \varphi_y(x) H_B f(x) - C_f J(\varphi) \right| \leqslant \varepsilon \|f\| J(\varphi).$$

Since ε can be made arbitrarily small

$$\lim_{y \to -\infty} \int_{\mathfrak{G}} \varphi_y(x) H_B f(x) \, dx$$

exists as desired.

Suppose next that $\mathfrak{G}/\mathfrak{G}_1$ is not compact. Then Theorem 5.5 is applicable. We can assume that $\mathfrak{G} = R \oplus C$, $\mathfrak{G}_1 = C$, and the conclusion of Theorem 5.5 holds. It follows that for $f \in \Phi^+$

$$\lim_{t \to \infty} \int_{\mathfrak{G}} \varphi_{-mt+y}(x) P^t H_B f(x) \, dx$$

exists uniformly for $y \in C$. The function φ can be chosen to have support far enough to the left so that

$$P^t H_B = H_B$$

for x in the support of φ_{-mt+y}. Then

$$\lim_{t\to\infty} \int_{\mathfrak{G}} \varphi_{-mt+y}(x) H_B f(x)\, dx$$

exists uniformly for $y \in C$, which implies that

$$\lim_{y\to-\infty} \int_{\mathfrak{G}} \varphi_y(x) H_B f(x)\, dx$$

exists where here $y \in \mathfrak{G}$. This completes the proof of the theorem.

Proof of Theorem 10.5. — In the non-singular case $\mathfrak{G}_1 = \mathfrak{G}$. The discrete time form of this result is easily shown. The reduction from continuous time to discrete time follows by the same technique used in proving Theorem 10.4.

11. Global Time Dependent Behaviour (Transient Case).

Throughout this section X_t will denote a transient i.d. process. Let B be a Borel set and let

$$E_B(t, A) = \int_{\mathfrak{G}} P_x(T_B \leqslant t, X_{T_B} \in A)\, dx.$$

Observe that for any Borel set A,

$$(11.1) \qquad P_x(T_B \leqslant t, X_{T_B} \in A) \leqslant P_x(T_A \leqslant t).$$

Now if A is relatively compact it follows from (3.18) that

$$\int_{\mathfrak{G}} P_x(T_A \leqslant t)\, dx < \infty.$$

From (11.1) we then see that $E_B(t, A) < \infty$ whenever A is relatively compact.

THEOREM 11.1. — *Let* B *be a co-transient set. Then for any relatively compact set* A *and any* h, $0 \leqslant h < \infty$,

$$(11.2) \qquad \lim_{t\to\infty} [E_B(t+h, A) - E_B(t, A)] = h\mu_B(A),$$

$$(11.3) \qquad \lim_{t\to\infty} E_B(t, A) t^{-1} = \mu_B(A),$$

and for any Borel set A, *and* $0 \leqslant h < \infty$,

$$(11.4) \quad \int_{B'} P_x(\tilde{T}_B = \infty) P_x(T_B \leqslant h;\ X_{T_B} \in A)\, dx = h\mu_B(A).$$

For a co-recurrent set B

$$\lim_{t\to\infty} [E_B(t+h, A) - E_B(t, A)] = 0$$

for any relatively compact set A.

Proof. — Let A be relatively compact. Then

$$E_B(t+h, A) - E_B(t, A) = \int_{\mathfrak{G}} P_x(t < T_B \leqslant t+h,\ X_{T_B} \in A)\, dx$$
$$\int_{\mathfrak{G}} dx \int_{\mathfrak{G}} P_x(T_B > t,\ X_t \in dy) P_y(T_B \leqslant h,\ X_{T_B} \in A)$$
$$= \int_{\mathfrak{G}} P_y(\tilde{T}_B > t) P_y(T_B \leqslant h,\ X_{T_B} \in A)\, dy.$$

By dominated convergence we then see that

$$(11.5) \quad \lim_{t\to\infty} [E_B(t+h, A) - E_B(t, A)] = \int_{B'} P_y(\tilde{T}_B = \infty) P_y(T_B \leqslant h,\ X_{T_B} \in A)\, dy.$$

From (11.5) it follows easily that

$$\int_{B'} P_y(\tilde{T}_B = \infty) P_y(T_B \leqslant h,\ X_{T_B} \in A)\, dy$$

is a linear function of h so we can write

$$(11.6) \quad \int_{B'} P_y(\tilde{T}_B = \infty) P_y(T_B \leqslant h,\ X_{T_B} \in A) = h\gamma_B(A).$$

If B is co-recurrent, then $P_y(\tilde{T}_B = \infty) = 0$ a.e. so $\gamma_B(A) = 0$. Suppose now that B is co-transient. Then $\gamma_B(A)$ is a Radon measure and we shall now prove that $\gamma_B(A) = \mu_B(A)$. To this end suppose first that B is relatively compact. It follows easily from (11.5) and (11.6) that

$$\lim_{t\to\infty} \frac{E_B(t, A)}{t} = \gamma_B(A),$$

and thus by an easy Abelian argument

$$(11.7) \quad \lim_{\lambda \downarrow 0} \lambda \int_0^\infty e^{-\lambda t} E_B(dt, A) = \gamma_B(A).$$

But $\lambda \int_0^\infty e^{-\lambda t} E_B(dt, A) = \mu_B^\lambda(A)$, and thus by Proposition 8.3 for any Borel set A such that $\mu_B(|\partial A|) = 0$, $\mu_B(A) = \gamma_B(A)$ and thus $\gamma_B(A) = \mu_B(A)$ for all Borel sets A. Thus (11.2) holds for B a relatively compact set. Now let B be any co-transient set and let B_n be relatively compact sets such that $B_n \uparrow B$. Then $P_y(\tilde{T}_{B_n} = \infty) \to P_y(\tilde{T}_B = \infty)$, and for any $f \in C_c^+$, $E_y[f(X_{T_{B_n}}); T_{B_n} \leqslant h] \to E_y[f(X_{T_B}); T_B \leqslant h]$. If A is compact and contains the support of f, then

$$E_y[f(X_{T_{B_n}}); T_{B_n} \leqslant h] P_y(\tilde{T}_{B_n} = \infty) \leqslant \|f\|_\infty P_y(T_A \leqslant h)$$

and thus by dominated convergence,

$$\begin{aligned}\lim_{n\to\infty} h(\mu_{B_n}, f) &= \lim_{n\to\infty} \int_{\mathfrak{G}} E_y[f(X_{T_{B_n}}); T_{B_n} \leqslant h] P_y(\tilde{T}_{B_n} = \infty)\, dy \\ &= \int_{\mathfrak{G}} E_y[f(X_{T_B}); T_B \leqslant h] P_y(\tilde{T}_B = \infty)\, dy.\end{aligned}$$

Now by Proposition 8.7 $(\mu_{B_n}, f) \to (\mu_B, f)$ for $f \in C_c$ and thus $(\gamma_B, f) = (\mu_B, f)$ for all such f and thus $\gamma_B = \mu_B$. Hence (11.2) and (11.4) are valid for B any co-transient set. Finally (11.3) follows easily from (11.2).

Remark. — If B is a co-transient, non-relatively compact set then $= E_B(t + h, A) - E_B(t, A)] \to \infty$, $t \to \infty$ if $\mu_B(A) = \infty$. However if $\mu_B(A) < \infty$ and A is not relatively compact it may be false that $E_B(t + h, A) - E_B(t, A) \to \mu_B(A)h$. In particular for a non-relatively compact set B,

$$E_B(t + h, \bar{B}) - E_B(t, \bar{B})$$

can be ∞ for all $t > 0$ but yet $C(B) < \infty$.

We now turn our attention to the last hitting time of B.

DEFINITION. — *Let* B *be a transient set. The last hitting time* W_B *of* B *is* $W_B = \sup\{t \geqslant 0 : X_t \in B\}$ *if* $T_B < \infty$. *If* $T_B = \infty$ *then* W_B *is undefined.*

PROPOSITION 11.1 — *Let* $r \in L_1(\mathfrak{G})$ *and let* B *be a transient set. Then for any* $t \geqslant 0$, $P_r(W_B = t) = 0$.

Proof. — It suffices to take $r \in L_1^+(\mathfrak{G})$. If

$$P_x(W_B = 0, T_B < \infty) \neq 0$$

then x must be an irregular point of B that is in B. By Proposition 3.1 the collection of such points has zero Haar measure. Now suppose $t > 0$. Then $P_r(W_B > t)$ is right continuous. To see that it is also left continuous observe that

$$(11.8)\quad \begin{aligned} P_r(W_B > t-h;\, T_B < \infty) &= \int_{\mathfrak{G}} P_r^{t-h}(dx) P_x(V_B < \infty) \\ &= \int_{\mathfrak{G}} P_x(V_B < \infty) \tilde{P}^{t-h} r(x)\, dx \\ &= \int_{\mathfrak{G}} \Phi_B(x) \tilde{P}^{t-h} r(x)\, dx = (\tilde{\mu}_B, \tilde{G}\tilde{P}^{t-h} r). \end{aligned}$$

Since $\tilde{G}\tilde{P}^{t-h} r \leqslant \tilde{G} r$ and $(\tilde{\mu}_B, \tilde{G} r) = (r, \Phi_B) < \infty$ we see by dominated convergence that the right hand side of (11.8) converges to

$$(\mu_B, \tilde{G}\tilde{P}^t r) = (\Phi_B, \tilde{P}^t r) = \int_{\mathfrak{G}} P_x(V_B < \infty) \tilde{P}^t r(x)\, dx = P_r(W_B > t).$$

Next we will find the distribution of the last hitting place in B.

Proposition 11.2. — *Let* B *be a transient set and let* $r \in L_1(\mathfrak{G})$. *Then for any Borel set* A

$$(11.9)\quad P_r(X_{W_B -} \in A;\, T_B < \infty) = \int_A \tilde{\mu}_B(dz) \tilde{G} r(z).$$

Proof. — For $T_B < \infty$ and $W_B \geqslant t$, $t > 0$, define $Y_t = X_{W_B - t}$. Let $Y_t = \Delta$ elsewhere. For $t = 0$ define $Y_0 = X_{W_B -}$ if $T_B < \infty$ and take $Y_0 = \Delta$ elsewhere. Take $f(\Delta) = 0$. It suffices to prove (11.9) for $r \in C_c^+$. We can write for $f \in C_c^+$,

$$\begin{aligned} E_r\left\{\int_0^\infty f(Y_t) e^{-\lambda t}\, dt;\, T_B < \infty\right\} &= E_r\left\{\int_0^{W_B} f(X_{W_B - t}) e^{-\lambda t}\, dt;\, T_B < \infty\right\} \\ &= E_r\left\{\int_0^{W_B} f(X_s) e^{-\lambda(W_B - s)}\, ds;\, T_B < \infty\right\} \\ &= \int_{\mathfrak{G}} \int_0^\infty P_r(X_s \in dy) E_y[e^{-\lambda W_B};\, T_B < \infty]\, ds \\ &= \int_{\mathfrak{G}} r(x)\, dx \int_{\mathfrak{G}} G(x, dy) E_y[e^{-\lambda W_B};\, T_B < \infty] f(y)\, dy \\ &= \int_{\mathfrak{G}} \tilde{G} r(y) E_y[e^{-\lambda W_B};\, T_B < \infty] f(y)\, dy. \end{aligned}$$

In other words

$$(11.10)\quad E_r\left\{\int_0^\infty f(Y_t)e^{-\lambda t}\,dt\right\} = \int_{\mathfrak{G}} \tilde{G}r(y)f(y)E_y[e^{-\lambda W_B};\ T_B < \infty]\ dy.$$

Now for $t > 0$

$$\int_0^\infty P_y(W_B \geqslant t)e^{-\lambda t}\,dt = G^\lambda \Phi_B(y)$$

and thus

$$E_y(e^{-\lambda W_B};\ T_B < \infty) = \Phi_B - \lambda G^\lambda \Phi_B.$$

Using Theorem 3.2 and the resolvent equation we can write for any $g \in \Phi^+$

$$(11.11)\quad \int_{\mathfrak{G}} g(y)E_y[e^{-\lambda W_B};\ T_B < \infty]\,dy = (\Phi_B, g) - (\lambda G^\lambda \Phi_B, g) = (\tilde{\mu}_B\tilde{G}, g) - \lambda(\tilde{\tilde{\mu}}_B\tilde{G}\tilde{G}^\lambda, g) = (\tilde{\tilde{\mu}}_B, \tilde{G}^\lambda g).$$

