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PROJECTIVE INVARIANTS
OF AN ORTHOGONAL ENNUPLE
IN A FINSLER SPACE

by H. D. PANDE (*)

1. Introduction.

We consider an $n$-dimensional Finsler space $F_n$ with the fundamental metric function $F(x, \dot{x})$. This fundamental function is positive homogeneous of the first degree in $\dot{x}^i$, it is $>0$ for $\sum(\dot{x}^i)^2 \neq 0$ and the quadratic form $(\delta^2 F^2/\delta x^i \delta x^j)\xi^i \xi^j$ is positive definite in the variables $\xi^i$. The metric tensor is given by

$$g_{ij}(x, \dot{x}) = \frac{1}{2} \delta_{ij} \partial^2 F^2(x, \dot{x}) \quad (1), \quad (2)$$

This tensor is symmetric in the indices $i, j$ and positive homogeneous of degree zero in $\dot{x}^i$. The contravariant components $g^{ij}(x, \dot{x})$ of the metric tensor is determined by

$$g^{ij}(x, \dot{x}) g_{jk}(x, \dot{x}) = \delta^i_k \quad \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{if } k \neq i \end{cases} \quad (1.2)$$

The covariant components of the unit vector along the direction of the element of support $(x^i, \dot{x}^i)$ are given by

$$l_i(x, \dot{x}) = \delta_i F(x, \dot{x}). \quad (1.3)$$

The covariant derivative of a vector $X^i(x, \dot{x})$, depending on the element of support, with respect to $x^k$ in the sense of

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(1) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

(2) Numbers in brackets refer to the references at the end of the paper.
Cartan is given by
\[(1.4) \quad X^i_j(x, \dot{x}) = (\delta_k X^i) - (\delta_j X^i) G^i_k + X^i \Gamma^i_j_k,\]
where
\[(1.5a) \quad G^i_k(x, \dot{x}) \overset{\text{def}}{=} \delta_k G^i(x, \dot{x}),\]
\[(1.5b) \quad 2G^i(x, \dot{x}) \overset{\text{def}}{=} \gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k,\]
\(\gamma^i_{jk}(x, \dot{x})\) being the Christoffel's symbols of second kind [1] and \(\Gamma^i_{jk}(x, \dot{x})\) are the Cartan's connection coefficients symmetric in their lower indices and homogeneous of degree zero in their directional arguments. We have [1]
\[(1.6) \quad G^i_{jk}(x, \dot{x}) \dot{x}^j = \Gamma^i_{jk}(x, \dot{x}) \dot{x}^j = G^i_k(x, \dot{x}),\]
where \(G^i_{jk}(x, \dot{x}) \overset{\text{def}}{=} \delta_k G^i_j((x, \dot{x}).\)

Let \(\lambda_{(a)} (a = 1, 2, \ldots, n)\) be the unit tangents of \(n\)-congruences of an orthogonal ennuple. The subscript « \(a\) » in the paranthesis simply distinguishes one congruence from the other. The covariant and contravariant components of \(\lambda_{(a)}\) will respectively be denoted by \(\lambda^i_{(a)}\) and \(\lambda_i^{(a)}\). Since \(n\)-congruences are mutually orthogonal, we have [2]
\[(1.7) \quad g_{ij}(x, \dot{x}) \lambda^i_{(a)} \lambda^j_{(b)} = \delta_{ab},\]
where the Kronecker delta \(\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}\). We have the Ricci coefficients of rotation, given by [2, 3]
\[(1.8) \quad Y_{abc}(x, \dot{x}) \overset{\text{def}}{=} \lambda^i_{(a)} \lambda^j_{(b)} \lambda^c_{(c)},\]
where the symbol \(\dagger\) denotes the covariant derivative with respect to \(\dot{x}^k\) in the sense of Cartan and
\[(1.9) \quad \mu^i_{(m)}(x, \dot{x}) \overset{\text{def}}{=} \sum_h Y_{mh} \lambda^i_{(h)}\).

The geometric entities \(\mu^i_{(m)}(x, \dot{x})\) are called the first curvature vector of a curve of congruence in Finsler space [3].