Applying this to $g = \tilde{G}r.f$ and using (11.10) we find that

$$E_r\left\{\int_0^\infty f(Y_t)e^{-\lambda t}\,dt;\ T_B < \infty\right\} = \int_{\mathfrak{G}} \tilde{\tilde{\mu}}_B\tilde{G}^\lambda\,(dy)\tilde{G}r(y)f(y) = \int_{\mathfrak{G}} \tilde{\tilde{\mu}}_B\,(dz)\int_{\mathfrak{G}} \tilde{G}^\lambda(z, dy)\tilde{G}r(y)f(y).$$

Thus for a.e. $t \geqslant 0$

$$(11.12)\quad E_r[f(Y_t);\ T_B < \infty] = \int_{\mathfrak{G}} \tilde{\tilde{\mu}}_B\,(dz)\int_{\mathfrak{G}} \tilde{P}^t(z, dy)\tilde{G}r(y)f(y).$$

Now the right hand side of (11.12) is continuous for $t > 0$ and right continuous at $t = 0$. Indeed, for any $t \geqslant 0$

$$\int_{\mathfrak{G}} \tilde{P}^t(z, dy)\tilde{G}r(y)f(y)\,dy \leqslant \|f\|_\infty \tilde{G}r(z)$$

and $(\tilde{\tilde{\mu}}_B, \tilde{G}r) = (r, \Phi_B) < \infty$, so the statement follows by dominated convergence. Also the left hand side is right continuous at $t = 0$. To see this note that if $W_B > 0$ then $W_B > t$ for t sufficiently small and thus as $f \in C_c$, $[f(X_{W_B - t});\ W_B > t;\ T_B < \infty] \to [f(X_{W_B - 0});\ W_B > 0,\ T_B < \infty]$. Hence by bounded convergence,

$$\lim_{t\downarrow 0} E_r[f(Y_t);\ T_B < \infty] = \lim_{t\downarrow 0} E_r[f(X_{W_B - t});\ W_B > t,\ T_B < \infty] = E_r[f(X_{W_B -});\ W_B > 0,\ T_B < \infty].$$

It now follows from (11.12) and the fact that both sides are right continuous at 0 that

$$(11.13)\quad P_r(X_{W_B-} \in dy,\ W_B > 0,\ T_B < \infty) = \tilde{G}r(y)\tilde{\tilde{\mu}}_B\,(dy).$$

By Proposition 11.1 $P_r(W_B > 0) = 1$, so (11.9) follows from (11.13).

The following interesting and useful identity plays an important role in finding the asymptotic behaviour of the last hitting time.

THEOREM 11.2. — *Let* B *be a transient set. Then for any* $t \geqslant 0$ *and any Borel set* A

$$(11.14)\quad \int_{\mathfrak{G}} P_x(W_B \leqslant t,\ X_{W_B-} \in A;\ \ T_B < \infty)\,dx = t\tilde{\tilde{\mu}}_B(A).$$

Proof. — Let K_n be relatively compact and $K_n \uparrow \mathfrak{G}$. Now using Proposition 11.2 we see that

$$\begin{aligned}\int_{K_n} P_x(W_B > t,\ X_{W_B-} \in A;\ T_B < \infty)\,dx &\\ &= \int_{K_n} dx \int_{\mathfrak{G}} P^t(x, dy)P_y(X_{W_B-} \in A;\ T_B < \infty)\\ &= \int_{\mathfrak{G}} \tilde{P}^t(y, K_n)P_y(X_{W_B-} \in A;\ T_B < \infty)\,dy\\ &= \int_A \tilde{\tilde{\mu}}_B\,(dz) \int_{\mathfrak{G}} \tilde{G}(z, dy)\tilde{P}^t(y, K_n).\end{aligned}$$

Thus again using Proposition 11.2

$$\begin{aligned}(11.15)\quad &\int_{K_n} [P_x(X_{W_B-} \in A,\ T_B \leqslant t)\,dx\\ &= \int_{K_n} [P_x(X_{W_B-} \in A, T_B < \infty) - P_x(W_B > t,\ X_{W_B-} \in A, T_B < \infty)]\,dx\\ &= \int_A \tilde{\tilde{\mu}}_B\,(dz)[\tilde{G}(z, K_n) - \tilde{G}\tilde{P}^t(z, K_n)] = \int_A \tilde{\tilde{\mu}}_B\,(dz) \int_0^t \tilde{P}^s(z, K_n)\,ds.\end{aligned}$$

Equation (11.14) now follows from (11.15) by letting $n \to \infty$.

12. Asymptotic Behaviour of P^tH_Bf.

Throughout this section X_t will be a transient process that satisfies Condition 1. Recall that whenever this condition is satisfied $\Sigma = \mathfrak{G}$. Let $h \in C_c^+$ be such that $J(h) = 1$ and set $r(t) = \int_t^\infty (h,\ P^sh)\,ds$. Then for any $f \in \Phi^*$, $R^tf \sim J(f)r(t)$

and $\tilde{R}^t f \to J(f)r(t)$. The function $r(t)$ will serve as a reference time function to measure how fast $P^t H_B f$ goes to zero. Our main result is the following.

THEOREM 12.1. — *Let* $B \in \mathcal{B}$ *and let* f *be a bounded measurable function. Then for* $\varphi \in C_c$

$$\lim_{t\to\infty} \frac{(\varphi, P^t H_B f)}{r(t)} = J(\varphi)(\mu_B, f). \tag{12.1}$$

If moreover $B \in \mathcal{B}^*$ *and* f *is in* $C(\mathfrak{G})$, *or, in the non-singular case, is any bounded measurable function, then*

$$\lim_{t\to\infty} \frac{P^t H_B f(x)}{r(t)} = (\mu_B, f) \tag{12.2}$$

where the convergence is uniform on compacts.

The proof of (12.1) and (12.2) proceed in the same way so we will only prove (12.2). The same arguments will prove (12.1) provided x is replaced by φ. The strategy of the proof is as follows. First we show that (12.1) holds for $f \in C(\mathfrak{G})$. Next we show that the analogue of (12.2) holds for the random walk X_n, $n = 0, 1, 2, \ldots$ By suitable approximations we then show that the desired limits exist in the stated generality. Finally the identification of the limit is carried out via the special case of the smoothed result (12.1) for $f \in C(\mathfrak{G})$.

To get started on the proof we first show that (12.1) holds for $f \in C(\mathfrak{G})$.

LEMMA 12.1. — *Let* $B \in \mathcal{B}$ *and* $\varphi \in C_c$ *and* $f \in C(\mathfrak{G})$. *Then*

$$\lim_{t\to\infty} \frac{(\varphi, P^t H_B f)}{r(t)} = J(\varphi)(\mu_B, f).$$

Proof. — Since $\overline{B}$ is compact it follows from (3.19) that there is a compact set K, $|\partial K| = 0$, such that

$$\Phi_B(x) \leqslant 2G(x, K).$$

Thus for any compact set C, $|\partial C| = 0$,

$$\sup_{x\in C} \frac{|P^t H_B 1(x)|}{r(t)} \leqslant \frac{2R^t(0, K - C)}{r(t)}$$

so by Theorem 5.3

$$(12.3)\quad \varlimsup_{t\to\infty} \sup_{x\in C} \frac{P^t H_B 1(x)}{r(t)} \leqslant \lim_{t\to\infty} \frac{2R^t(0, K - C)}{r(t)} = 2|K - C| < \infty.$$

Thus there is a finite measure γ_B and a sequence $t_n \to \infty$ such that the measures $(\varphi, P^{t_n} H_B)/r(t_n)$ converge weakly to γ. In particular, as $G\psi \in C(\mathfrak{G})$ if $\psi \in C_c$ we see that

$$\lim_n \frac{(\varphi, P^{t_n} H_B G\psi)}{r(t_n)} = (\gamma, G\psi).$$

But by Proposition 3.3

$$(\varphi, P^t H_B G\psi) = (\tilde{P}^t\varphi, H_B G\psi) = (\tilde{H}_B \tilde{G} \tilde{P}^t \varphi, \psi) = (\tilde{H}_B \tilde{R}^t \varphi, \psi)$$

and thus by Theorems 5.3 and 8.1

$$\lim_{t\to\infty} \frac{(\varphi, P^t H_B G\psi)}{r(t)} = \lim_{t\to\infty} \frac{(\tilde{H}_B \tilde{R}^t \varphi, \psi)}{r(t)} = J(\varphi)(\tilde{\Phi}_B, \psi) = J(\varphi)(\mu_B, G\psi).$$

Thus $J(\varphi)(\mu_B, G\psi) = (\gamma, G\psi)$ and as the potential of a finite measure determines the measure we see that $J(\varphi)\mu_B = \gamma$. The standard weak compactness argument now completes the proof of the lemma.

We will now show that the analogue of (12.2) holds for the random walk $\{X_n, n \geqslant 0\}$. To this end let H'_B and G' denote the quantities H_B and G for the random walk and let e_B denote the equilibrium measure of B for the random walk.

LEMMA 12.2. — *Let* $B \in \mathcal{B}$ *in the non-singular case and let* $B \in \mathcal{B}$ *have* $|\partial B| = 0$ *in general. Assume* f *is bounded and continuous a.e. in general or just bounded and measurable in the non-singular case. Then uniformly in* x *on compacts,*

$$\lim_n \frac{P^n H'_B f(x)}{r(n)} = (e_B, f).$$

Proof. — Let $V'_B = \inf\{n > 0 : X_n \in B\}$ ($= 0$ if no such n) and let $P^B(x, dy) = P_x(X_{V'_B} \in dy,\ V'_B < \infty)1_B(x)$. Then

$$P^n H'_B f = H'_B - \sum_{j=0}^{n-1} P^j(I_B - P^B)f.$$

Since $P^n H'_B f \to 0$ as $n \to \infty$, we see that

$$H'_B f = G'(I_B - P^B)f. \tag{12.4}$$

Also

$$\begin{aligned}(1, (I_B - P^B)f) &= (1_B, (I_B - P^B)f)\\ &= (1_B, f) - (1_B, P^B f) = (f, 1_B) - (f, \tilde{P}^B 1_B)\\ &= \int_B P_x(\tilde{V}'_B = \infty)f(x)\,dx = (e_B, f).\end{aligned}$$

From (12.4) we see that

$$P^n H'_B f = P^n G'(I_B - P^B)f = \left(\sum_{j=n+1}^{\infty} P^j\right)(I_B - P^B)f. \tag{12.5}$$

If $|\partial B| = 0$ then by Lemma 2.1 of [8] $(I_B - P^B)f$ is continuous a.e. when $f \in C(\mathfrak{G})^+$. By Theorem 5.3, uniformly in x on compacts, $\left(\sum_{j=n+1}^{\infty} P^j\right)(I_B - P^B)f(x) \sim r(n)(1,\ (I_B - P^B)f)$. This establishes the lemma.

LEMMA 12.3. — *Let $B \in \mathcal{B}^*$ and let $f \in C(\mathfrak{G})$ in general or be just bounded and measurable in the non-singular case. Then*

$$\lim_{t\to\infty} \frac{P^t H_B f(x)}{r(t)} = \gamma(f)$$

exists, is finite, and is independent of x. Moreover the convergence is uniform on compacts.

Proof. — By Proposition 9.2, if $B \in \mathcal{B}_4$, $H_B f(x)$ is continuous a.e. if $f \in C(\mathfrak{G})$. Let K be compact $K \supset B$ and $|\partial K| = 0$. Then by Lemma 12.2, uniformly, in x on compacts

$$P^{[t]} H'_B H_B f(x) \sim r(t)(e_K, H_B f). \tag{12.6}$$

Let

$$\tau = \inf\{t \geqslant T'_K : X_t \in B].$$

Then

$$H'_B H_B f = E_x[f(X_\tau);\ \tau < \infty]$$

and

$$(12.7)\quad |E_x[f(X_\tau);\ \tau < \infty] - E_x[f(X_{T_B}),\ T_B < \infty]| \leqslant 2\|f\| P_x(\tau > T_B) \leqslant 2\|f\| P_x(T'_K > T_B).$$

Now choose $B_1 \in \mathcal{B}_4$ such that $K \supset B_1 \supset B$ and such that $P_x(X_t \in B_1) \geqslant \frac{1}{2}$ for all $x \in \overline{B}$ and $0 \leqslant t \leqslant 1$. Then

$$P_x(T'_K > T_B) \leqslant 2P_x(X_\tau \in B_1).$$

Since $H_K H_B = H_B$ we see from (12.7) and the above that

$$(12.8)\quad |P^{[t]}[H'_K - H_K]H_B f(x)| \leqslant 4\|f\| P^{[t]} H'_K 1_{B_1}(x).$$

Since $B_1 \in \mathcal{B}_4, |\partial B_1| = 0$, the function 1_{B_1} is continuous a.e. Thus by Lemma 12.2 the right hand side of (12.8) is asymptotic to $4\|f\| r(t) e_K(B_1)$. Let $\varepsilon > 0$ be given. Then we can choose K such that $e_K(B_1) \leqslant \varepsilon$. Indeed

$$e_K(B_1) = \int_{B_1} P_x(V'_K = \infty)\, dx$$

and as $K \uparrow \mathfrak{G}$, $P_x(V'_K = \infty) \downarrow 0$. If K is so choosen then (12.8) shows that uniformly in x on compacts

$$(12.9)\quad \overline{\lim_{t\to\infty}} \left| \frac{P^{[t]} H'_K H_B f(x)}{r(t)} - \frac{P^{[t]} H_K H_B f}{r(t)} \right| \leqslant 4\|f\|\varepsilon.$$

It now follows from (12.9) and (12.6) and the fact that $H_K H_B = H_B$ that uniformly in x on compacts,

$$(12.10)\quad \lim_{t\to\infty} \frac{P^{[t]} H_B f}{r(t)} = \lim_{K \uparrow \mathfrak{G}} (e_K, H_B f) = \gamma(f)$$

exists and is independent of x. Now

$$(12.11)\quad |P^t H_B f - P^{[t]} H_B f| \leqslant 2\|f\| \int_{\mathfrak{G}} P^t(x, dy) P_y(T_B \leqslant 1).$$

By (3.18), $P_y(T_B \leqslant 1) \leqslant 2P_y(X_1 \in C)$ for some compact set C such that $|\partial C| = 0$. Thus we see that the right hand side of (12.11) is dominated by

$$2\|f\| P_x(X_{t+1} \in C). \quad \text{As} \quad (P_x(X_{t+1} \in C))/r(t) \to 0$$

as $t \to \infty$ uniformly in x on compacts, we see that uniformly in x on compacts

$$(12.12) \qquad \lim_{t\to\infty} \frac{P^t H_B f(x)}{r(t)} = \gamma(f).$$

This establishes the lemma.

Proof of Theorem 12.1. — It follows at once from Lemma 12.3 that for any $\varphi \in C_c$,

$$(12.13) \qquad \lim_{t\to\infty} \frac{(\varphi,\ P^t H_B f)}{r(t)} = J(\varphi)\gamma(f).$$

From (12.13) and Lemma 12.1 we see at once that for any $f \in C(\mathfrak{G})$, $\gamma(f) = (\mu_B, f)$. In the non-singular case f can be any bounded measurable function and we must show that $\gamma(f) = (\mu_B, f)$ for all bounded measurable functions f. From Lemma 12.3 it follows by a theorem of Nikodym (see [3], p. 160) that there is a finite measure γ such that $(\gamma, f) = \gamma(f)$. Since every finite measure on $\mathfrak{G}$ is regular and

$$(\gamma, f) = \gamma(f) = (\mu_B, f) \quad \text{for all} \quad f \in C(\mathfrak{G})$$

we see that $\gamma = \mu_B$. This completes the proof.

Corollary 12.1 — *Let* $B \in \mathcal{B}$. *Then for any* $f \in \Phi$ *and* $\varphi \in C_c$

$$(12.14) \quad \lim_{t\to\infty} \frac{(\varphi,\ P^t G_B f)}{r(t)} = J(\varphi) \int_{\mathfrak{G}} P_x(\tilde{T}_B = \infty) f(x)\, dx.$$

Moreover, if $B \in \mathcal{B}^*$ *and* $f \in \Phi^*$, *then uniformly in* x *on compacts,*

$$(12.15) \quad \lim_{t\to\infty} \frac{P^t G_B f(x)}{r(t)} = \int_B P_x(\tilde{T}_B = \infty) f(x)\, dx.$$

Proof. — Equations (12.14) and (12.15) follow at once from (12.1) and (12.2) and Theorem 5.3 via the relation

$$P^t G_B f = P^t G f - P^t H_B G f.$$

13. Asymptotic Behaviour of the Last Hitting Time.

Throughout this section we will assume that X_t is a transient process and that B is a relatively compact set. We will also assume that X_t satisfies Condition 1. Our purpose is to investigate the behaviour of $E_x[f(X_{W_B-});\ W_B > t,\ T_B < \infty]$ for large t. Throughout this section $r(t)$ will be the reference time function introduced in § 12.

PROPOSITION 13.1. — *Let* $B \in \mathcal{B}$, $f \in \Phi$, *and* $\varphi \in C_c$. *Then*

$$(13.1)\quad E_\varphi[f(X_{W_B-});\ W_B > t,\ T_B < \infty] \sim J(\varphi)(\tilde{\bar{\mu}}_B, f)r(t),\quad t \to \infty.$$

Proof. — Using Proposition 11.2 we can write

$$\begin{aligned} E_\varphi[f(X_{W_B-});\ W_B > t,\ T_B < \infty] & \\ &= P^t_\varphi(dy)E_y[f(X_{W_B-});T_B < \infty] \\ &= \int_{\mathfrak{G}} \tilde{\bar{\mu}}_B\ (dz)f(z)\tilde{G}\tilde{P}^t\varphi(z) \\ &= \int_{\mathfrak{G}} \tilde{\bar{\mu}}_B\ (dz)f(z)\tilde{R}^t\varphi(z). \end{aligned}$$

Equation (13.1) now follows from Theorem 5.3.

We will now show that the unsmoothed version of this result holds for nice sets in general or arbitrary sets in the non-singular case. To this end we will need the following.

LEMMA 13.1. — *Let* $f \in \Phi^*$. *Then uniformly in* x *on compacts and uniformly in* τ *on compact time intervals,*

$$(13.2)\qquad \lim_{K\uparrow\mathfrak{G}}\ \overline{\lim_{t\to\infty}}\ \frac{1}{r(t)}\int_{K'} R^t(x,\ dy)P^\tau f(y) = 0.$$

Proof. — By Theorem 5.3 we know that uniformly in $(x,\ \tau)$ on compacts,

$$\lim_{t\to\infty} \frac{R^{t+\tau}f(x)}{r(t)} = J(f),$$

and as

$$R^{t+\tau}f(x) = \int_{\mathfrak{G}} R^t(x,\ dy)P^\tau f(y)$$

we see that uniformly in $(x,\ \tau)$ on compacts

$$J(f) = \lim_{t\to\infty} \frac{1}{r(t)} \int_{\mathfrak{G}} R^t(x,\ dy)P^\tau f(y).$$

But also uniformly in $(x,\ \tau)$ on compacts,

$$(13.4)\quad \lim_{K\uparrow\mathfrak{G}}\lim_{t\to\infty}\int_K \frac{R^t(x,\ dy)}{r(t)}\, P^\tau f(y) = \lim_{K\uparrow\mathfrak{G}}\int_K P^\tau f(y)\ dy = J(f).$$

Equation (13.2) now follows from (13.3) and (13.4).

THEOREM 13.1. — *Let* $B \in \mathcal{B}^*$ *and let* $f \in \Phi^*$. *Then uniformly in* x *on compacts,*

$$(13.5)\quad \lim_{t\to\infty} \frac{E_x[f(X_{W_B-}); W_B > t, T_B < \infty]}{r(t)} = (\tilde{\tilde{\mu}}_B, f).$$

Proof. — It suffices to prove the theorem for $f \in (\varphi^*)^+$. Observe that for $\tau > 0$

$$(13.6)\quad E_x[f(X_{W_B-}), \tau < W_B \leqslant \tau + 1] = \int_{\mathfrak{G}} P^\tau(x, dy) E_y[f(X_{W_B-}); W_B \leqslant 1, T_B < \infty]$$

and thus

$$(13.7)\quad \int_t^\infty E_x[f(X_{W_B-}), \tau < W_B \leqslant \tau + 1]\, d\tau = \int_{\mathfrak{G}} R^t(x, dy) E_y[f(X_{W_B-}); W_B \leqslant 1, T_B < \infty].$$

Now by Theorem 9.2 $E_y[f(X_{W_B-}); W_B \leqslant 1, T_B < \infty]$ is continuous a.e. y when $B \in \mathcal{B}_4$ and $f \in C_c$, and thus by Theorem 5.3 and Theorem 11.2

$$(13.8)\quad \lim_{K\uparrow\mathfrak{G}} \lim_{t\to\infty} \int_K \frac{R^t(x, dy)}{r(t)} E_y[f(X_{W_B-}); W_B \leqslant 1, T_B < \infty] = \int_{\mathfrak{G}} E_y[f(X_{W_B-}); W_B \leqslant 1, T_B < \infty]\, dy = ({}_B\tilde{\tilde{\mu}}, f).$$

Now observe that

$$E_y[f(X_{W_B-}); W_B \leqslant 1, T_B < \infty] \leqslant \|f\|_\infty P_y(T_B \leqslant 1)$$

and by (3.19) for some compact set C, $|\partial C| = 0$,

$$P_y(T_B \leqslant 1) \leqslant 2P_y(X_1 \in C).$$

Thus by Lemma 13.1

$$\lim_{K\uparrow\mathfrak{G}} \overline{\lim_{t\to\infty}} \frac{1}{r(t)} \int_{K'} \frac{R^t(x, dy)}{r(t)} E_y[f(X_{W_B-}); W_B \leqslant 1, T_B < \infty]\, dy \leqslant 2\|f\| \lim_{K\uparrow\mathfrak{G}} \overline{\lim_{t\to\infty}} \int_{K'} \frac{R^t(x, dy)}{r(t)} P^1(y, C) = 0.$$

We have thus shown that the right hand side of (13.7) is asymptotic to $r(t)(\tilde{\tilde{\mu}}_B, f)$, the convergence being uniform in x on compacts. Now

$$\int_t^\infty E_x[f(X_{W_B-}); \tau < W_B \leqslant \tau + 1]\, d\tau = \int_t^{t+1} E_x[f(X_{W_B-}); W_B > s, T_B < \infty]\, ds$$

and so as $(r(t+1))/r(t) \to 1$, we see that uniformly in x on compacts,

$$\overline{\lim_{t\to\infty}} \frac{E_x[f(X_{W_B-}); W_B > t; T_B < \infty]}{r(t)}$$
$$= \overline{\lim_{t\to\infty}} \frac{E_x[f(X_{W_B-}); W_B > t+1; T_B < \infty]}{r(t+1)} \frac{r(t+1)}{r(t)} \leqslant (\mu_B, f)$$

and similarly

$$(\tilde{\mu}_B, f) \leqslant \underline{\lim_{t\to\infty}} \frac{E_x[f(X_{W_B-}); W_B > t]}{r(t)}$$

Thus (13.5) holds.

There is a strengthening of Theorem 13.1 that is possible when for some $t > 0$ the group $\mathfrak{G}_t$ is $\mathfrak{G}$.

THEOREM 13.2. — *Let* $B \in \mathcal{B}^*$ *and* let $f \in \Phi^*$. *Assume that for some* $t > 0$ *the group* $\mathfrak{G}_t$ *is* $\mathfrak{G}$. *Then for any* $h > 0$, *uniformly in* x *on compacts,*

$$E_x[f(X_{W_B-}); t < W_B \leqslant t+h] \sim p(t)(\mu_B, f)h,$$

where $p(t) = \int_{\mathfrak{G}} P^t h(x) h(x)\, dx$, $h \in C_c^+$, $J(h) = 1$.