2. Projective transformation.

The equation of a geodesic
\[(2.1) \quad \frac{d^2 x^i}{ds^2} + \Gamma^i_{jk}(x, \dot{x}) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0\]
assumes the following form by the transformation of its parameter $s$ to $t$ [4]:

$$\dot{x}^i \left( \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i(x, \dot{x}) \dot{x}^j \dot{x}^k \right) - \dot{x}^i \left( \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i(x, \dot{x}) \dot{x}^j \dot{x}^k \right) = 0,$$

where

$$\Gamma_{jk}^i(x, \dot{x}) = \Gamma_{kj}^i(x, \dot{x}).$$

The equation (2.2) remains unchanged if we replace the Cartan's connection coefficient $\Gamma_{jk}^i(x, \dot{x})$ by a new symmetric coefficient $\Gamma_{jk}^i(x, \dot{x})$, given by [6]

$$\Gamma_{jk}^i(x, \dot{x}) \equiv \Gamma_{jk}^i(x, \dot{x}) + 2\delta_{ij}p_k + p_{jk}\dot{x}^i,$$

where $p_k(x, \dot{x})$ is a covariant vector, positively homogeneous of degree zero in its directional arguments and

$$p_{jk}(x, \dot{x}) \equiv \delta_{jk}p_k(x, \dot{x}).$$

Définition 2.1. — Let $F_n$ and $\bar{F}_n$ be two spaces with fundamental tensor $g_{ij}(x, \dot{x})$ and $\bar{g}_{ij}(x, \dot{x})$ at the corresponding points. Then the spaces are said to be in geodesic correspondence if their geodesics are the same and we shall call (2.4) a "projective change" of the Cartan's function $\Gamma_{jk}^i(x, \dot{x})$.

Contracting (2.4) with respect to the indices $i$ and $j$, we get

$$\Gamma_{ik}^i(x, \dot{x}) = \Gamma_{ik}^i(x, \dot{x}) + (n + 1)p_k(x, \dot{x}).$$

Differentiating (2.6) with respect to $\dot{x}^i$, we obtain

$$\delta_i \Gamma_{ik}^i(x, \dot{x}) = \delta_i \Gamma_{ik}^i(x, \dot{x}) + (n + 1)p_{ik}(x, \dot{x}).$$

3. Projective invariants.

Theorem 3.1. — If $\lambda^i_{(a)}(x)$ and $\lambda^{(a)\dot{i}}(x)$ are the contravariant and covariant components of an orthogonal ennule, then the following geometric entities are invariant under the projective change:

$$\Lambda^i_k(x, \dot{x}) \equiv \lambda^i_{(a)k} - \frac{1}{n + 1} \lambda^i_{(a)} \left\{ 2 \sum \lambda_{(b)m} \delta^i_j \lambda^{(b)\dot{m}}_{(a)\dot{k}} \right\},$$
Proof. — If we denote by $\lambda_{(a)k}$ the covariant derivative of $\lambda_{(a)}$ in the sense of Cartan for the connection coefficients $\Gamma_{jk}^i(x, \dot{x})$, then we have

$$\lambda_{(a)k} = \partial_t \lambda_{(a)} + \lambda_{(a)} \Gamma_{jk}^i.$$  

Hence we get in consequence of (2.4)

$$\lambda_{(a)k}^i - \lambda_{(a)i}^i = \lambda_{(a)} \{2 \delta_{(i}^j p^k) + p_{jk} \dot{x}^i \}.$$  

Multiplying (3.4) by $\lambda_{(a)i}$ throughout and summing with respect to $a$ and using the orthogonality condition (1.7), we obtain

$$\sum_a \lambda_{(a)k} (\lambda_{(a)k}^i - \lambda_{(a)i}^i) = (n + 1)p_k.$$  

Eliminating the vector $p_k(x, \dot{x})$ from equations (3.4) and (3.5), we get

$$\lambda_{(a)k}^i - \lambda_{(a)i}^i = \frac{1}{n + 1} \lambda_{(a)} \left[ \delta_{(i}^j \sum_b \lambda_{(b)m} (\lambda_{(b)k}^m - \lambda_{(b)i}^m) + \delta_{(i}^j \sum_b \lambda_{(b)m} (\lambda_{(b)j}^m - \lambda_{(b)i}^m) \right].$$

Again, with the help of (2.6), equation (3.5) yields

$$\sum_a \lambda_{(a)k} (\lambda_{(a)k}^i - \lambda_{(a)i}^i) = \Gamma_{jk}^{*i} - \Gamma_{jk}^{*i},$$

which gives us (3.2).

Theorem 3.2. — When $F^n$ and $F^n_1$ are in geodesic correspondence, we have the following geometric entities which are invariant under the projective change:

$$C_k(x, \dot{x}) \overset{\text{def}}{=} \lambda_{(a)k}^i - \frac{1}{n + 1} \lambda_{(a)} \{2 \delta_{(i}^j \Gamma_{jk}^{*i} + \dot{x}^i \dot{x}^j \Gamma_{jk}^{*i} \},$$