Proof. — The proof follows from the ratio theorem by an argument similar to that used to prove Theorem 13.1.The details will be omitted.

Remark. — By Proposition 5.3 the assumptions of Theorem 13.2 are always satisfied if the process is non-singular.

14. Asymptotic Behaviour of the First Hitting Time.

Throughout this section we will assume that X_t is a transient process. If Condition 1 is not satisfied and $f \in \Phi^*$, $R^t f = 0(e^{-\gamma t})$ for some constant $\gamma > 0$, so in this case $\int_0^\infty R^t f\, dt < \infty$. Suppose Condition 1 is satisfied. Then by Theorem 5.3 either $\int_0^\infty R^t f dt = \infty$ for all $f \in (\Phi^*)^+$ with $J(f) > 0$ or $\int_0^\infty R^t f\, dt < \infty$ for all such functions f. Once again $r(t)$ will be the reference time function introduced in § 12.

DEFINITION 14.1. — *A transient process is called weakly transient if* $\int_0^\infty r(t)\,dt = \infty$. *It is called strongly transient if* $\int_0^\infty r(t)\,dt < \infty$.

We now turn our attention to the asymptotic behaviour of $E_x[f(X_{T_B});\ t < T_B < \infty]$. We will first investigate

$$\int_0^\tau E_x[f(X_{T_B});\ t < T_B < \infty]\,dt.$$

THEOREM 14.1. — *If* X_t *is strongly transient, then for any* $f \in \Phi$ *and any set* $B \in \mathcal{B}$,

$$\int_0^\infty E_x[f(X_{T_B});\, t < T_B < \infty]\,dt = G_B H_B f(x). \tag{14.1}$$

On the other hand if X_t *is weakly transient then for* $B \in \mathcal{B}^*$ *and* $f \in \Phi^*$,

$$\int_0^\tau E_x[f(X_{T_B});\ t < T_B < \infty]\,dt \sim (\mu_B, f)P_x(T_B = \infty)\int_0^\tau r(t)\,dt \tag{14.2}$$

the convergence being uniform in x *on compacts.*

Proof. — It suffices to prove the theorem for $f \geqslant 0$. Assume X_t is strongly transient. By (3.19) we can find a compact set K such that $\Phi_B(x) \leqslant 2G(x, K)$ and thus

$$P^t H_B f \leqslant \|f\| P^t \Phi_B \leqslant R^t(x, K)\|f\|$$

and thus if X_t is strongly transient we see that

$$\int_0^\infty P^t H_B|f|\,dt < \infty.$$

Now

$$E_x[f(X_{T_B});\ t < T_B < \infty] = \int_{B'} P_x(T_B > t,\ X_t \in dy)H_B f(y) \tag{14.3}$$

and as the right hand side is dominated by $P^t H_B|f|$ we see that

$$\int_0^\infty dt \int_{B'} P_x(T_B > t,\ X_t \in dy)H_B f(y) = G_B H_B f(x) < \infty.$$

This establishes (14.1).

Now assume X_t is weakly transient, $f \in \Phi^*$ and $B \in \mathcal{B}^*$. From (14.3) we see that

$$\text{(14.4)}\quad \begin{aligned} &E_x[f(X_{T_B});\ t < T_B < \infty] \\ &\quad = P^t H_B f - \int_0^t \int_{\overline{B}} P_x(T_B \in ds,\ X_{T_B} \in dy) P^{t-s} H_B f(y) \end{aligned}$$

and thus

$$\text{(14.5)}\quad \begin{aligned} &\int_0^\tau E_x[f(X_{T_B});\ t < T_B < \infty]\ dt \\ &= \int_0^\tau P^t H_B f\, dt - \int_0^\tau \int_{\overline{B}} P_x(T_B \leqslant t,\ X_{T_B} \in dy) P^{\tau - t} H_B f(y)\ dt. \end{aligned}$$

Now a weakly transient process must satisfy Condition 1 and thus from Theorem 12.1

$$\int_0^\tau P^t H_B f(x)\ dt \sim (\mu_B, f) \int_0^\tau r(t)\ dt.$$

Also from that same theorem we see that given $\varepsilon > 0$ there is a t_0 such that for $t \geqslant t_0$ and all $x \in \overline{B}$

$$(1 - \varepsilon)(\mu_B, f) r(t) \leqslant P^t H_B f(x) \leqslant (1 + \varepsilon)(\mu_B, f) r(t).$$

Thus

$$\text{(14.6)}\quad \begin{aligned} &\int_{\overline{B}} \int_0^{\tau - t_0} P_x(T_B \leqslant t,\ X_{T_B} \in dy) P^{\tau - t} H_B f(y)\ dt \\ &\qquad \leqslant (1 + \varepsilon)(\mu_B, f) \int_0^{\tau - t_0} P_x(T_B \leqslant t) r(\tau - t)\ dt \\ &\qquad \leqslant (1 + \varepsilon)(\mu_B, f) \int_0^\tau P_x(T_B \leqslant t) r(\tau - t)\ dt. \end{aligned}$$

Since $P_x(T_B \leqslant t) \uparrow P_x(T_B < \infty)$ a simple summability argument shows that

$$\text{(14.7)}\quad \lim_{\tau \uparrow \infty} \frac{\int_0^\tau P_x(T_B \leqslant t) r(\tau - t)\ dt}{\int_0^\tau r(t)\ dt} = P_x(T_B < \infty).$$

Also

$$\begin{aligned} &\int_{\overline{B}} \int_{\tau - t_0}^\tau P_x(T_B \leqslant t,\ X_{T_B} \in dy) P^{\tau - t} H_B f(y)\ dt \\ &\qquad \leqslant \|f\| \int_{\overline{B}} \int_{\tau - t_0}^\tau P_x(T_B \leqslant t,\ X_{T_B} \in dy) P^{\tau - t} H_B 1(y) \\ &\qquad \leqslant \|f\| \int_{\tau - t_0}^\tau P_x(T_B \leqslant t)\ dt \leqslant \|f\| t_0 P_x(T_B \leqslant \tau) \leqslant \|f\| t_0. \end{aligned}$$

Thus

$$(14.8)\quad \lim_{\tau\to\infty} \frac{\int_{\bar{B}}\int_{\tau-t_0}^{\tau} P_x(T_B \leqslant t,\ X_{T_B} \in dy)P^{\tau-t}H_B f(y)}{\int_0^{\tau} r(t)\,dt} = 0.$$

It follows from (14.6), (14.7) and (14.8) that

$$(14.9)\quad \overline{\lim_{\tau\to\infty}} \frac{\int_{\bar{B}}\int_0^{\tau} P_x(T_B \leqslant t,\ X_{T_B} \in dy)P^{\tau-t}H_B f(y)}{\int_0^{\tau} r(t)\,dt} \leqslant (1+\varepsilon)(\mu_B, f)P_x(T_B < \infty).$$

Similar arguments show that

$$(14.10)\quad \underline{\lim_{\tau\to\infty}} \frac{\int_{\bar{B}}\int_0^{\tau} P_x(T_B \leqslant t,\ X_{T_B} \in dy)P^{\tau-t}H_B f(y)}{\int_0^{\tau} r(t)\,dt} \geqslant (1-\varepsilon)(\mu_B, f)P_x(T_B < \infty).$$

The desired result (15.2) now follows from (14.5), (14.9), (14.10) and Theorem 12.1.

We will now obtain the second term in the asymptotic expansion of $E_B(t, A)$ for a set $B \in \mathcal{B}$. For this purpose it is more convenient to deal with functions rather than sets. Define $E_B^t f$ by

$$E_B^t f = \int_{\mathfrak{G}} E_B(t, dy)f(y).$$

Theorem 14.2. — *If X_t is strongly transient then for any relatively compact set B and any $f \in \Phi^+$,*

$$(14.11)\quad E_B^t f - t(\mu_B, f) \uparrow \int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty)H_B f(x)\,dx < \infty.$$

On the other hand, if X_t is weakly transient, then for $f \in (\Phi^*)^+$ and $B \in \mathcal{B}^*$

$$(14.12)\quad E_B^t f - t(\mu_B, f) \sim C(B)(\mu_B, f)\int_0^t r(\tau)\,d\tau.$$

Proof. — Suppose X_t is strongly transient. By Theorem 11.1 we can write

$$t(\mu_B, f) = \int_{\mathfrak{G}} P_x(\tilde{T}_B = \infty)E_x[f(X_{T_B}); T_B \leqslant t]\,dx.$$

(The integral over $\mathfrak{G}$ rather than B' is permissible since

$P_x(\tilde{T}_B = \infty) = 0$ for $x \in B$). Hence

$$(14.13)\quad E_B^t f - t(\mu_B, f) = \int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty) E_x[f(X_{T_B}); T_B \leqslant t]\ dx.$$

From (14.13) it is clear that $E_B^t f - t(\mu_B, f)$ is increasing with t increasing. Also

$$|E_x[f(X_{T_B}); T_B \leqslant t]| \leqslant \|f\| P_x(T_B \leqslant t).$$

Using Proposition 3.6 and Theorem 8.1 we see that there is a compact set K such that for all $t > 0$

$$\begin{aligned}(14.14)\quad \int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty) P_x(T_B \leqslant t)\ dx &= \int_{\bar{B}} \mu_B(dy) \int_{\mathfrak{G}} G(y, dx) P_x(T_B \leqslant t) \\ &\leqslant 2 \int_{\bar{B}} \mu_B(dy) \int_0^{t+1} R^s(y, K)\ ds.\end{aligned}$$

Since the process is strongly transient,

$$\int_{\bar{B}} \mu_B(dy) \int_0^\infty R^s(y, K)\ ds < \infty.$$

Consequently,

$$\begin{aligned}\lim_{t\to\infty} \int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty) E_x[f(X_{T_B}); T_B \leqslant t)\ dx &= \int_{\mathfrak{G}} P_x(\tilde{T}_B < \infty) H_B f(x)\ dx < \infty\end{aligned}$$

as desired.

Assume now that X_t is weakly transient and let f and B be as stated in the hypothesis. Using (11.4) and our duality relations we can write

$$\begin{aligned} E_B^{t+1} f - E_B^t f - (\mu_B, f) &= \int_{\mathfrak{G}} E_x[f(X_{T_B}); t < T_B \leqslant t+1]\ dx - (\mu_B, f) \\ &= \int_{\mathfrak{G}} \int_{\mathfrak{G}} P_x(T_B > t, X_t \in dy) E_y[f(X_{T_B}), T_B \leqslant 1]\ dx - (\mu_B, f) \\ &= \int_{B'} P_y(\tilde{T}_B > t) E_y[f(X_{T_B}); T_B \leqslant 1] - (\mu_B, f) \\ &= \int_{B'} P_y(t < \tilde{T}_B < \infty) E_y[f(X_{T_B}); T_B \leqslant 1]\ dy.\end{aligned}$$

In other words

$$\begin{aligned}(14.15)\quad E_B^{t+1} f - E_B^t f - (\mu_B, f) &= \int_{B'} P_y(t < \tilde{T}_B < \infty) E_y[f(X_{T_B}); T_B \leqslant 1]\ dy\end{aligned}$$

and thus

$$\int_0^\tau [(E_B^{t+1}f - E_B^t f) - (\mu_B, f)]\,dt = \int_0^\tau dt \int_{B'} P_y(t < \tilde{T}_B < \infty)E_y[f(X_{T_B}); T_B \leqslant 1]\,dy.$$

Observe that

$$\begin{aligned}\left|\int_0^\tau [E_B^{t+1}f - E_B^t f - (\mu_B, f)]\,dt - [E_B^\tau f - \tau(\mu_B, f)]\right| &\leqslant E_B^1|f| + |E_B^{\tau+1}f - E_B^\tau f| \\ &\leqslant E_B^1|f| + \|f\| \int_{\mathfrak{G}} P_x(\tilde{T}_B > \tau)P_x(T_B \leqslant 1)\,dx \\ &\to E_B^1|f| + \|f\|C(B) < \infty \text{ as } \tau \to \infty,\end{aligned}$$

and thus as $\int_0^\infty r(t)\,dt = \infty$ we see that

$$(14.17)\quad \lim_{t\to\infty} \frac{1}{\int_0^\tau r(t)\,dt}\left|\int_0^\tau [E_B^{t+1}f - E_B^t f - (\mu_B, f)]\,dt - [E_B^\tau f - \tau(\mu_B, f)]\right| = 0.$$

Thus to establish (14.12) it suffices then to show that

$$(14.18)\quad \lim_{t\to\infty} \frac{\int_0^\tau [E_B^{t+1}f - E_B^t f - (\mu_B, f)]\,dt}{\int_0^\tau r(t)\,dt} = C(B)(\mu_B, f).$$

Let K be compact. Then by Theorem 14.1

$$\lim_{t\to\infty} \frac{\int_0^\tau dt \int_K P_y(t < \tilde{T}_B < \infty)E_y[f(X_{T_B}); T_B \leqslant 1]\,dy}{\int_0^\tau r(t)\,dt} = C(B)\int_K P_y(\tilde{T}_B = \infty)E_y[f(X_{T_B}); T_B \leqslant 1]\,dy.$$

By (11.4) as $K \uparrow \mathfrak{G}$ the right hand side converges to $C(B)(\mu_B, f)$. Thus to establish the result we need to show that

(14.19)

$$\lim_{K\uparrow\mathfrak{G}} \overline{\lim_{t\to\infty}} \frac{\int_0^\tau dt \int_{K'} P_y(t < \tilde{T}_B < \infty)E_y[f(X_{T_B}); T_B \leqslant 1]\,dy}{\int_0^\tau r(t)\,dt} = 0.$$

Now

$$E_y[f(X_{T_B}); T_B \leqslant 1] \leqslant \|f\|P_y(T_B \leqslant 1)$$

so it suffices to show

$$\lim_{K\uparrow\mathfrak{G}} \overline{\lim_{t\to\infty}} \frac{\int_0^\tau dt \int_{K'} P_y(t < \tilde{T}_B < \infty) P_y(T_B \leqslant 1)\, dy}{\int_0^\tau r(t)\, dt} = 0.$$

Now using Theorem 8.1

$$\begin{aligned}\int_{K'} P_y(t < \tilde{T}_B < \infty) P_y(T_B \leqslant 1)\, dy &= \int_{\mathfrak{G}} \tilde{P}^t \tilde{\Phi}_B(y) P_y(T_B \leqslant 1) 1_{K'}(y)\, dy \\ &= \int_{\mathfrak{G}} \tilde{\Phi}_B(x) \int_{K'} P^t(x, dy) P_y(T_B \leqslant 1)\, dx \\ &= \int_B \mu_B\,(dz) \int_{K'} R^t(z, dy) P_y(T_B \leqslant 1).\end{aligned}$$

By (3.18) we can find a compact set A, $|\partial A| = 0$, such that for all $y \in \mathfrak{G}$, $P_y(T_B \leqslant 1) \leqslant 2P^1(y, A)$. Hence

$$\int_{K'} P_y(t < \tilde{T}_B < \infty) P_y(T_B \leqslant 1)\, dy \leqslant 2 \int_{\bar{B}} \mu_B\,(dz) \int_{K'} R^t(z, dy) P^1(y, A).$$

We then know by Lemma 13.1 that uniformly on compacts,

$$\lim_{K\uparrow\mathfrak{G}} \overline{\lim_{t\to\infty}} \frac{\int_{K'} R^t(z, dy) P_y(T_B \leqslant 1)}{r(t)} = 0.$$

Since $\int_0^\infty r(t)\, dt = \infty$ it follows that uniformly on compacts,

$$\lim_{K\uparrow\mathfrak{G}} \overline{\lim_{t\to\infty}} \frac{\int_0^\tau dt \int_{K'} R^t(z, dy) P_y(T_B \leqslant 1)}{\int_0^\tau r(t)\, dt} = 0.$$

This completes the proof.

Examples show that, in general, Theorem 14.2 is the best one can do on the asymptotic behaviour of

$$P_x(t < T_B < \infty,\ X_{T_B} \in A).$$

However, under more restrictive conditions, stronger results are possible.

THEOREM 14.3. — *Let* $B \in \mathcal{B}^*$ *and* $f \in \Phi^*$. *If*

$$\sup_{t>0} r(t)/r(2t) < \infty,$$

then, uniformly in x on compacts,

(14.20) $$E_x[f(X_{T_B}),\ t < T_B < \infty] \sim P_x(T_B = \infty)(\mu_B, f)r(t).$$

Also for any $h > 0$ and $f \in \Phi^*$ for $B \in \mathcal{B}^*$,

(14.21) $$E_B^{t+h}f - E_B^t f - h(\mu_B, f) \sim C(B)(\mu_B, f)r(t).$$

Proof. — It suffices to consider $f \geqslant 0$. From (14.4) we know that

$$E_x[f(X_{T_B});\ t < T_B < \infty] = P^t H_B f - \int_0^t \int_{\bar{B}} P_x(T_B \in ds,\quad X_s \in dy)P^{t-s}H_B f(y).$$

By Theorem 12.1

$$P^t H_B f(x) \sim (\mu_B, f)r(t)$$

and also by Theorem 12.1

$$\lim_{\tau\to\infty}\lim_{t\to\infty} \frac{\int_{\bar{B}}\int_0^\tau P_x(T_B \in ds,\ X_{T_B} \in dy)P^{t-s}H_B f(y)}{r(t)} = \lim_{\tau\to\infty} P_x(T_B \leqslant \tau)(\mu_B,\ f) = P_x(T_B < \infty)(\mu_B,\ f).$$

Thus to establish (14.20) we need to show that

(14.22) $$\lim_{\tau\to\infty}\overline{\lim_{t\to\infty}} \frac{1}{r(t)} \int_\tau^t P_x(T_B \in ds,\ X_s \in dy)P^{t-s}H_B f(y) = 0.$$

To this end, decompose $\int_\tau^t$ as $\int_\tau^t = \int_\tau^{t/2} + \int_{t/2}^{t-\tau} + \int_{t-\tau}^t$.

Now

$$\int_{\bar{B}}\int_\tau^{t/2} P_x(T_B \in ds,\ X_s \in dy)P^{t-s}H_B f(y) \leqslant \|f\| \int_{\bar{B}}\int_\tau^{t/2} P_x(T_B \in ds,\ X_s \in dy)P^{t-s}\Phi_B.$$

Since $P^t\Phi_B$ is decreasing as t increases, $P^{t-s}\Phi_B \leqslant P^{t/2}\Phi_B$, $\tau < s \leqslant t/2$. Thus

$$\int_{\bar{B}}\int_\tau^{t/2} P_x(T_B \in ds,\ X_s \in dy)P^{t-s}H_B f(y) \leqslant \|f\| \int_{\bar{B}} P_x(\tau < T_B \leqslant t/2,\ X_{T_s} \in dy)P^{t/2}\Phi_B(y).$$

It follows from Theorem 12.1 and the fact that $r(t)$ is decreasing and strictly positive for all $t \geqslant 0$ that there is a constant $\alpha < \infty$ such that $P^t\Phi_B \leqslant \alpha r(t)$, $t \geqslant 0$. Thus

$$\lim_{\tau\to\infty} \overline{\lim_{t\to\infty}} \frac{1}{r(t)} \int_{\bar{B}} \int_{\tau}^{t/2} P_x(T_B \in ds,\ X_s \in dy) P^{t-s} H_B f(y)$$
$$\leqslant \lim_{\tau\to\infty} \sup_{t>0} \frac{r(t/2)}{r(t)}\ \alpha \|f\| C(B) P_x(\tau < T_B < \infty) = 0.$$

Now, again using the monotonicity of $P^t\Phi_B$, we find that

$$\int_{\bar{B}} \int_{t/2}^{t-\tau} P_x(T_B \in ds,\ X_s \in dy) P^{t-s} H_B f(y)$$
$$\leqslant \int_{\bar{B}} \|f\| P_x(t/2 < T_B \leqslant t-\tau,\ X_{T_B} \in dy) P^{\tau}\Phi_B(y)$$
$$\leqslant \|f\| \alpha r(\tau) P_x(t/2 < T_B < \infty).$$

But

$$P_x(t/2 < T_B < \infty) \leqslant P^{t/2}\Phi_B(x) \leqslant \alpha r(t/2).$$

Hence

$$\lim_{\tau\to\infty} \overline{\lim_{t\to\infty}} \frac{1}{r(t)} \int_{t/2}^{t-\tau} P_x(T_B \in ds,\ X_s \in dy) P^{t-s} H_B f(y)$$
$$\leqslant \lim_{\tau\to\infty} \|f\| \alpha^2 \sup_{t>0} \frac{r(t/2)}{r(t)}\, r(\tau) = 0.$$

Finally,

$$\int_{\bar{B}} \int_{t-\tau}^{t} P_x(T_B \in ds,\ X_s \in dy) P^{t-s} H_B f(y) \leqslant \|f\| P_x(t-\tau < T_B \leqslant t).$$

Once again using the fact that there is an $A \in \mathcal{B}$, $|\partial A| = 0$, such that $P_x(T_B \leqslant \tau) \leqslant 2P(X_\tau \in A)$ we see that

$$P_x(t-\tau < T_B \leqslant t)$$
$$\leqslant 2 \int_{B'} P_x(T_B > t-\tau,\ X_{t-\tau} \in dy) P_y(X_\tau \in A)$$
$$\leqslant 2 \int_{\mathfrak{G}} P_x(X_{t-\tau} \in dy) P_y(X_\tau \in A) = 2P_x(X_t \in A).$$

But

$$\lim_{t\to\infty} \frac{P_x(X_t \in A)}{r(t)} = 0$$

so

$$\lim_{t\to\infty} \frac{1}{r(t)} \int_{t-\tau}^{t} \int_{\bar{B}} P_x(T_B \in ds,\ X_s \in dy) P^{t-s} H_B f(y) = 0.$$

This establishes (14.20).

To establish (14.21) note that by the same computation that yielded (14.15) we can show that

$$E_B^{t+h}f - E^t f - h(\mu_B, f)$$
$$= \int_{B'} P_y(t < \tilde{T}_B < \infty) E_y]f(X_{T_B}); T_B \leqslant h] \, dy$$

Using (14.20) one easily shows that

$$\lim_{K \uparrow \mathfrak{S}} \lim_{t \to \infty} \int_K \frac{P_y(t < \tilde{T}_B < \infty) E_y[f(X_{T_B}); T_B \leqslant h]}{r(t)} = C(B)(\mu_B, f)h$$

so to establish (14.21) it must be shown that

$$\lim_{K \uparrow \mathfrak{S}} \overline{\lim_{t \to \infty}} \frac{1}{r(t)} \int_{K'} P_y(t < \tilde{T}_B < \infty) E_y[f(X_{T_B}); T_B \leqslant h] \, dy = 0.$$

But this follows from Lemma 13.1, (3.18), and the estimate

$$\int_{K'} P_y(t < \tilde{T}_B < \infty) E_y[f(X_{T_B}); T_B \leqslant h]$$
$$\leqslant \|f\| \int_{K'} \tilde{P}^t \tilde{\Phi}_B(y) P_y(T_B \leqslant h) \, dy$$
$$= \|f\| \int_{\bar{B}} \mu_B \, (dz) \int_{K'} R^t(z, dy) P_y(T_B \leqslant h)$$
$$\leqslant 2\|f\| \int_{\bar{B}} \tilde{\tilde{\mu}}_B \, (dz) R^{t+1}(z, dy) P^h(y, A)$$

for a suitable set $A \in \mathfrak{B}$, $|\partial A| = 0$. This establishes the theorem.

15. Behavior Along the Path.

Throughout this section X_t will be a transient i.d. process.