and

$$C_k^{*i}(x, \dot{x}) = \lambda_{(a)k}^i - \frac{1}{n + 1} \lambda_{(a)} \{ \sum_b 2 \lambda_{(b)m} \delta_{(i}^j (\lambda_{(b)i}^m) + \dot{x}^i \dot{x}^j \Gamma_{jk}^{*i} \}. $$
Proof. — Using equations (2.6), (2.7) and (3.4), we get
\[
\lambda_{(o)k} - \lambda_{(o)\kappa} = \frac{1}{n+1} \lambda_{(o)} \left[ \sum_b \delta_j (\Gamma_{j\gamma}^{\gamma})^k - \Gamma_{j\gamma}^{\gamma} \right] + \delta_k (\lambda_{(b)j} - \lambda_{(b)\kappa}) + \frac{\lambda_{(o)} \delta_j (\Gamma_{j\gamma}^{\gamma})^k - \Gamma_{j\gamma}^{\gamma}}{n+1},
\]
which yields the result (3.8).

Again, eliminating \( p_k(x, \dot{x}) \) and \( p_{\kappa}(x, \dot{x}) \) from equations (2.7), (3.4) and (3.5), we obtain
\[
\lambda_{(o)k} - \lambda_{(o)\kappa} = \frac{1}{n+1} \lambda_{(o)} \sum_b \delta_j (\lambda_{(b)j} - \lambda_{(b)\kappa}) + \frac{\lambda_{(o)} \delta_j (\Gamma_{j\gamma}^{\gamma})^k - \Gamma_{j\gamma}^{\gamma}}{n+1},
\]
which gives us (3.9).

Theorem 3.3. — When \( F_n \) and \( F_n^\gamma \) are in geodesic correspondence, we have the following projective invariant geometric entities:
\[
S_{abc}(x, \dot{x}) = \frac{1}{n+1} \left( \delta_{ab} \lambda_{(o)}^{\gamma} \Gamma_{j\gamma}^{\gamma} + \delta_{be} \lambda_{(o)}^{\gamma} \Gamma_{j\gamma}^{\gamma} \right)
\]
and
\[
S_{a\gamma}(x, \dot{x}) = \frac{1}{n+1} \left( \delta_{ca} Y_{b\gamma} - \delta_{ca} Y_{b\gamma} \right) - \frac{1}{n+1} \left( \delta_{ca} Y_{b\gamma} - \delta_{ca} Y_{b\gamma} \right) - \frac{1}{n+1} \left( \delta_{ca} Y_{b\gamma} - \delta_{ca} Y_{b\gamma} \right),
\]
and
\[
S_b(x, \dot{x}) = \frac{1}{n+1} \left( \delta_{ab} \lambda_{(c)}^{\gamma} \Gamma_{j\gamma}^{\gamma} + \delta_{bc} \lambda_{(c)}^{\gamma} \Gamma_{j\gamma}^{\gamma} \right),
\]
where \( Y_{abc} \) are Ricci coefficients of rotation.

Proof. — Multiplying (3.10) by the product \( \lambda_{(b)l} \lambda_{(c)}^{\gamma} \) and using the orthogonality relation (1.7), we get
\[
Y_{abc} = \frac{1}{n+1} \left( \delta_{ab} \lambda_{(c)}^{\gamma} \Gamma_{j\gamma}^{\gamma} + \delta_{bc} \lambda_{(c)}^{\gamma} \Gamma_{j\gamma}^{\gamma} \right) + \lambda_{(c)}^{\gamma} \lambda_{(b)j} \delta_j (\Gamma_{j\gamma}^{\gamma})^k = Y_{abc} - \frac{1}{n+1} \left( \delta_{ab} \lambda_{(c)}^{\gamma} \Gamma_{j\gamma}^{\gamma} \right) + \delta_{bc} \lambda_{(c)}^{\gamma} \Gamma_{j\gamma}^{\gamma} + \lambda_{(c)}^{\gamma} \lambda_{(b)j} \delta_j (\Gamma_{j\gamma}^{\gamma})^k,
\]
where the projectively transformed Ricci coefficients of rotation are given by

\[(3.16) \quad \mathring{Y}_{abc} \overset{\text{def}}{=} \lambda_{(a)}^i \mathring{\lambda}_{(b)}^i \lambda_{(c)}^i.\]

Similarly, multiplying (3.11) by the product \(\lambda_{(a)}^i \lambda_{(c)}^i\) and using the orthogonal relation (1.7), we obtain (3.13).

Again, multiplying (3.7) by \(\lambda_{(a)}^i\) and making use of (1.7), we get

\[(3.17) \quad \sum_a \mathring{Y}_{aab} - \lambda_{(b)}^k \mathring{\Gamma}_{\gamma k}^{ae} = \sum_a Y_{aab} - \lambda_{(b)}^k \Gamma_{\gamma k}^{ae},\]

which shows that \(S_b(x, \mathring{x})\) are invariant under the projective change.

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