For Brownian motion on R^d, $d \geqslant 3$, it is well known that for any $f \in C_c$ and any $g \in C_c$ such that $J(g) \neq 0$,

$$\lim_{x \to \infty} \frac{Gf_x(y)}{Gg_x(0)} = \frac{J(f)}{J(g)}. \tag{15.1}$$

Examples show however that even for quite nice i.d. process the ratio's $Gf_x(y)/Gg_x(0)$ in general have no limit as $x \to \infty$. Note however that whenever these ratios have a limit then for any $x \in \mathfrak{G}$,

$$\text{(15.2)} \qquad P_x\left[\lim_{t\to\infty} \frac{Gf_{X_t}(y)}{Gg_{X_t}(0)} = \frac{J(f)}{J(g)}\right] = 1.$$

Now it is plausable that (15.2) may hold in wide generality even if the limits in (15.1) fail to exist. Indeed, from our results in § 10 we know that for any type II process with $m > 0$, $\lim_{z\to+\infty} Gf_z(y) = J(f)$, and as $P_x[X_t \to +\infty] = 1$ for all $x \in \mathfrak{G}$ it follows easily that (15.2) holds for any type II process. We will now show that this is true for *any* i.d. process.

THEOREM 15.1. — *Let $f \in \Phi$ and let $g \in \Phi$ have $J(g) \neq 0$. Then for a.e. $x \in \mathfrak{G}$,*

$$\text{(15.3)} \qquad P_x\left[\lim_{t\to\infty} \frac{Gf_{X_t}(y)}{Gg_{X_t}(0)} = \frac{J(f)}{J(g)}\right] = 1.$$

Moreover, if f and $g \in C_c$, then (15.3) *holds for all $x \in \mathfrak{G}$.*

To prove the theorem we will need several lemmas.

LEMMA 15.1. — *There is a function $r \in C(\mathfrak{G}) \cap L_1(\mathfrak{G})$, $r > 0$, $J(r) = 1$ and $\tilde{G}r > 0$ and continuous.*

Proof. — Such an r is easily constructed using Urysohn's lemma and the second countability of $\mathfrak{G}$. We omit the details.

LEMMA 15.2. — *Let r be as in Lemma 15.1 and assume the process X_t is started with density r. Let B be a relatively compact set having positive capacity and define Y_t, $t \geqslant 0$ as follows: $Y_0 = X_{W_B-}$ if $T_B < \infty$ and $Y_0 = \Delta$ elsewhere. For $t > 0$ take $Y_t = X_{W_B - t}$ if $W_B > t$ and $T_B < \infty$ and take $Y_t = \Delta$ elsewhere. Then Y_t is a sub-Markov process on $[T_B < \infty]$ having transition operator $Q^t(x, dy)$ given by*

$$Q^t(x, dy) = \frac{\tilde{P}^t(x, dy)\tilde{G}r(y)}{\tilde{G}r(x)}$$

and initial measure $\tilde{\tilde{\mu}}_B\,(dz)\tilde{G}r(z)$.

Proof. — A function defined on $\mathfrak{G}$ will be extended to $\mathfrak{G} \cup \{\Delta\}$ be defining $f(\Delta) = 0$.

Let f_k, $0 \leqslant k \leqslant n$ be in C_c^+ and let

$$0 \leqslant h_1 < h_2 < \cdots < h_n < \infty.$$

Then

$$\int_0^\infty \mathrm{E}_r\{f_0(\mathrm{Y}_t)f_1(\mathrm{Y}_{t+h_1}) \ldots f_n(\mathrm{Y}_{t+h_n})\}e^{-\lambda t}\, dt$$
$$= \int_0^\infty \mathrm{E}_r\{f_0(\mathrm{X}_{\mathrm{W_B}-t})f_1(\mathrm{X}_{\mathrm{W_B}-t-h_1}) \ldots$$
$$f_n(\mathrm{X}_{\mathrm{W_B}-t-h_n});\ \mathrm{W_B} > t + h_n,\ \mathrm{T_B} < \infty\}e^{-\lambda t}\, dt$$
$$= \mathrm{E}_r\left\{\int_0^{\mathrm{W_B}-h_n} f_n(\mathrm{X}_s)f_{n-1}(\mathrm{X}_{s+h_n-h_{n-1}}) \ldots\right.$$
$$\left. f_0(\mathrm{X}_{s+h_n})e^{-\lambda(\mathrm{W_B}-h_n-s)}\, ds;\ \mathrm{T_B} < \infty\right\}$$
$$= \int_{\mathfrak{G}} r(y)\, dy \int_{\mathfrak{G}} \ldots \int_{\mathfrak{G}} \int_0^\infty ds \mathrm{P}^s(y, x_n)f_n(x_n)$$
$$.\ \mathrm{P}^{h_n-h_{n-1}}(x_n, dx_{n-1})f_{n-1}(x_{n-1}) \ldots \mathrm{P}^{h_1}(x_1, dx_0)f_0(x_0)$$
$$.\ \mathrm{E}_{x_0}(e^{-\lambda \mathrm{W_B}};\ \mathrm{T_B} < \infty)$$
$$= \int_{\mathfrak{G}} r(y)\, dy \int_{\mathfrak{G}} \ldots \int_{\mathfrak{G}} \mathrm{G}(y, dx_n)f_n(x_n)$$
$$.\ \mathrm{P}^{h_n-h_{n-1}}(x_n, dx_{n-1})f_{n-1}(x_{n-1}) \ldots \mathrm{P}^{h_1}(x, dx_0)$$
$$.f(x_0)\mathrm{E}_{x_0}[e^{-\lambda \mathrm{W_B}};\ \mathrm{T_B} < \infty].$$

Now $\int_{\mathfrak{G}} r(y)\mathrm{G}(y, dx_n)\, dy$ has density $\tilde{\mathrm{G}}r(x_n)$, so applying our duality relations we see that

$$(15.4) \quad \int_0^\infty \mathrm{E}_r\{f_0(\mathrm{Y}_t) \ldots f_n(\mathrm{Y}_{t+h_n})e^{-\lambda t}\, dt\}$$
$$= \int_{\mathfrak{G}} \ldots \int_{\mathfrak{G}} dx_0 \mathrm{E}_{x_0}(e^{-\lambda \mathrm{W_B}};\ \mathrm{T_B} < \infty)f_0(x_0)\tilde{\mathrm{P}}^{h_1}(x_0, dx_1)f_1(x_1) \ldots$$
$$\tilde{\mathrm{P}}^{h_n-h_{n-1}}(x_{n-1},\ dx_n)f_n(x_n)\tilde{\mathrm{G}}r(x_n).$$

By (11.11) we see that

$$\mathrm{E}_{x_0}(e^{-\lambda \mathrm{W_B}};\ \mathrm{T_B} < \infty)\, dx_0 = \tilde{\tilde{\mu}}_{\mathrm{B}}\tilde{\mathrm{G}}^\lambda(dx_0)$$

and thus the right hand side of (15.4) can be written as

$$(15.5) \quad \int_{\mathfrak{G}} \ldots \int_{\mathfrak{G}} \tilde{\tilde{\mu}}_{\mathrm{B}}\tilde{\mathrm{G}}^\lambda(dx_0)f_0(x_0)\tilde{\mathrm{P}}^{h_1}(x_0, dx_1)f_1(x_1) \ldots$$
$$\tilde{\mathrm{P}}^{h_n-h_{n-1}}(x_{n-1}, dx_n)f_n(x_n)\tilde{\mathrm{G}}r(x_n).$$

Thus for a.e. t

$$(15.6) \quad \mathrm{E}_r[f_0(\mathrm{Y}_t) \ldots f_n(\mathrm{Y}_{t+h_n})] = \int_{\mathfrak{G}} \tilde{\tilde{\mu}}_{\mathrm{B}}\tilde{\mathrm{P}}^t\ (dx_0)f_0(x_0)\psi(x_0)$$

where

$$\psi(x_0) = \int_{\mathfrak{G}} \ldots \int_{\mathfrak{G}} \tilde{\mathrm{P}}^{h_1}(x_0, dx_1) \ldots$$
$$\tilde{\mathrm{P}}^{h_n-h_{n-1}}(x_{n-1}, dx_n)f_1(x_1) \ldots f_n(x_n)\tilde{\mathrm{G}}r(x_n).$$

Since $\tilde{\mathrm{G}}r(x)$ is continuous so is $\psi(x)$. Hence the right hand side of (15.6) is right continuous for $t \geqslant 0$. Arguing as the proof of Proposition 11.2 we can show that the left hand side of (15.6) is right continuous at 0 and thus from (15.6) we see that

$$(15.7)\quad E_r[f_0(Y_0)f_1(Y_{h_1}) \dots f_n(Y_{h_n})] = \int \tilde{\mu}_B\,(dx_0)f_0(x_0)\psi(x_0)$$
$$= \int_{\mathfrak{G}} \tilde{\mu}_B\,(dx_0)f_0(x_0) \int_{\mathfrak{G}} \dots \int_{\mathfrak{G}} \tilde{P}^{h_1}(x_0, dx_1) \dots \tilde{P}^{h_n-h_{n-1}}(x_{n-1}, dx_n)f_1(x_1) \dots f_n(x_n)\tilde{G}r(x_n).$$

Using the fact that $\tilde{G}r > 0$ we can rewrite (15.7) as

$$(15.8)\quad E_r[f_0(Y_0) \dots f_n(Y_{h_n})]$$
$$= \int_{\mathfrak{G}} \cdots \int_{\mathfrak{G}} \tilde{\mu}_B\,(dx_0)\tilde{G}r(x_0)Q^{h_1}(x_0, dx_1) \dots Q^{h_n-h_{n-1}}(x_{n-1}, dx_n) \cdot f_0(x_0) \dots f_n(x_n).$$

This establishes the lemma.

LEMMA 15.3. — *Let r and Y_t be as in Lemma 15.2. For $f \in \Phi$ define $K_rf(x)$ by*

$$K_rf(x) = \frac{\tilde{G}f(x)}{\tilde{G}r(x)}.$$

Then

$$(15.9)\qquad P_r[\lim_{t\to\infty} K_rf(X_t) = J(f)] = 1.$$

The proof of this lemma uses the clever Martingale argument of Hunt [6].

Proof. — It suffices to consider $f \geqslant 0$. Let $s < t$. Then

$$E\{K_rf(Y_t)|Y_u, u \leqslant s\} = E\{K_rf(Y_t)|Y_s\}$$
$$= \int_{\mathfrak{G}} Q^{t-s}(Y_s, dx)K_rf(x) = \int_{\mathfrak{G}} \frac{\tilde{P}^{t-s}(Y_s, dx)\tilde{G}r(x)}{\tilde{G}r(Y_s)} \frac{\tilde{G}f(x)}{\tilde{G}r(x)}$$
$$= \int_{\mathfrak{G}} \frac{\tilde{P}^{t-s}(Y_s, dx)\tilde{G}f(x)}{\tilde{G}r(Y_s)} \leqslant \frac{\tilde{G}f(Y_s)}{\tilde{G}r(Y_s)} = K_rf(Y_s).$$

It follows from this that $K_rf(Y_t)$ is a supermartingale. Let $0 \leqslant a < b < \infty$. The number of downcrossings of (a, b) by $K_rf(X_t)$ on $[0, W_B]$ is the same as the number of upcrossings of (a, b) by $K_r(Y_t)$. Let $U(a, b)$ denote this number of upcrossings. Then by the upcrossing inequality for supermartingales we see that

$$(b - a)EU(a, b) \leqslant E[K_rf(Y_0)].$$

Letting $D(a, b)$ be the number of downcrossings of (a, b) by $K_r f(X_t)$ we see that

$$(b - a)ED(a, b) \leqslant E_r[K_r f(Y_0)] = E_r[K_r f(X_{W_B -}); T_B < \infty].$$

But using Proposition 11.2

$$(15.10) \quad E_r[K_r f(X_{W_B -}); T_B < \infty] = \int_{\mathfrak{G}} \tilde{\mu}_B(dx)\tilde{G}r(x)K_r f(x)\,dx$$
$$= \int_B \tilde{\mu}_B(dx)\tilde{G}f(x)$$
$$= (f, \Phi_B).$$

Thus

$$(b - a)E_r D(a, b) \leqslant (f, \Phi_B).$$

As $B \uparrow \mathfrak{G}$, $W_B \uparrow \infty$ and $T_B \downarrow 0$ so that the number of downcrossings of (a, b) by $K_r f(X_t)$ for $t \in (0, \infty)$ is bounded in expectation by $(b - a)^{-1}J(f)$. It follows that $K_r f(X_t)$ has a limit a.s. as $t \to \infty$. Denote this limit by ξ. Hence $P(\lim_{t\to\infty} K_r f(X_t) = \xi) = 1$. Then for integer n

$$P_r(\lim_{n\to\infty} K_r f(X_n) = \xi) = 1.$$

But then ξ is measurable on the σ-field of sets invariant under a finite permutation of coordinates $X_n - X_{n-1}$, $n = 1, 2, \ldots$ so by the Hewitt-Savage $0 - 1$ law for some constant $\alpha(f)$ (that may depend also on r) $P_r(\xi = \alpha(f)) = 1$ and thus

$$(15.11) \qquad P_r(\lim_{t\to\infty} K_r f(X_t) = \alpha(f)) = 1.$$

From (15.10) we see that

$$(15.12) \quad \lim_{B \uparrow \mathfrak{G}} E_r[K_r f(X_{W_B -}); T_B < \infty] = \lim_{B \uparrow \mathfrak{G}} (f, \Phi_B) = J(f).$$

Since $W_B \uparrow \infty$ as $B \uparrow \mathfrak{G}$ we see from (15.11) and (15.12) that

$$(15.13) \quad \alpha(f) = E_r[\lim_{t\to\infty} K_r f(X_t)] = \lim_{t\to\infty} E_r K_r f(X_t).$$

It follows from (15.12) and (15.13) that $\alpha(f) = J(f)$ and thus

$$P_r\,[\lim_{t\to\infty} K_r f(X_t) = J(f)] = 1$$

as desired.

We may now prove the theorem.

Proof of Theorem 15.1. — For any x and $y \in \mathfrak{G}$

$$\tilde{G}f_y(x) = \tilde{G}f(x-y) = \tilde{G}f_{-x}(-y)$$

and thus

$$\tilde{G}f_{-y}(X_t) = \tilde{G}f_{-X_t}(y) = \tilde{G}f_{\tilde{X}_t}(y). \tag{15.14}$$

Hence applying Lemma 15.3 to the reverse process for the function f_{-y} we see that

$$P_r\left[\lim_{t\to\infty}\frac{Gf_{X_t}(y)}{Gr_{X_t}(0)} = J(f)\right] = 1.$$

By taking ratio's it follows from this that if $J(g) \neq 0$ then

$$P_r\left[\lim_{t\to\infty}\frac{Gf_{X_t}(y)}{Gg_{X_t}(0)} = \frac{J(f)}{J(g)}\right] = 1. \tag{15.15}$$

Finally, as $r > 0$ we see from (15.15) that (15.3) holds for a.e. x.

We will now show that for f and $g \in C_c$ we can strengthen the a.e. x to all x. From (15.3) we know that

$$P_0\left[\lim_{t\to\infty}\frac{G(-X_t, A+x)}{G(-X_t, B+x)} = \frac{|A|}{|B|}\right] = 1, \quad \text{a.e.} \quad x$$

for any two relatively compact sets such that $|B| > 0$. Assume now that A and B are also such that $|\partial A| = |\partial B| = 0$. By Propositions 2.1 and 2.2 of [11] we can then find compact sets A_n and B_n, open sets A'_n and B'_n and symmetric neighborhoods S_n of 0 such that

$$\begin{gathered} A_n + S_n \subset A \subset A'_n + S_n \\ B_n + S_n \subset B \subset B'_n + S_n \end{gathered}$$

and $|A'_n - A_n| < \dfrac{1}{n}$, $|B'_n - B_n| < \dfrac{1}{n}\cdot$ Hence there are points x_n and $y_n \in S_n$ such that

$$P_0\left[\lim_{t\to\infty}\frac{G(-X_t, A_n+x_n)}{G(-X_t, B'_n+x_n)} = \frac{|A_n|}{|B'_n|}\right] = 1$$

and

$$P_0\left[\lim_{t\to\infty}\frac{G(-X_t, A'_n+y_n)}{G(-X_t, B_n+y_n)} = \frac{|A'_n|}{|B_n|}\right] = 1.$$

But as

$$\frac{G(-X_t, A_n + x_n)}{G(-X_t, B'_n + x_n)} \leqslant \frac{G(-X_t, A)}{G(-X_t, B)} \leqslant \frac{G(-X_t, A'_n + y_n)}{G(-X_t, B_n + y_n)}$$

we see that

$$P_0\left[\frac{|A_n|}{|B'_n|} \leqslant \varliminf_{t\to\infty} \frac{G(-X_t, A)}{G(-X_t, B)} \leqslant \varlimsup_{t\to\infty} \frac{G(-X_t, A)}{G(-X_t, B)} \leqslant \frac{|A'_n|}{|B_n|}\right] = 1.$$

Letting $n \to \infty$ we see that

$$P_0\left[\lim_{t\to\infty} \frac{G(-X_t, A)}{G(-X_t, B)} = \frac{|A|}{|B|}\right] = 1.$$

Since $|\partial(A + x)| = |\partial A|$ it follows that for any $x \in \mathfrak{G}$

$$\text{(15.16)} \qquad P_x\left[\lim_{t\to\infty} \frac{G(-X_t, A)}{G(-X_t, B)}\right] = 1.$$

But, keeping B fixed (15.16) asserts that the measures $G(-X_t, \cdot)/G(-X_t, B)$ converge weakly a.s. P_x to $|\cdot|/|B|$. It follows that for any $f \in C_c$

$$\text{(15.17)} \qquad P_x\left[\lim_{t\to\infty} \frac{Gf(-X_t)}{G(-X_t, B)} = \frac{J(f)}{|B|}\right] = 1,$$

and thus by taking ratios we see that (15.3) holds for all x whenever $f \in C_c$ and $g \in C_c$. This completes the proof.

Corollary 15.1. — *Let f and $\varphi \in \Phi$ and let $g \in C_c$, $J(g) \neq 0$. Then for all $x \in \mathfrak{G}$,*

$$\text{(15.18)} \qquad P_x\left[\lim_{t\to\infty} \frac{(Gf_{X_t}, \varphi)}{Gg_{X_t}(0)} = \frac{J(f)J(\varphi)}{J(g)}\right] = 1.$$

Proof. — Note that

$$\int_{\mathfrak{G}} Gf_{X_t}(y)\varphi(y)\, dy = \int_{\mathfrak{G}} G(0, dz) \int_{\mathfrak{G}} f_{X_t}(z + y)\varphi(y)\, dy$$
$$= G\psi_{X_t}(0)$$

where $\psi(x) = \int_{\mathfrak{G}} f(x + y)\varphi(y)$. Using the fact that $\psi \in C_c$ we see that (15.18) follows from Theorem 15.1.

COROLLARY 15.2. — *Let* B *be a relatively compact set. For any* $f \in \Phi$ *and* $g \in \Phi$, $J(g) \neq 0$

$$(15.19) \quad P_x\left[\lim_{t\to\infty} \frac{G_B f_{X_t}(y)}{G g_{X_t}(0)} = \frac{J(f)}{J(g)} P_y(T_B = \infty)\right] = 1 \text{ a.e. } x \in \mathfrak{G}.$$

Moreover if f *and* $g \in C_c$ *then* (15.19) *holds for all* $x \in \mathfrak{G}$. *In addition if* $f \in C_c$, $g \in C_c$ *and* $\varphi \in \Phi$ *then for* $x \in \mathfrak{G}$,

$$(15.20) \quad P_x\left[\lim_{t\to\infty} \frac{(G_B f_{X_t}, \varphi)}{G g_{X_t}(0)} = \frac{J(f)}{J(g)} \int_{\mathfrak{G}} P_y(T_B = \infty)\varphi(y)\, dy\right] = 1.$$

Proof. — It suffices to prove these results for $f \geqslant 0$. Given such an f we can find an $r \in C_c^+$, $J(r) \neq 0$ such that

$$\sup_{y \in \overline{B}} f_{-y}(x) \leqslant r(x).$$

But then for any $y \in \overline{B}$,

$$\begin{aligned} 0 \leqslant G f_{X_t}(y) &= G f_{-y}(-X_t) \\ &= \int_{\mathfrak{G}} G(-X_t, dz) f_{-y}(z) \leqslant G r(-X_t) = G r_{X_t}(0). \end{aligned}$$

Thus if $G r_{X_t}(0) > 0$ we see that

$$(15.21) \qquad 0 \leqslant \frac{G f_{X_t}(y)}{G r_{X_t}(0)} \leqslant 1$$

and also

$$(15.22) \quad \frac{G f_{X_t}(y)}{G r_{X_t}(0)} - \int_{\overline{B}} H_B(y, dz) \frac{G f_{X_t}(z)}{G r_{X_t}(0)} = \frac{G_B f_{X_t}(y)}{G r_{X_t}(0)}.$$

The first two assertions of the corollary now follow from (15.22) and Theorem 15.1. Similarly, the last assertion of the corollary follows from (15.21), (15.22), Theorem 15.1 and Corollary 15.1.

COROLLARY 15.3. — *Let* B *be a relatively compact set. Let* $f, \varphi \in \Phi$ *and let* $g \in \Phi$, $J(g) \neq 0$. *Then for a.e.* $x \in \mathfrak{G}$

$$(15.23) \quad P_x\left[\lim_{t\to\infty} \frac{(f_{-X_t}, G_B\varphi)}{\hat{G} g(X_t)} = \frac{J(f)}{J(g)} \int_{\mathfrak{G}} P_y(\hat{T}_B = \infty)\varphi(y)\, dy\right] = 1.$$

Moreover (15.23) *holds for all* $x \in \mathfrak{G}$ *if* f *and* $g \in C_c$.

Proof. — Using our duality relations we see that

$$(f_{-X_t}, G_B\varphi) = (\varphi, \tilde{G}_B f_{-X_t}) = (\varphi, \tilde{G}_B f_{\tilde{X}_t}).$$

The assertions of the theorem now follow from Corollary 15.2.

An unsmoothed version of (15.23) will be given in Theorem 15.3.

For Brownian motion on R^d, $d \geqslant 3$ it is well-known that for any $B \in \mathcal{B}$ and any $f \in C_c$

$$\lim_{x \to \infty} E_x[f(X_{T_B}) | T_B < \infty] = \frac{(\mu_B, f)}{C(B)}.$$

Examples show that in general an i.d. process does not have such limits existing. Our next result will be to show that such limits always exist if we go to infinity along the path of the reverse process. We first state a smoothed version of this result. The unsmoothed version (valid for sets in $\mathcal{B}_4$) will be taken up after this.

THEOREM 15.2. — *Let* B *be any relatively compact set and let* $f, g \in C_c$, $J(g) \neq 0$ *and let* $\varphi \in C(\mathfrak{G})$. *Then for all* $x \in \mathfrak{G}$,

$$(15.24) \quad P_x \left[\lim_{t \to \infty} \frac{(f_{-X_t}, H_B\varphi)}{\tilde{G}g(X_t)} = \frac{J(f)}{J(g)} (\mu_B, \varphi) \right] = 1$$

and if $C(B) > 0$ *then for all* $x \in \mathfrak{G}$,

$$(15.25) \quad P_x \left[\lim_{t \to \infty} E f_{-X_t}[\varphi(X_{T_B}) | T_B < \infty] = \frac{(\mu_B, \varphi)}{C(B)} \right] = 1.$$

Proof. — It suffices to prove the theorem for functions f, φ and g that are non-negative. Henceforth in the proof we will assume this is the case. Now

$$(f_{-X_t}, H_B G\varphi) = (f_{-X_t}, G\varphi) - (f_{-X_t}, G_B\varphi)$$

and so by Corollaries 15.1 and 15.3 we see that for all $x \in \mathfrak{G}$,

$$P_x \left[\lim_{t \to \infty} \frac{(f_{-X_t}, H_B G\varphi)}{\tilde{G}g(X_t)} = \frac{J(f)}{J(g)} (\tilde{\Phi}_B, \varphi) \right] = 1$$

and so by Theorem 8.1, for all $x \in \mathfrak{G}$,

$$(15.26) \quad P_x \left[\lim_{t \to \infty} \frac{(f_{-X_t}, H_B G\varphi)}{\tilde{G}g_{-X_t}(0)} = \frac{J(f)}{J(g)} (\mu_B, G\varphi) \right] = 1.$$

Since $f \in C_c^+$ and $\overline{B}$ is compact we can find $r \in C_c^+$ such that $J(r) > 0$ and such that

$$\sup_{y \in \overline{B}} f_{-y}(x) \leqslant r(x)$$

and thus for all $y \in \overline{B}$,

$$0 \leqslant \tilde{G} f_{-X_t}(y) = G f_{-y}(X_t) \leqslant \tilde{G} r(X_t).$$

Now by Theorem 8.1 and the above

$$(15.27) \quad (f_{-X_t}, \Phi_B) = (\tilde{\mu}_B, \tilde{G} f_{-X_t}) \leqslant \tilde{C}(B) \tilde{G} r(X_t)$$

and thus if $\tilde{G} r(X_t) > 0$

$$0 \leqslant \frac{(f_{-X_t}, \Phi_B)}{\tilde{G} r(X_t)} \leqslant \tilde{C}(B).$$

Thus by dominated convergence and Theorem 15.1, for all $x \in \mathfrak{G}$,

$$(15.28) \qquad P_x \left[\lim_{t \to \infty} \frac{(f_{-X_t}, \Phi_B)}{\tilde{G} r(X_t)} = \frac{J(f)}{J(r)} \tilde{C}(B) \right] = 1.$$

Consequently there is a subsequence $t_n(\omega) \to \infty$ and a finite measure $\gamma(\omega, dy)$ supported on $\overline{B}$ such that for any $\psi \in C(\mathfrak{G})$,

$$(15.29) \qquad P_x \left[\lim_{t \to \infty} \frac{(f_{-X_{t_n}}, H_B \psi)}{\tilde{G} r(X_{t_n})} = (\gamma, \varphi) \right] = 1.$$

In particular, $G\varphi \in C(\mathfrak{G})$ whenever $\varphi \in C_c$ and thus from (15.29) and (15.26) we see that for all $x \in \mathfrak{G}$,

$$(15.30) \qquad P_x \left[(\gamma, G\varphi) = \frac{J(f)}{J(r)} (\mu_B, G\varphi) \right] = 1.$$

Since $\mathfrak{G}$ is 2'nd countable $C_c^+(\mathfrak{G})$ is separable and so we can find a sequence $\{\varphi_n\}$, $\varphi_n \in C_c^+$ such that any $\varphi \in C_c^+$ can be uniformly approximated by a subsequence of the φ_n all of whose supports are contained in some fixed compact set (depending on φ of course). From (15.30) it follows that

$$P_x \left[(\gamma, G\varphi_n) = \frac{J(f)}{J(r)} (\mu_B, G\varphi_n), \; n = 1, 2, \ldots \right] = 1$$

and thus by the usual 3 epsilon argument for any $\varphi \in C_c^+$

$$P_x\left[(\gamma, G\varphi) = \frac{J(f)}{J(r)}(\mu_B, G\varphi)\right] = 1.$$

Consequently, by Theorem 6.1 (for $\lambda = 0$) we see that

$$P_x\left[\gamma = \frac{J(f)}{J(r)}\mu_B\right] = 1. \tag{15.31}$$

If there was another subsequence of the measures

$$f_{-X_t}H_B/\tilde{G}r(X_t)$$

that converged weakly a.s. P_x the same argument would show the limit measure to be $[J(f)/J(r)]\mu_B$. Thus we have shown that for any $\psi \in C(\mathfrak{G})$

$$P_x\left[\lim_{t\to\infty}\frac{(f_{-X_t}, H_B\psi)}{\tilde{G}r(X_t)} = \frac{J(f)}{J(r)}(\mu_B, \psi)\right] = 1.$$

Equation (15.24) follows at once from this by taking ratios in using Theorem 15.1. Finally if $C(B) > 0$ then (15.25) follows from (15.22) by applying it to the functions φ and 1 and taking ratios. This completes the proof.

In general the smoothed results in Theorem 15.2 and in Corollary 15.3 are the best that are possible for arbitrary sets in $\mathcal{B}$. We will now show that the smoothing can be dropped for sets in $\mathcal{B}_4$.

THEOREM 15.3. — *Let* $B \in \mathcal{B}_4$ *and let* $\varphi \in C(\mathfrak{G})$. *Then for* $g \in C_c$, $J(g) \neq 0$ *and all* $x \in \mathfrak{G}$,

$$P_x\left[\lim_{t\to\infty}\frac{H_B\varphi(-X_t)}{\tilde{G}g(X_t)} = \frac{(\mu_B, \varphi)}{J(g)}\right] = 1 \tag{15.33}$$

and

$$P_x\left[\lim_{t\to\infty}E_{-X_t}[\varphi(X_{T_B})|T_B < \infty] = \frac{(\mu_B, \varphi)}{C(B)}\right] = 1. \tag{15.34}$$

Also, for any $\varphi \in C_c$ *and* $g \in C_c$ *such that* $J(g) \neq 0$,

$$P_x\left[\lim_{t\to\infty}\frac{G_B\varphi(-X_t)}{\tilde{G}g(X_t)} = \frac{1}{J(g)}\int_{\mathfrak{G}} P_y(\tilde{T}_B = \infty)\varphi(y)\,dy\right] = 1 \tag{15.35}$$

for all $x \in \mathfrak{G}$.

Proof. — It suffices to consider φ and g non-negative. Let A be compact and let $A \subset \mathring{B}$. By Proposition 2.1 of [11] we can then find a symmetric open neighborhood S of 0 such that $A - S \subset \mathring{B}$. Thus for any $y \in S$, $T_{A-y} \geqslant T_{\mathring{B}}$ and so for $f \in C_C^+$

$$(15.36) \quad E_x \int_{T_{A-y}}^{\infty} f(X_t + y)\, dt \leqslant E_x \int_{T_B}^{\infty} f_{-y}(X_t)\, dt.$$

Suppose f is such that $Gf(x) > 0$ for all $x \in \overline{B}$. Then given $\varepsilon > 0$ there is an open neighborhood N of 0, $N \subset S$ and $|\partial N| = 0$ such that for $y \in N$ and $x \in \overline{B}$,

$$(15.37) \quad (1 - \varepsilon)Gf(x) \leqslant Gf_{-y}(x) \leqslant (1 + \varepsilon)Gf(x).$$

For such an f then we see from (15.36) and (15.37) that for all $x \in \mathfrak{G}$,

$$\int_N H_A Gf(x + y)\, dy \leqslant (1 + \varepsilon)|N|\, H_B Gf(x).$$

From Theorem 15.2 we then obtain

$$P_x\left[\frac{(\mu_A, Gf)}{J(g)} \leqslant (1 + \varepsilon) \lim_{t \to \infty} \frac{H_B Gf(-X_t)}{\tilde{G}g(X_t)}\right] = 1$$

and as ε is arbitrary we see that for any $f \in C_C^+$ such that $Gf > 0$ on $\overline{B}$

$$(15.38) \quad P_x\left[\frac{(\mu_A, Gf)}{J(g)} \leqslant \lim_{t \to \infty} \frac{H_B Gf(-X_t)}{Gg(X_t)}\right] = 1.$$

Similarly, if U is a relatively compact open set, $U \supset \overline{B}$ and $Gf > 0$ on $\overline{B}$ then by an essentially the same argument we obtain

$$(15.39) \quad P_x\left[\overline{\lim_{t \to \infty}} \frac{H_B Gf(-X_t)}{\tilde{G}g(X_t)} \leqslant \frac{(\mu_U, Gf)}{J(g)}\right] = 1.$$

Let A_n, $n \geqslant 1$ be compact subsets of $\mathring{B}$, $A_1 \subset A_2 \subset \cdots$, $\bigcup_n A_n = \mathring{B}$ and let U_n, $n \geqslant 1$ be relatively compact open sets, $U_1 \supset U_2 \supset \cdots$, $\bigcap_n U_n = \bigcap_n \overline{U}_n = \overline{B}$. It then follows from (15.38) and (15.39) that

$$(15.40)\quad P_x\left[\frac{(\mu_{\mathring{B}}, Gf)}{J(g)} \leqslant \varliminf_{t\to\infty} \frac{H_B Gf(-X_t)}{\tilde{G}g(X_t)} \leqslant \varlimsup_{t\to\infty} \frac{H_B Gf(-X_t)}{\tilde{G}g(X_t)} \leqslant \frac{(\mu_{\mathring{B}}, Gf)}{J(g)}\right] = 1.$$

Now by Theorem 8.1 $(\mu_{\mathring{B}}, Gf) = (f, \tilde{\Phi}_{\mathring{B}})$ and $(\mu_{\overline{B}}, Gf) = (f, \Phi_{\overline{B}})$. Since $B \in \mathcal{B}_4$, $P_x(\tilde{T}_{\overline{B}} = \tilde{T}_{\mathring{B}}) = 1$ a.e. x and thus $\tilde{\Phi}_{\overline{B}} = \tilde{\Phi}_{\mathring{B}}$ a.e. But then $\tilde{\Phi}_B = \tilde{\Phi}_{\mathring{B}} = \tilde{\Phi}_{\overline{B}}$ a.e. and so by the uniqueness of the equilibrium measure $\mu_{\overline{B}} = \mu_{\mathring{B}} = \mu_B$. Thus from (15.40) we see that for any $f \in C_c$ such that $Gf > 0$ on $\overline{B}$

$$(15.41)\quad P_x\left[\lim_{t\to\infty} \frac{H_B Gf(-X_t)}{\tilde{G}g(X_t)} = \frac{(\mu_B, Gf)}{J(g)}\right] = 1.$$

Actually, (15.41) holds for all $f \in C_c^+$ because if $Gf > 0$ on $\overline{B}$ and $\varphi \in C_c^+$ then $G(f + \varphi) > 0$ on $\overline{B}$. Hence (15.41) holds for f and $f + \varphi$ and therefore for φ.

We will now show that (15.33) holds by a weak compactness argument using (15.41) to identify the limit function. To this end let U be a relatively compact open set such that $U \supset \overline{B}$. By Proposition 2.1 of [11] we can find a symmetric open neighborhood N of 0 such that $\overline{B} + N \subset U$. Then for any $y \in N$, $\Phi_B(x) \leqslant \Phi_U(x + y)$. Let $h \in C_c^+$, $J(h) = 1$ have support contained in N. Then

$$\Phi_B(-X_t) \leqslant \int_{\mathfrak{G}} \Phi_U(y - X_t)h(y)\, dy = (h_{-X_t}, \Phi_U)$$

and so by Theorem 15.2 for any $x \in \mathfrak{G}$,

$$(15.42)\qquad P_x\left[\varlimsup_{t\to\infty} \frac{\Phi_B(-X_t)}{\tilde{G}g(X_t)} \leqslant \frac{C(U)}{J(g)}\right] = 1.$$

It follows from (15.42) that there is a sequence $t_n(\omega) \to \infty$ and a finite measure $\gamma(\omega, dy)$ supported on $\overline{B}$ such that for any $\varphi \in C(\mathfrak{G})$,

$$(15.43)\qquad P_x\left[\lim_{n\to\infty} \frac{H_B\varphi(-X_{t_n})}{\tilde{G}g(X_{t_n})} = (\gamma, \varphi)\right] = 1$$

for all $x \in \mathfrak{G}$. Equation (15.33) now follows from (15.41) and (15.43) by essentially the same argument used to prove the smoothed version in Theorem 15.2. We will omit these details. Equation (15.34) is a direct consequence of (15.33) and the fact that $C(B) > 0$ for any $B \in \mathcal{B}_4$. Let $r \in C_c$, $J(r) = 1$. Then

$$(\tilde{G}g_{-X_t}, r) = (g_{-X_t}, Gr) = (g, Gr_{X_t})$$

and thus by Corollary 15.1 and Theorem 15.1 for all $x \in \mathfrak{G}$,

$$P_x\left[\lim_{t\to\infty} \frac{G\varphi(-X_t)}{\tilde{G}g_{-X_t}(0)} = \lim_{t\to\infty} \frac{G\varphi_{X_t}(0)}{(g,\ Gr_{x_t})} \frac{(\tilde{G}g_{-X_t}, r)}{Gg_{-X_t}(0)} = \frac{J(\varphi)}{J(g)}\right] = 1.$$

Equation (15.35) now follows from this relation and (15.33) via the first passage relation. This completes the proof.

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